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Scale Invariance Explains Quadratic Damping: Case of Insect Wings Flapping

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Abstract

In many application areas, the dynamics of a system is described by a harmonic oscillator with quadratic damping. In particular, this equation describes the flapping of the insect wings and is, thus, used in the design of very-small-size unmanned aerial vehicles. The fact that the same equation appears in many application areas, from stability of large structures to insect flight, seems to indicate that this equation follows from some fundamental principles. In this paper, we show that this is indeed the case: this equation can be derived from natural invariance requirements.

Keywords: quadratic damping, insect flight, nano AUVs, scale invariance

1 Formulation of the Problem

Quadratic damping is ubiquitous. The dynamics of an ideal harmonic oscillator, without damping, is described by the following differential equation:

$$\ddot{x} = -f(x) + F(t), \tag{1}$$

where $F(t)$ is the acceleration caused by an outside force applied to this oscillator, and $f(x)$ is an acceleration caused by the deviation x from the stable state. In the first approximation, we have Hooke's law $f(x) = k_1 \cdot x/M$ for some coefficient k_1, where M is the object's mass. A more accurate description requires the use of higher order terms. For example, in many practical situations, it is efficient to use $f(x) = (k_1 \cdot x + k_3 \cdot x^3)/M$.

In reality, there is damping. In many cases, damping is linear (proportional to the velocity). For this type of damping, the differential equation describing the dynamics of the oscillators takes the following form:

$$\ddot{x} = -b \cdot \dot{x} - f(x) + F(t), \tag{2}$$

for some value $b > 0$.

In other cases, however, the damping is a quadratic function of velocity:

$$\ddot{x} = -b \cdot \dot{x} \cdot |\dot{x}| - f(x) + F(t). \tag{3}$$

Such cases are ubiquitous in mechanics; see, e.g., [1, 2, 3, 4, 5, 13, 18, 19]. In particular, the equation (3) describes how, when the insect flies, the angle x between the insect's wing surface and the horizontal plane changes with time; see, e.g., [6, 8, 12, 14, 15, 17]. This equation is also used in the design of very-small-size unmanned aerial vehicles (UAV) patterned after the insect wings; see, e.g., [9, 10, 11, 12, 14, 15, 17, 20, 21].

Why quadratic damping. The fact that the same quadratic damping equation appears in many different situations, from stability of large-scale structures to insect flight and nano size UAVs seems to indicate there should be a fundamental explanation for this equation, an explanation not depending on the specifics of the corresponding dynamical situation.

In this paper, we show that indeed, this equation can be derived from the natural invariance requirements.

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2 Main Result: Scale Invariance Explains Quadratic Damping

Scale invariance: a brief reminder. In describing the physical world, we want to describe how physical quantities change. The way we describe this change is by describing these quantities in terms of numbers. However, the corresponding number depends on what measuring unit we use. For example, if we previously measured time in hours and then switch to seconds, then all numerical values describing time will multiply by 60. In general, if we replace the original measuring unit with a unit which is $\lambda > 0$ times smaller than the previous one, then each numerical value t is replaced by a new value $\widetilde{t} = \lambda \cdot t$.

The selection of a measuring unit is purely a matter of convenience, so it makes sense to require that the physical formula do not change if we simply change the corresponding measuring unit. Of course, if we change the unit for one quantity – e.g., time – we should also appropriately change a measuring unit for related quantities – e.g., for velocity, otherwise, e.g., the formula $v = d/t$ that describe the average velocity v as ratio of distance d and time t will stop working if we only change the unit for time but not for velocity. The corresponding invariance is known as *scale invariance.*

Comment. While many physical processes are scale invariant, there are certain cases where there is a preferred unit of time. For example, in quantum field theory, there is a so-called Planck time $\sqrt{\hbar \cdot G/c^5} \approx 5.4 \cdot 10^{-44}$ sec, where $\hbar$ is the Planck's constant (the main constant in quantum physics), G is the gravitational constant (that describes the strength of the gravitational interaction), and c is the speed of light; see, e.g., [7, 16]. However, this time interval is so much smaller than time intervals in which we are interested that we can safely ignore the existence of this unit and assume that all the equations are indeed scale-invariant.

What does scale invariance mean for the damping formula. Let us consider the general case of possible dampings, i.e., the formula

$$\ddot{x} = -D(\dot{x}) - f(x) + F(t), \tag{4}$$

for some function $D(v)$.

The values $f(x)$ and $F(t)$ are beyond our control, so we will be dealing only with the acceleration and with the damping terms. In effect, for the purpose of finding out which dampings are physically reasonable, we will be thus considering the following simplified equation:

$$\ddot{x} = -D(\dot{x}). \tag{5}$$

This equation contains two physical quantities: angle and time. For angles, there is a fixed unit – e.g., a radian, so scaling does not make physical sense. The only quantity to which it makes sense to apply scaling is time t. It is therefore reasonable to require that the equation (5) not change if we re-scale time, i.e., if, for some positive constant $\lambda > 0$, we replace t with $\widetilde{t} = \lambda \cdot t$.

The angular velocity $\dot{x}$ is, in effect, the ratio of changes in angle Δx and time Δt: $\dot{x} = \Delta x/\Delta t$. When we replace t with $\widetilde{t} = \lambda \cdot t$ and thus, replace Δt with $\Delta\widetilde{t} = \lambda \cdot \Delta t$, we thus get a re-scaled value of the velocity $\widetilde{\dot{x}} = \Delta x/\Delta\widetilde{t}$ which is related to the original value by the formula

$$\widetilde{\dot{x}} = \frac{\Delta x}{\Delta\widetilde{t}} = \frac{\Delta x}{\lambda \cdot \Delta t} = \frac{1}{\lambda} \cdot \frac{\Delta x}{\Delta t} = \frac{1}{\lambda} \cdot \dot{x}. \tag{6}$$

Similarly, the second derivative $\ddot{x}$ – which is, in effect, the ratio between changes in velocity $\Delta\dot{x}$ and in time Δt – gets transformed into

$$\widetilde{\ddot{x}} = \frac{\Delta\widetilde{\dot{x}}}{\Delta\widetilde{t}} = \frac{\frac{1}{\lambda} \cdot \Delta\dot{x}}{\lambda \cdot \Delta t} = \frac{1}{\lambda^2} \cdot \frac{\Delta\dot{x}}{\Delta t} = \frac{1}{\lambda^2} \cdot \ddot{x}. \tag{7}$$

Scale invariance means that if the formula (5) was true in the original time scale, then it will be true in the new time scale, i.e., that we will have

$$\widetilde{\ddot{x}} = -D\left(\widetilde{\dot{x}}\right). \tag{8}$$

Substituting expressions (6) and (7) into the formula (8), we conclude that for all $\lambda > 0$, we have:

$$\frac{1}{\lambda^2} \cdot \ddot{x} = -D\left(\frac{1}{\lambda}\dot{x}\right). \tag{9}$$

Substituting the expression (5) for $\ddot{x}$ into this formula, we conclude that for all $\dot{x}$ and for all λ, we have

$$-\frac{1}{\lambda^2} \cdot D(\dot{x}) = -D\left(\frac{1}{\lambda}\dot{x}\right). \tag{10}$$

In particular, for every positive real number v, we can take $\dot{x} = 1$ and $\lambda = 1/v$. In this case, $1/\lambda = v$, $1/\lambda^2 = v^2$, and thus, the formula (10) takes the form

$$D(v) = b_+ \cdot v^2, \tag{11}$$

where we denoted $b_+ \stackrel{\text{def}}{=} D(1)$.

Similarly, for every negative real number $v < 0$, we can take $\dot{x} = -1$ and $\lambda = -1/v$. In this case, $1/\lambda = -v$, $1/\lambda^2 = v^2$, and thus, the formula (10) takes the form

$$D(v) = b_- \cdot v^2, \tag{12}$$

where we denoted $b_+ \stackrel{\text{def}}{=} D(-1)$.

Additional symmetry. The formulas (11) and (12) almost explain the formula for quadratic damping – except that in that formula, there is only one parameter b while here, we have two different parameters b_+ and b_-. To finalize the derivation of the quadratic damping formula, we need to take into account one more physical invariance – related to the fact that, in general, while the absolute value of the angle is absolute and cannot be meaningfully changed by scaling, the sign of the angle is rather arbitrary: it depends on what direction we select. It is therefore reasonable to require that the general formula (5) not change if we simply change the sign of the angle.

If we change the sign of the angle, i.e., multiply all the angle values by -1, then all the values of the angular difference Δx will also be multiplied by -1 and thus, the numerical value of the angular velocity $\dot{x} = \Delta x/\Delta t$ will be multiplied by -1. Similarly, since all the values of the difference in velocities $\Delta\dot{x}$ will be multiplied by -1, the numerical value of the angular acceleration $\ddot{x} = \Delta\dot{x}/\Delta t$ will also be multiplied by -1.

So, this invariance means that if the formula (5) was true, then it should remain true if we replace $\dot{x}$ with $-\dot{x}$ and $\ddot{x}$ and $-\ddot{x}$, i.e., we should have

$$-\ddot{x} = -D(-\dot{x}). \tag{13}$$

Substituting the expression (5) for $\ddot{x}$ into this formula, we conclude that

$$D(\dot{x}) = -D(-\dot{x}),$$

i.e., equivalently, that

$$D(-\dot{x}) = -D(\dot{x}). \tag{14}$$

In particular, for each $\dot{x} = v > 0$, from using the expression (11) for $D(v)$, we conclude that

$$D(-v) = -D(v) = -b_+ \cdot v^2, \tag{15}$$

i.e., by comparing with the formula (12), that $b_- = -b_+$.

The formulas (11) and (15) can be combined into a single formula

$$D(\dot{x}) = b_+ \cdot \dot{x} \cdot |\dot{x}|, \tag{16}$$

which is exactly the damping formula that we wanted to explain. Thus, the quadratic damping formula can indeed by derived from scale invariance, with $b = b_+$.

Comment. The only minor thing that we are missing in this derivation is the requirement that $b = b_+ > 0$. This requirement is easy to explain: if we have $b < 0$, then an object moving with some velocity $v > 0$ will start accelerating in the same direction, thus increasing its kinetic energy. Such a self-acceleration contradicts the energy conservation law. In contrast, when $b < 0$, the object decelerates and loses kinetic energy – which is exactly what damping is about.

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References

[1] Andronov, A.A., and C.E. Chaikin, *Theory of Oscillations*, Palala Press, 2018.

[2] Andronov, A.A., Vitt, A.A., and S.E. Khaikin, *Theory of Oscillations*, Nauka Publ., Moscow, 1981 (in Russian).

[3] Bogoliubov, N.N., and Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Nauka Publ., Moscow, 1974 (in Russian).

[4] Cvetićanin, L., Vibrations of the nonlinear oscillator with quadratic nonlinearity, *Physica A*, vol.341, pp.123–128, 2004.

[5] Cvetićanin, L., Oscillator with string quadratic damping force, *Publications de l'Institut Mathématique*, vol.85, no.99, pp.119–130, 2009.

[6] Dickinson, M.H., Lehmann, F.O., and S.P. Sane, Wing rotation and the aerodynamic basis of insect flight, *Science*, vol.284, no.5422, pp.1954–1960, 1999.

[7] Feynman, R., Leighton, R., and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.

[8] Griffin, C.D., Browning, P., Hamburg, S., Cox, J., Huebsch, W., Katzner, E., and T. Katzner, Free flight observations and aerodynamic analysis for biologically-inspired optimization, *Proceedings of the 55th AIAA Aerospace Sciences Meeting*, Grapevine, Texas, January 9–13, 2017.

[9] Hassanalian, M., and A. Abdelkefi, Effective design of flapping wing actuation mechanisms: theory and experiments, *Proceedings of the 54th AIAA Aerospace Sciences Meeting*, San Diego, California, January 4–8, 2016.

[10] Hassanalian, M., Abdelkefi, A., Wei, M., and S. Ziaei-Rad, A novel methodology for wing sizing of bio-inspired flapping wing micro air vehicles: theory and prototype, *Acta Mechanica*, vol.228, no.3, pp.1097–1113, 2017.

[11] Hassanalian, M., Khaki, H., and M. Khosrawi, A new method for design of fixed wing micro air vehicle, *Journal of Aerospace Engineering*, vol.229, pp.837–850, 2014.

[12] Hassanalian, M., Throneberry, G., and A. Abdelkefi, Wing shape and dynamic twist design of bio-inspired nano air vehicles for forward flight purposes, *Aerispace Science and Engineering*, vol.68, pp.518–529, 2017.

[13] Magnus, K., *Schwingungen*, Teubner, Stuttgart, 1997.

[14] Shimoyama, L., Miura, H., Suzuki, K., and Y. Ezura, Insect-like microrobots with external skeletons, *IEEE Control Systems Magazine*, vol.13, no.1, pp.37–41, 1993.

[15] Tanaka, H., Hoshino, K., Matsumoto, K., and I. Shimoyama, Flight dynamics of a butterfly-type ornithopter, *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems IROS'2005*, Edmonton, Alberta, Canada, pp.2706–2711, August 2–6, 2005.

[16] Thorne, K.S., and R.D. Blandford, *Modern Classical Physics: Optics, fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2017.

[17] Throneberry, G., Hassanalian, M., and A. Abdelkefi, Optimal design of insect wing shape for hovering nano air vehicles, *Proceedings of the Joint 58th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference and 25th AIAA/AHS Adaptive Structures Conference*, Grapevine, Texas, January 9–13, 2017.

[18] Timoshenko, S., *Vibration Problems in Engineering*, Blurb, San Francisco, California, 2019.

[19] Weaver Jr., W., and S.P. Timoshenko, *Vibration Problems in Engineering*, Wiley, New York, 2013.

[20] Whitney, J.P., and R.J. Wood, Conceptual design of flapping-wing micro air vehicles, *Bioinspirtion and Biomimetics*, vol.7, no.3, Paper 036001, 2012.

[21] Wood, R.J., Finio, B., Karpelson, M., Ma, K., Pérez-Arancibia, N.O., Sreetharan, P.S., Tanaka, H., and J.P. Whitney, Progress on 'pico' air vehicles, *International Journal of Robotic Research*, vol.31, no.11, pp.1292–1302, 2012.