Avoiding Einstein-Podolsky-Rosen (EPR) Paradox: Towards a More Physically Adequate Description of a Quantum State

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Abstract

The famous EPR paradox shows that if we describe quantum particles in the usual way – by their wave functions – then we get the following seeming contradiction. If we entangle the states of the two particles, then move them far away from each other, and measure the state of the first particle, then the state of the second particle immediately changes – which contradicts to special relativity, according to which such immediate-action-at-a-distance is not possible. It is known that, from the physical viewpoint, this is not a real paradox: if we measure any property of the second particle, the results will not change whether we perform the measurement on the first particle or not. What the above argument shows is that the usual wave function description of a quantum state does not always adequately describe the corresponding physics. In this paper, we propose a new, more physically adequate description of a quantum state, a description in which there is no EPR paradox: measurements performed at the first particle does not change the state of the remote second one.

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Keywords: Einstein-Podolsky-Rosen paradox, quantum physics, density operator

1 EPR Paradox: A Brief Reminder

Quantum physics: a brief reminder. In quantum physics, a state of a particle is described by a complex-valued *wave function* $\psi(x)$ for which

$$\int |\psi(x)|^2 \, dx = 1;$$

see, e.g., [2, 7]. A state of an N-particle system is similarly described by a complex-valued function $\psi(x_1, \cdots, x_N)$. In particular, when we have a collection of N independent particles with wave functions $\psi_i(x_i)$, then their joint state is described by the product

$$\psi(x_1, \cdots, x_N) = \prod_{i=1}^{N} \psi_i(x_i).$$

When we measure the location of a particle, then we can get different locations x with different probabilities. For each spatial region U, the probability that the resulting location will be in U is equal to $\int_U |\psi(x)|^2 \, dx$. The probability to be found anywhere should be equal to 1 – this explains the above restriction on the integral of $|\psi(x)|^2$. In this sense, the expression $|\psi(x)|^2$ serves as the probability density of the corresponding probability distribution (see, e.g., [6]).

For independent particles, this implies that

$$|\psi(x_1, \cdots, x_N)|^2 = \prod_{i=1}^{N} |\psi_i(x_i)|^2,$$

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i.e., that indeed, in accordance with the meaning of independence, the probabilities to find different particles at different locations are independent random events.

Notion of a Hilbert space. In contrast to classical physics, where a state of a particle is characterized by finitely many parameters – e.g., its coordinates and velocity – to fully describe the state of a quantum particle, we need to describe infinitely many values $\psi(x)$. In this sense, a wave function can be viewed as an infinite-dimensional vector with infinitely many different components $\psi(x)$. In these terms, the expression $\int |\psi(x)|^2 \, dx$ becomes a natural infinite-dimensional analogue of the usual formula for the square $\|a\|^2 = \sum\limits_{i=1}^{n} a_i^2$ of the length $\|a\|$ of a finite-dimensional vector $a = (a_1, \cdots, a_n)$. In view of this analogy, we can also consider a formula

$$\langle \psi_1, \psi_2 \rangle = \int \psi_1(x) \cdot \psi_2^*(x) \, dx$$

similar to the usual dot (scalar) product $\langle a, b \rangle = \sum\limits_{i=1}^{n} a_i \cdot b_i^*$ of the two complex-valued vectors $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, where for each complex number $z = p + q \cdot i$ (i $\overset{\text{def}}{=} \sqrt{-1}$), z^* denotes its complex conjugate $z^* = a - b \cdot i$. With this definition, the linear space of all the complex-valued functions becomes an infinite-dimensional analogue of the n-dimensional Euclidean space. This analogue is known as the *Hilbert space*.

Quantum measurements: example. According to quantum theory, the measurement not only returns the value x of the measured quantity – it also changes the state of the corresponding particle into a new state in which the particle is located at the point x with probability 1.

The coordinate measurement can be described as follows. For each point x, we consider the space S_x of all functions located on exactly this point x (strictly speaking, this has to be a generalized function (see, e.g., [3, 8]) – equal to infinity at 0 and to 0 everywhere else). To each of these spaces, we associate a value – the coordinates of this point. Different such spaces are *mutually orthogonal*, in the sense that if $\psi_1 \in S_{x_1}$ and $\psi_2 \in S_{x_2}$, then $\langle \psi_1, \psi_2 \rangle = 0$. Indeed, if $\psi_1(x)$ if only different from 0 when $x = x_1$ and $\psi_2(x)$ is only different from 0 when $x = x_2$, this means that the product $\psi_1(x) \cdot \psi_2^*(x)$ is always equal to 0, and thus integrating this expression indeed results in $\langle \psi_1, \psi_2 \rangle = 0$. Every function $\psi(x)$ can be represented as a linear combination of such functions – it is sufficient to combine a function f_x from each space S_x with the coefficient proportional to $\psi(x)$.

After measurement, we get one of the values x, and the original state ψ turns into the state proportional to the projection $\pi_x(\psi)$ of the state ψ on the space S_x. The probability to get each value x is proportional to the square $\|\pi_x(\psi)\|^2$ of this projection.

Quantum measurements: a general description. The above reformulation enables us to describe the general measurement process in quantum mechanics. In this general process, we select a sequence of linear subspaces S_i of the original Hilbert space. To each space, we associate a value λ_i. These subspaces must be mutually orthogonal – if $\psi_1 \in S_{i_1}$ and $\psi_2 \in S_{i_2}$ for some $i_1 \neq i_2$, then we should have $\langle \psi_1, \psi_2 \rangle = 0$. Also, every element of the Hilbert space must be representable as a linear combination of functions from S_i.

When we measure the corresponding quantity for a system in a state ψ, then the state transforms into the projection $\pi_i(\psi)$ of this state on one of the subspaces S_i, and the corresponding value λ_i is returned as the measurement result. The probability of this value is equal to

$$p_i = \|\pi_i(\psi)\|^2. \tag{1}$$

It should be mentioned that since the states S_i are mutually orthogonal, for the original state $\psi = \sum\limits_{i} \pi_i(\psi)$, Pythagoras theorem implies that

$$1 = \|\psi\|^2 = \sum\limits_{i} \|\pi_i(\psi)\|^2,$$

so the sum $\sum\limits_{i} p_i$ of the corresponding probabilities is indeed equal to 1.

Einstein-Podolsky-Rosen (EPR) paradox: a brief reminder. For two independent particles in states $\psi_i(x_i)$, the joint state is a product $\psi_1(x_1) \cdot \psi_2(x_2)$. When particles are not independent (= *entangled*), we can

have a more complex joint state, such as the state

$$\frac{1}{\sqrt{2}} \cdot (|0_1\rangle \cdot |0_2\rangle + |1_1\rangle \cdot |1_2\rangle)$$

used in quantum computing (see, e.g., [5]), where $|0_1\rangle$ denotes the 0-state of the 1st particle.

When we measure the state of Particle 1, then, according to the above description of measurement in quantum physics, the joint state changed into a projection:

- either onto the space of all the states in which the first particle is in the 0 state,

- or onto the space of all the state in which the first particle is in the 1 state.

One can check that, as a result, we get either $|0_1\rangle \cdot |0_2\rangle$ or into $|1_1\rangle \cdot |1_2\rangle$.

Thus, the state of the second particle – as described by the wave function – immediately changed too.

When the particles are separated, this action-at-a-distance seems to contradict special relativity, according to which all speeds are limited by the speed of light c; this is the essence of the EPR paradox.

2 EPR Paradox: Bohr's Explanation

The Nobelist Niels Bohr – one of the main pioneers of quantum physics – explained that from the physical viewpoint, there is no paradox:

- while the wave function indeed changes immediately,

- this process cannot be used for faster-than-light communication: the results of measurements performed on Particle 2 does not change when we perform the measurements on Particle 1.

Remaining challenge. However, from the mathematical viewpoint, the EPR paradox remains a challenge.

The EPR paradox shows that the notion of a wave function, while convenient for computations and predictions, is not always the most physically adequate description of a quantum state. How can we get a more physically adequate mathematical description, a description that would not lead to any seeming paradox?

What we do in this paper. In this paper, we propose such a more adequate description, a description in which the EPR paradox stops being a paradox: no matter what we do with the first particle, the state of the second particle does not change.

Comment. The results of this paper were first announced in [1].

3 Towards a More Adequate Description of a Quantum State

From wave functions to a density operator. To explain our proposal, we need to recall another notion from quantum mechanics – the notion of the density operator ρ.

As we have mentioned, the probability of each measurement result i is described by the formula (1). Since the spaces S_i are mutually orthogonal, the projections to these spaces are also mutually orthogonal, i.e., $\langle \pi_j(\psi), \pi_i(\psi) \rangle = 0$ for all $i \neq j$. Since $\psi = \sum\limits_{j=1}^{n} \psi_j(\psi)$, we thus conclude that

$$\langle \pi_i(\psi), \psi \rangle = \sum_j \langle \pi_i(\psi), \pi_j(\psi) \rangle = \langle \pi_i(\psi), \pi_i(\psi) \rangle = \|\pi_i(\psi)\|^2 = p_i,$$

so

$$p_i = \langle \pi_i(\psi), \psi \rangle. \tag{2}$$

One can easily check that projection is a linear operation: projection of the sum of two vectors is equal to the sum of their projections. We know how linear functions look like, so the projection has the form

$$(\pi_i(\psi))(x) = \int A(x, y) \cdot \psi(y) \, \mathrm{d}y$$

for some coefficients $A(x, y)$. Thus, by definition of the scalar product,

$$p_i = \langle \pi_i(\psi), \psi \rangle = \int A(x, y) \cdot \psi(y) \cdot \psi^*(x) \, dx dy,$$

i.e.,

$$p_i = \int A(x, y) \cdot \rho(x, y) \, dx dy,$$

where we denoted

$$\rho(x, y) \stackrel{\text{def}}{=} \psi^*(x) \cdot \psi(y). \qquad (3)$$

This function $\rho(x, y)$ is known as the *density operator*.

The density operator uniquely determines the probability of each measurement result – and thus, provides an alterative way to describing the state of a quantum system.

Density operator: notations. In the finite case, when we have a finite-dimensional vector $a = (a_1, \cdots, a_n)$ instead of an infinite-dimensional vector $\psi(x)$, the corresponding matrix $A_{ij} = a_i^* \cdot a_j$ is known as the *tensor product* $a^* \otimes a$ of the complex conjugate vector $a^* \stackrel{\text{def}}{=} (a_1^*, \cdots, a_n^*)$ and the original vector a. Because of this analogy, the above expression (3) is also denoted by $\psi^* \otimes \psi$.

For two matrices A and B, the expression $\sum_{i,j} A_{ij} \cdot B_{ij}$ is equal to the trace (sum of the diagonal elements) $\mathrm{Tr}(C) \stackrel{\text{def}}{=} \sum_i C_{ii}$ of the product matrix $C = AB$ defined the usual way $C_{ij} = \sum_k A_{ik} \cdot B_{kj}$. In view of this analogy, for two functions $A(x, y)$ and $\rho(x, y)$ of two variables – i.e., for the two infinite-dimensional analogues of matrices – we can similarly define

$$\mathrm{Tr}(A\rho) \stackrel{\text{def}}{=} \int A(x, y) \cdot \rho(x, y) \, dx dy;$$

then, the formula for the probability p_i gets a simple form

$$p_i = \mathrm{Tr}(A\rho).$$

What if we first perform another measurement? Suppose that before performing a current measurement, we perform another measurement, as a result of which, instead of the original state φ, we get different states φ_j with probability q_j. For each of these states φ_j, the probability of getting the current measurement result λ_i is equal to $\mathrm{Tr}(A\widetilde{\rho}_j)$, where $\widetilde{\rho}_j \stackrel{\text{def}}{=} \varphi_j^* \otimes \varphi_j$. Thus, the overall probability of getting the result λ_i can be computed by using the formula of the complete probability:

$$p_i = \sum_j q_j \cdot \mathrm{Tr}(A\widetilde{\rho}_j),$$

i.e., equivalently, as $p_i = \mathrm{Tr}(A\rho)$, where we denoted

$$\rho = \sum_j q_i \cdot \widetilde{\rho}_j = \sum_j q_j \cdot (\varphi_j^* \otimes \varphi_j). \qquad (4)$$

We can thus say that after the measurement, the original state $\varphi^* \otimes \varphi$ gets transformed into a new state (4).

Towards a resulting proposal. Let us first consider the case of a 2-particle system – a particular case of which is the EPR system. In this case, the wave function depends both on the coordinates x_1 of the first particle and on the coordinates x_2 on the second particle, i.e., it depends on a 6-dimensional variable $x = (x_1, x_2)$. Thus, the probability of a measurement result takes the form

$$\mathrm{Tr}(A\rho) = \int A(x_1, x_2, y_1, y_2) \cdot \rho(x_1, x_2, y_1, y_2) \, dx_1 dx_2 dy_1 dy_2.$$

When we perform a measurement, all our instruments are located in some spatially bounded region U. Thus, all the measurements that we perform only take into account the values $A(x_1, x_2, y_1, y_2)$ corresponding to points $x_i, y_i \in U$.

- If this function A depends on all four coordinates x_1, x_2, y_1, and y_2, then the resulting probabilities depend only on the values $\rho(x_1, x_2, y_1, y_2)$ corresponding to $x_i, y_i \in U$.

- If the function A depends only on the coordinates x_1 and y_1 of the first particle, then the probability depends on the integrals

$$\rho_1(x_1, y_1) \stackrel{\text{def}}{=} \int \rho(x_1, x_2, y_1, y_2) \, \mathrm{d}x_2 \mathrm{d}y_2.$$

- Finally, if the function A depends only on the coordinates x_2 and y_2 of the second particle, then the probability depends on the integrals

$$\rho_2(x_2, y_2) \stackrel{\text{def}}{=} \int \rho(x_1, x_2, y_1, y_2) \, \mathrm{d}x_1 \mathrm{d}y_1.$$

The integrals corresponding to the second and the third cases represent *marginal* density operators ρ corresponding to each particle – a concept very similar to marginal probability distributions (see, e.g., [6]).

Thus, to describe the results of all U-localized measurements performed on the 2-particle system, we need to consider the following:

- the density operator $\rho_{|U}$ restricted to the area U, and

- the marginal density operators $\rho_{i|U}$ restricted to U.

In general, for any system of N particles described by the density operator $\rho(x_1, \cdots, x_N, y_1, \cdots, y_N)$, for any nonempty subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$, we can consider the marginal distribution

$$\rho_I(x_{i_1}, \cdots, x_{i_k}) = \int \rho(x_1, \cdots, x_N, y_1, \cdots, y_N) \, \mathrm{d}x_{j_1} \cdots \mathrm{d}x_{j_{N-k}},$$

where $j_1, \ldots, j_{N-k}$ are all the indices which are not in I. For each density operator $\rho(x_1, \cdots, x_N, y_1, \cdots, y_N)$ and for each spatial area U, we can consider its *restriction* $\rho_{|U}$ which is defined only for $x_i, y_i \in U$.

In these terms, we can formulate the resulting proposal.

The resulting proposal. Let U be the area in which all our measuring equipment is located. Then, to describe the results of all related measurements on an N-particle system, it is sufficient to know the restriction $\rho_{|U}$ of the density operator and the restrictions $\rho_{I|U}$ of all the marginal density operators on this area U.

Our proposal is thus to describe the state of the system *not* by the original wave function or by the original density operator, but by these *restricted* operators.

Please note that in this proposal, the description of the state is uniquely determined by the probabilities of different U-located measurements. Thus, since (as we have mentioned) in the EPR setting, the probabilities of different results of measuring the second particle do not change whether we measure the first particle or not, the state of the second particle also does not change if, outside the area U, we perform measurements on the first particle. In other words, what we propose is exactly the alternative description of quantum states that enables us to avoid the EPR paradox – and is, in this sense, more physically adequate.

Terminological comment. One can easily check that when we limit ourselves to a subregion $U' \subset U$, then the restriction $\rho_{|U'}$ is equal to a naturally defined restriction of $\rho_{|U}$: $\rho_{|U'} = (\rho_{|U})_{|U'}$. In mathematical terms, this property means that the corresponding restrictions $\rho(U)$ form a *presheaf* (it is actually a sheaf; see, e.g., [4, 9]). In these terms, what we propose is replacing density operators with sheafs of density operators. The resulting formalism may be somewhat more complicated mathematically – but it is more physically adequate.

Acknowledgments

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