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A Weak Contraction in a Fuzzy Metric Space

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Abstract

In this work our aim is to establish a weakened version of a contraction mapping principle in a fuzzy metric space with a partial ordering. The weak contraction is primarily considered on specific chains. When it is considered on the whole space it generalizes a fuzzy contraction mapping theorem. The result is supported with an example.

Keywords: fuzzy metric space, weak contraction, topology, partial ordering, Cauchy sequence, fixed point

1 Introduction and Preliminaries

The purpose of this paper is to generalize a fuzzy Banach contraction mapping principle to coincidence point and common fixed point results in a fuzzy metric space with a partial order. In the sequel we prove a weak contraction mapping theorem by way of weakening the contraction. Fuzzy metric space has been defined in a number of ways. It is the inherent flexibility of fuzzy concepts that makes possible the fuzzification of the notion of metric spaces in more than one inequivalent ways.

In particular, a notion of a fuzzy metric space was introduced by Kramosil and Michalek [22]. George and Veeramani modified the definition of Kramosil and Michalek for topological reasons [14]. They proved that the topology induced by such a fuzzy metric space, is a Hausdorff topology. Actually, such a topology is metrizable as one can find, for instance, in [12, Theorem 1]. There are several fixed point results established in this fuzzy metric space. Some instances of these works are in [6, 11, 15, 16, 17, 25, 32, 39, 41, 42, 43]. Fuzzy fixed results are more versatile than the regular metric fixed point results. This is due to the flexibility which the fuzzy concepts inherently posses. For example, Banach's contraction mapping principle has been extended in fuzzy metric spaces in two inequivalent ways in [15, 17]. Today fuzzy fixed theory has a developed literature and can be regraded as subject in its own right.

Attempts for generalizing the Banach's contraction mapping principle had been therefor a long period of time. Still today it remains an active branch of fixed point theory. The works noted in [2, 3, 13, 24, 37] are some references from this area of research. One such generalization is the weak contraction principle which was first introduced by Alber et al [1] in Hilbert spaces and later adapted to complete metric spaces by Rhoades [29]. A weak contraction mapping is intermediate to a contraction mapping and a nonexpansive mapping. Later on, several authors created a number of results using weak inequalities, that is, the inequalities of the type used in [4, 7, 9, 10, 12, 19]. These results are fixed and coincidence point results some of which further generalize the weak contraction while others are independent results. Fixed point theory in partially ordered metric spaces is of relatively recent origin. An early result in this direction is due to Turinici [38] in which fixed point problems were studied in partially ordered uniform spaces. Later, this branch of fixed point theory has developed through a number of works included in which are some results involving weak contractive inequalities. Some instances of these works are in references [8, 26, 27, 28]. Weak contraction in partially ordered metric spaces was studied amongst others by Harjani et al [19] and Sauco et al [35].

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One reason for considering partially ordered metric space in fixed point theory of contractive mappings is that the contraction condition is restricted only to pairs of points which are connected by the order relation. In fact it has been observed in the proofs of several fixed point theorems including that of Banach's contraction mapping principle that the assumed contraction inequality is not required to be satisfied for arbitrary pairs of points from the metric space. Thus it is possible to restrict the set of pairs of points appropriately for the above purpose. The introduction of partial order does the job. There is also an alternative approach for the same purpose which is by use of admissibility conditions [34]. We do not consider this approach here.

The same fact described above was also perceived in the fuzzy fixed point theory as well. This has resulted into several fixed point related results in partially ordered fuzzy metric spaces. Some instances of these works are [6, 31, 33, 40].

Our work is motivated by the above considerations. Putting the above mentioned concepts together we establish a weak contraction mapping principle in the framework of fuzzy metric spaces with a partial order defined on it in which the weak contraction condition is assumed only for pairs of points connected by the partial order relation. The weak inequality is postulated by using two control functions. In metric spaces there are several uses of control functions in similar results [5, 10, 13]. Compared with these results, in the present work control functions need to satisfy some different conditions which are suitable for fuzzy metric spaces. These conditions are instrumental in the proof of the main result obtained here.

Definition 1.1 ([18, 36]). A binary operation $* : [0,1]^2 \to [0,1]$ is called a continuous $t-$ norm if the following properties are satisfied :
(i) * is associative and commutative,
(ii) $a * 1 = a$ for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Some examples of continuous $t-$ norm are $a *_1 b = \min\{a, b\}$, $a *_2 b = ab$ and $a *_3 b = \max\{a + b - 1, 0\}$. Several aspects of the theory of $t-$ norms with examples are given comprehensively by Klement al. in their book [21].

George and Veeramani in their paper [14] introduced the following definition of fuzzy metric space. We will be concerned only with this definition of fuzzy metric space.

Definition 1.2([14]) The $3-$ tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non- empty set, $*$ is a continuous $t-$ norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:
(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.

Definition 1.3([14]) Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$ and $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ and radius r is defined by $B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}$. A subset $A \subset X$ is called open if for each $x \in A$, there exists $t > 0$ and $0 < r < 1$, such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology and is called the topology on X induced by the fuzzy metric M. This topology is metrizable as we indicated above.

Example 1.4([14]) Let X be the set of all real numbers and d be the Euclidean metric on X. Let $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. For each $t > 0$ and $x, y \in X$, let $M(x, y, t) = t/(t + d(x, y))$. Then $(X, M, *)$ is a fuzzy metric space.

Example 1.5 Let (X, d) be a metric space and ψ be an increasing and continuous function of $(0, \infty)$ into $(0, 1]$ such that $\lim_{t\to\infty} \psi(t) = 1$. Three typical examples of these functions are

$$\begin{aligned}\psi(t) &= \frac{t}{t+1},\\ \psi(t) &= \sin\left(\frac{2t}{2t+1}\right)\end{aligned}$$

and

$$\psi(t) = (1 - e^{-t}).$$

Let $*$ be any continuous $t-$ norm. For each $t > 0$ and $x, y \in X$, let $M(x, y, t) = \psi(t)^{d(x,y)}$. Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.6([14]) Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ iff $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence iff for each ϵ satisfying $0 < \epsilon < 1$ and each $t > 0$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be a complete fuzzy metric space.

The following lemma was proved by Grabiec[10] for fuzzy metric spaces defined by Kramosil et al. The proof is also applicable to the fuzzy metric space given in Definition 1.2.

Lemma 1.7 ([15]) Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is non decreasing for all $x, y \in X$.

Lemma 1.8 (Lṕez and Romaguera [23]) M is continuous function on $X^2 \times (0, \infty)$.

Let $(X, \preceq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be monotone increasing if for any $x_1, x_2 \in X, x_1 \preceq x_2$ implies $fx_1 \preceq fx_2$.

Orbit of a function $f : X \to X$ at the point $x \in X$ is the set $\{x, fx, f^2x, f^3x, \ldots, f^nx, \ldots\}$ where f^nx is the n-th iterate of f.

Definition 1.9 Let $(X, M, *)$ be a complete fuzzy metric space. Let C be a subset of X. Let $f : C \to C$ be a self mapping which satisfies the following inequality:

$$\psi(M(fx, fy, t)) \leq \psi(M(x, y, t)) - \phi(M(x, y, t)) \tag{1.1}$$

where $x, y \in X, t > 0$ and $\psi, \phi : (0, 1] \to [0, \infty)$ are two functions such that

(i) ψ is continuous and monotone decreasing with $\psi(t) = 0$ if and only if $t = 1$,

(ii) ϕ is lower semi continuous with $\phi(s) = 0$ if and only if $s = 1$.

Then f is called a weak contraction on C.

The following two lemmas were established in [33]. We use them for deriving our main results in the next section.

Lemma 1.10 ([33]) If $*$ is a continuous t-norm, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha$, $\gamma_n \to \gamma$ as $n \to \infty$, then

$$\overline{\lim_{k\to\infty}} (\alpha_k * \beta_k * \gamma_k) = \alpha * \overline{\lim_{k\to\infty}} \beta_k * \gamma$$

and

$$\underline{\lim_{k\to\infty}} (\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim_{k\to\infty}} \beta_k * \gamma.$$

Lemma 1.11 ([33]) Let $\{f(k, .) : (0, \infty) \to (0, 1], k = 0, 1, 2, \ldots\}$ be a sequence of functions such that $f(k, .)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim_{k\to\infty}} f(k, t)$ is a left continuous function in t and $\underline{\lim_{k\to\infty}} f(k, t)$ is a right continuous function in t.

2 Main Results

Theorem 2.1 Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete fuzzy metric space which has the property that whenever $\{x_n\}$ is a monotone increasing sequence in X converging to a point z, it follows that $x_n \preceq z$ for all n. Let $f : X \to X$ be a mapping with the monotone property on X. Let there exist $x_0 \in X$ such that $x_0 \preceq fx_0$. Then the orbit $O(x_0, f)$ is a chain with x_0 as the least element. If f is weakly contractive on every chain containing $O(x_0, f)$ with x_0 as the least element, then f has a fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be the sequence defined by $x_{n+1} = fx_n = f^nx_0, \;\; n \geq 1$. Since f has monotone property on X, by the fact that $x_0 \preceq fx_0 = x_1$, it follows that $x_1 = fx_0 \preceq fx_1 = x_2$. Repeating the above process we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \cdots . \tag{2.1}$$

Then $O(f, x_0)$ is a chain with x_0 as the least element and, therefore, by a condition of our theorem, f is weakly contractive on $O(f, x_0)$.

Putting $x = x_{n-1}$ and $y = x_n$ in (1.1) we obtain, for all $n \geq 1, t > 0$,

$$\psi(M(fx_{n-1}, fx_n, t)) \leq \psi(M(x_{n-1}, x_n, t)) - \phi(M(x_{n-1}, x_n, t)),$$

that is,

$$\psi(M(x_n, x_{n+1}, t) \leq \psi(M(x_{n-1}, x_n, t) - \phi(M(x_{n-1}, x_n, t)). \tag{2.2}$$

From the above inequality, for all $n \geq 1, t > 0$,

$$\psi(M(x_n, x_{n+1}, t) \leq \psi(M(x_{n-1}, x_n, t)$$

which, by the monotone decreasing property of ψ, implies that for all $n \geq 1, t > 0$,

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t),$$

that is, $\{M(x_n, x_{n+1}, t)\}$ is a monotone increasing sequence in X. This sequence being bounded above by 1, for every choice of $t > 0$, there exists $a(t)$ such that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = a(t) \leq 1. \tag{2.3}$$

Now taking $n \to \infty$ in (2.2) we obtain, for all $t > 0, \psi(a(t)) \leq \psi(a(t)) - \phi(a(t))$, which is a contradiction unless $a(t) = 1$, for all $t > 0$.

Thus we conclude that for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{2.4}$$

Next we suppose, if possible, that $\{x_n\}$ is not a Cauchy sequence in X. Then there exists some $\epsilon > 0$ and some λ with $0 < \lambda < 1$, for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with

$$n(k) > m(k) > k \tag{2.5}$$

such that

$$M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1 - \lambda) \tag{2.6}$$

for all positive integer k.

We may choose the $n(k)$ as the smallest integer exceeding $m(k)$ for which (2.6) holds. Then, for all positive integer k,

$$M(x_{m(k)}, x_{n(k)-1}, \epsilon) > (1 - \lambda). \tag{2.7}$$

Then, for all $k \geq 1, 0 < s < \epsilon/2$, we obtain,

$$\begin{aligned} (1-\lambda) &\geq M(x_{m(k)}, x_{n(k)}, \epsilon) \\ &\geq M(x_{m(k)}, x_{m(k)-1}, s) * M(x_{m(k)-1}, x_{n(k)-1}, \epsilon - 2s) * M(x_{n(k)-1}, x_{n(k)}, s). \end{aligned} \tag{2.8}$$

Let

$$h_1(t) = \overline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, t), \quad t > 0. \tag{2.9}$$

Taking limit supremum on both sides of (2.8), using (2.4), and the properties of M and $*$, by Lemma 1.10, we obtain

$$(1-\lambda) \geq 1 * \overline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon - 2s) * 1 = h_1(\epsilon - 2s). \tag{2.10}$$

Since M is bounded with range in [0,1], continuous and, by Lemma 1.7, monotone increasing in the third variable t, it follows by an application of Lemma 1.11 that h_1 as given in (2.9) is continuous from the left. In view of the above fact, letting $s \to 0$ in (2.10), we obtain

$$\overline{\lim_{k\to\infty}} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \leq (1-\lambda). \tag{2.11}$$

Let

$$h_2(t) = \underline{\lim}_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t), \quad t > 0. \tag{2.12}$$

Again, for all $k \geq 1, s > 0$,

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon + s) &\geq M(x_{m(k)-1}, x_{m(k)}, s) * M(x_{m(k)}, x_{n(k)-1}, \epsilon) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, s) * (1-\lambda). \end{aligned} \tag{2.13}$$

Taking limit infimum as $k \to \infty$ in (2.13), by virtue of (2.4), we obtain

$$\begin{aligned} h_2(\epsilon + s) &= \underline{\lim}_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon + s) & (2.14) \\ &\geq \underline{\lim}_{k\to\infty} M(x_{m(k)-1}, x_{m(k)}, s) * (1-\lambda) \\ &= 1 * (1-\lambda) = (1-\lambda). & (2.15) \end{aligned}$$

Since M is bounded with range in [0,1], continuous and, by Lemma 1.7, monotone increasing in the third variable t, it follows by an application of Lemma 1.11 that h_2 as given in (2.12) is continuous from the right. Taking $s \to 0$ in the above inequality (2.14), we obtain

$$\underline{\lim}_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) \geq (1-\lambda). \tag{2.16}$$

The inequalities (2.11) and (2.15) jointly imply that

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) = (1-\lambda). \tag{2.17}$$

Again by (2.6),

$$\overline{\lim_{k\to\infty}} M(x_{m(k)}, x_{n(k)}, \epsilon) \leq (1-\lambda). \tag{2.18}$$

Also for all $k \geq 1, s > 0$, we obtain

$$M(x_{m(k)}, x_{n(k)}, \epsilon + 2s) \geq M(x_{m(k)}, x_{m(k)-1}, s) * M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) * M(x_{n(k)-1}, x_{n(k)}, s).$$

Taking limit infimum as $k \to \infty$ in the above inequality, using (2.4), (2.16) and the properties of M and $*$, by Lemma 1.10, we obtain

$$\underline{\lim}_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \epsilon + 2s) \geq 1 * \underline{\lim}_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, \epsilon) * 1 = (1-\lambda).$$

Since M is bounded with range in [0,1], continuous and, by Lemma 1.7, monotone increasing in the third variable t, it follows by an application of Lemma 1.11 that $\underline{\lim}_{k\to\infty} M(x_{m(k)}, x_{n(k)}, t)$ is continuous function of t from the right.

Taking $s \to 0$ in the above inequality, and using Lemma 1.11, we obtain

$$\underline{\lim}_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \epsilon) \geq (1-\lambda). \tag{2.19}$$

Combining (2.17) and (2.18), we obtain

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \epsilon) = (1-\lambda). \tag{2.20}$$

Putting $x = x_{m(k)-1}, y = x_{n(k)-1}$ and $t = \epsilon$ in (1.1) we obtain

$$\begin{aligned}\psi(M(x_{m(k)}, x_{n(k)}, \epsilon)) &= \psi(M(fx_{m(k)-1}, fx_{n(k)-1}, \epsilon))\\ &\leq \psi(M(x_{m(k)-1}, x_{n(k)-1}, \epsilon)) - \phi(M(x_{m(k)-1}, x_{n(k)-1}, \epsilon)).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (2.16), (2.19), continuity of ψ and the lower semi-continuity of ϕ, we obtain

$$\psi(1-\lambda) \leq \psi(1-\lambda) - \phi(1-\lambda),$$

which is a contradiction since $\phi(1-\lambda) \neq 0$.

Thus it is established that $\{x_n\}$ is a Cauchy sequence and hence it is convergent to a point z in the complete fuzzy metric space X, that is,

$$\lim_{n\to\infty} x_n \quad = \quad z \in X \tag{2.21}$$

Next we show that z is fixed point of f.

By our construction, $\{x_n\}$ is a monotone increasing sequence and hence, by our assumption of the theorem, $x_n \preceq z$ for all $n \geq 0$.

If follows that $\{x_n : n \geq 0\} \cup \{z\}$ is a chain containing $O(f, x_0)$ with x_0 as the least element. By a condition of the theorem, f is a weak contraction on this chain. Then for all $n \geq 1$, $t > 0$,

$$\begin{aligned}\psi(M(x_n, fz, t)) &= \psi(M(fx_{n-1}, fz, t))\\ &\leq \psi(M(x_{n-1}, z, t)) - \phi(M(x_{n-1}, z, t)).\end{aligned}$$

Letting $n \to \infty$ in the above inequality, using the properties of ψ, ϕ and (2.20), we obtain

$$\psi(\lim_{n\to\infty} M(x_n, fz, t)) \leq \psi(1) - \phi(1) = 0,$$

that is, by a property of ψ,

$$\lim_{n\to\infty} M(x_n, fz, t) \quad = \quad 1 \quad \text{for all} \quad t > 0,$$

which implies that

$$x_n \to fz \quad as \quad n \to \infty. \tag{2.22}$$

Since the topology in the fuzzy metric space is a Hausdorff topology, we conclude from (2.20) and (2.22) that $z = fz$.

This completes the proof of the theorem.

The fixed point in the above theorem is not in general unique as the example in the next section demonstrates. In the following theorem we also show that when the weak contraction is defined over the whole space, the fixed point is unique.

Theorem 2.2 Let $(X, M, *)$ be a fuzzy metric space and $f : X \to X$ be a weak contraction on X as in the Definition 1.9. Then f has a unique fixed point.

Proof. The existence of a fixed point is established by following the same argument as in the proof of Theorem 2.1 except that we need not have to consider the partial order relation here. The uniqueness of the fixed point is established as following.

Let z_1 and z_2 be two fixed points of f, that is, $fz_1 = z_1$ and $fz_2 = z_2$. Then, from (1.1), for all $t > 0$,

$$\begin{aligned}\psi(M(z_1, z_2, t)) &= \psi(M(fz_1, fz_2, t))\\ &\leq \psi(M(z_1, z_2, t)) - \phi(M(z_1, z_2, t)),\end{aligned}$$

which is a contradiction unless $M(z_1, z_2, t) = 1$, which implies that $z_1 = z_2$.

Thus the fixed point is unique.

Corollary 2.3 (Fuzzy Banach Contraction mapping Theorem [14]) Let $f : X \to X$, where $(X, M, *)$ is a complete fuzzy metric space, be such that

$$\frac{1}{M(fx, fy, t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right) \tag{2.23}$$

where $x, y \in X, t > 0$ and $0 < k < 1$. Then T has a unique fixed point.

Proof. Let

$$\psi(s) = \frac{1-s}{s}$$

and

$$\phi(s) = \frac{(1-k)(1-s)}{s}$$

where $0 < s \leq 1$. Then we see that with the above choices of ψ and ϕ, (2.23) implies (1.1) for all $x, y \in X$ and $t > 0$.

The Corollary 2.3 then follows by an application of the Theorem 2.2.

Remark 2.4 Corollary 2.3 is a generalization of a fuzzy contraction principle established by Gregori et al. [17].

3 Example

In this section we provide an illustration of our main result.

Example 3.1 Let $X = [0, \infty)$ and $x \preceq y$ if $x \geq y$. Let

$$M(x, y, t) = e^{-\frac{|x-y|}{t}}$$

where $x, y \in X$ and $t > 0$. Let $*$ be any continuous $t-$ norm. Let $f(x) = x^2$. Then f is a monotone function.

We consider any element $x_0 = c$ of X where $0 < c < 1/4$. Then $O(f, x_0) = \{c, c^2, c^4, \ldots\}$. Let

$$\psi(s) = \frac{1-s}{s} \quad \text{and} \quad \phi(s) = \frac{1}{s} - \frac{1}{\sqrt{s}}.$$

In the following we verify that with the above choices of ψ, ϕ and x_0, the inequality (1.1) is satisfied on any chain containing $O(f, x_0)$ with x_0 as the least element. This will be established if we show that (1.1) is satisfied on any maximal chain containing $O(f, x_0)$ with $x_0 = c$ as the least element.

In the present case there is only one maximal chain, namely, the set $[0, c]$. Let $x, y \in [0, c]$. Without loss of generality we assume $x \prec y$, that is, $x > y$. The case where $x = y$ is trivial from the properties of ψ and ϕ. Then (1.1) is written as

$$\psi(e^{-\frac{(x^2-y^2)}{t}}) \leq \psi(e^{-\frac{(x-y)}{t}}) - \phi(e^{-\frac{(x-y)}{t}})$$

which is the same as

$$\begin{aligned} e^{\frac{(x^2-y^2)}{t}} - 1 \quad &\leq \quad e^{\frac{(x-y)}{t}} - 1 - \phi(e^{-\frac{(x-y)}{t}}) \\ &= \quad e^{\frac{(x-y)}{t}} - 1 - e^{\frac{(x-y)}{t}} + e^{\frac{(x-y)}{2t}}, \end{aligned}$$

that is, the same as

$$(e^{\frac{(x^2-y^2)}{t}}) \leq e^{\frac{(x-y)}{2t}}.$$

Now $x + y \leq 1/2$ by our choice of $c < 1/4$. Then, for all $t > 0$

$$\begin{aligned} e^{\frac{(x^2-y^2)}{t}} &= e^{\frac{(x-y)(x+y))}{t}} \\ &\leq e^{\frac{(x-y)}{2t}} \\ &\leq e^{\frac{(x-y)(x+y))}{2t}}, \end{aligned}$$

that is, the inequality (1.1) is satisfied in this case. Then, by an application of Theorem 2.1, we obtain a fixed point of f. Here we obtain $x = 0$ as a fixed point.

Remark 3.2 Here fixed point of the function is not unique, $x = 1$ being another fixed point. Also Corollary 2.3 is not applicable to this example since this is not a contraction on the whole space. Following Remark 2.4 we conclude that Theorem 2.1 actually improves the result of Gregori et.al [17].

4 Conclusions

In our study we use arbitrary continuous t-norms. It is an interesting question whether the result is valid with some non-continuous t-norms as well. Particularly *Hadžić* type t-norm [18] can be considered for this purpose. The weak contraction is intermediate to a contraction and a non-expansive mapping. In metric spaces the introduction of weak contraction by Rhoades [29] has opened a new chapter in metric fixed point theory with a large number of papers following it. It is expected that such a development paralleling that in metric spaces is also possible in fuzzy metric spaces. Moreover, the weak contractive type results have many applications as, for instances, in the works [20, 30]. Such applications in problems like those of fuzzy integral equations, etc. are supposed to be in order in future works.

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