

# How to Define “And”- and “Or”-Operations for Intuitionistic and Picture Fuzzy Sets

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## Abstract

The traditional fuzzy logic does not distinguish between the cases when we know nothing about a statement  $S$  and the cases when we have equally convincing arguments for  $S$  and for its negation: in both cases, we assign the degree 0.5 to such a statement  $S$ . This distinction is provided by intuitionistic fuzzy logic, when to describe our degree of confidence in a statement  $S$ , we use two numbers that characterize our degree of confidence in  $S$  and in “not  $S$ ”. An even more detailed distinction is provided by picture fuzzy logic, in which we differentiate between cases when we are still trying to understand the truth value and cases when we are no longer interested. The question is how to extend “and”- and “or”-operations to these more general logics. In this paper, we provide a general idea for such extension, an idea that explain several extensions that have been proposed and successfully used.

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## 1 Formulation of the Problem

**Traditional fuzzy logic: brief reminder.** In the traditional fuzzy logic (see, e.g., [2, 4, 5, 6, 7, 8]), we describe our degree of confidence in a statement  $A$  by a number  $a \in [0, 1]$ , so that:

- 0 means no confidence,
- 1 means full confidence, and
- intermediate values describe intermediate degrees of confidence.

In practice, there is often a need to estimate the degree of confidence in composite statements like  $A \& B$  and  $A \vee B$ . There are many such statements, so it is not feasible to ask the experts about all of them. Instead, we must estimate our degree of confidence in  $A \& B$  and  $A \vee B$  based on our degrees of confidence  $a$  and  $b$  in the statements  $A$  and  $B$ . These estimates  $f_{\&}(a, b)$  and  $f_{\vee}(a, b)$  are known as “and”-operation ( $t$ -norm) and “or”-operation ( $t$ -conorm).

- The most widely used operations are  $f_{\&}(a, b) = \min(a, b)$  and  $f_{\vee}(a, b) = \max(a, b)$ .
- Another very popular choice is  $f_{\&}(a, b) = a \cdot b$  and  $f_{\vee}(a, b) = a + b - a \cdot b$ .

**Need for intuitionistic fuzzy logic.** In the traditional fuzzy logic, we do not distinguish between:

- the cases when we know nothing about  $A$  and

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- the cases when we have equally strong arguments for and against  $A$ .

In both types of cases, we assign  $a = 0.5$ . To make this distinction, we can use two degrees:

- the degree  $a_+$  of confidence in  $A$  and
- the degree of confidence  $a_-$  in  $\neg A$ ,

the degrees for which  $a_+ + a_- \leq 1$ . In this *intuitionistic fuzzy* approach (see, e.g., [1]):

- in the first case, we have  $a_+ = a_- = 0$ , and
- in the second case, we have  $a_+ = a_- = 0.5$ .

To define “and”- and “or”-operations to intuitionistic fuzzy sets, we can find  $f(a, b)$  for which  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$  always imply  $f_{\&}(a_+, b_+) + f(a_-, b_-) \leq 1$ .

**Picture fuzzy sets.** If  $a_+ + a_- < 1$ , this means that we do not have enough evidence for or against  $A$ .

- This may mean that we are still trying.
- This may mean that we are not interested in  $A$  at all.
- This may also mean that we are interested to some degree  $a_0$ , for which  $a_+ + a_- + a_0 \leq 1$ .

How do we define “and”- and “or”-operations on such *picture sets* (see, e.g., [3])? Again, a similar idea is to find a function  $f_0(a, b)$  for which  $a_+ + a_- + a_0 \leq 1$  and  $b_+ + b_- + b_0 \leq 1$  always imply

$$f_{\&}(a_+, b_+) + f(a_-, b_-) + f_0(a_0, b_0) \leq 1.$$

**Problem.** What operations should we define? In this paper, we use the above idea to describe possible “and”- and “or”-operations for intuitionistic and picture sets. These turn out to be the same operations that have been shown to be practically successful – but now we have a theoretical explanation for these heuristic operations.

## 2 Case of Intuitionistic Fuzzy Sets

**Definition 2.1.** Let  $f_{\&}(a, b)$  be an “and”-operation. We say that  $f(a, b)$  is an intuitionistic operation corresponding to  $f_{\&}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , then

$$f_{\&}(a_+, b_+) + f(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_-, b_-)$ , there exist values  $a_+$  and  $b_+$  for which  $a_+ + a_- \leq 1$ ,  $b_+ + b_- \leq 1$ , and

$$f_{\&}(a_+, b_+) + f(a_-, b_-) = 1.$$

**Proposition 2.1.** For each “and”-operation  $f_{\&}(a, b)$ , the corresponding intuitionistic operation has the form

$$f(a, b) = 1 - f_{\&}(1 - a, 1 - b).$$

**Corollary 2.1** For each “and”-operation ( $t$ -norm)  $f_{\&}(a, b)$ , the corresponding intuitionistic operation is an “or”-operation ( $t$ -conorm).

**Corollary 2.2.** For  $f_{\&}(a, b) = \min(a, b)$ , the corresponding intuitionistic operation is  $f(a, b) = \max(a, b)$ .

**Corollary 2.3.** For  $f_{\&}(a, b) = a \cdot b$ , the corresponding intuitionistic operation is  $f(a, b) = a + b - a \cdot b$ .

**Definition 2.2.** Let  $f_{\vee}(a, b)$  be an “or”-operation. We say that  $f(a, b)$  is an intuitionistic operation corresponding to  $f_{\vee}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , then

$$f_{\vee}(a_+, b_+) + f(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_0, b_0)$ , there exists values  $a_+$ ,  $b_+$ ,  $a_-$ , and  $b_-$  for which  $a_+ + a_- \leq 1$ ,  $b_+ + b_- \leq 1$ , and

$$f_{\vee}(a_+, b_+) + f(a_-, b_-) = 1.$$

**Proposition 2.2.** For each “or”-operation  $f_{\vee}(a, b)$ , the corresponding intuitionistic operation has the form

$$f(a, b) = 1 - f_{\vee}(1 - a, 1 - b).$$

**Corollary 2.4.** For each “or”-operation ( $t$ -conorm)  $f_{\vee}(a, b)$ , the corresponding intuitionistic operation is an “and”-operation ( $t$ -norm).

**Corollary 2.5.** For  $f_{\vee}(a, b) = \max(a, b)$ , the corresponding intuitionistic operation is  $f(a, b) = \min(a, b)$ .

**Corollary 2.6.** For  $f_{\vee}(a, b) = a + b - a \cdot b$ , the corresponding intuitionistic operation is  $f(a, b) = a \cdot b$ .

### 3 Case of Picture Fuzzy Sets

*Comment.* Based on the results of Section 2, we conclude that in intuitionistic fuzzy logic:

- if we start with an “and”-operation, we end up with an “or”-operation, and
- vice versa, if we start with an “or”-operation, we end up with an “and”-operation.

For each “and”-operation, we have a very specific “or”-operation and vice versa. However, for the purpose of generality, it makes sense to also consider the case when instead of a related pair of “and”- and “or”-operations, we have a general pair of such operations, with the only condition that whenever  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , then we have

$$f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-) \leq 1.$$

The dual “and”- and “or”-operations – as described by Propositions 2.1 and 2.2 – always satisfy this condition.

**Definition 3.1.** We say that an “and”-operation  $f_{\&}(a, b)$  and an “or”-operation are compatible if whenever  $a_+ + a_- \leq 1$  and  $b_+ + b_- \leq 1$ , we have

$$f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-) \leq 1.$$

**Definition 3.2.** Let  $f_{\&}(a, b)$  be an “and”-operation, and let  $f_{\vee}(a, b)$  be a compatible “or”-operation. We say that  $f_0(a, b)$  is a picture “and”-operation corresponding to  $f_{\&}(a, b)$  and  $f_{\vee}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_0 + a_- \leq 1$  and  $b_+ + b_- + b_0 \leq 1$ , then

$$f_{\&}(a_+, b_+) + f_0(a_0, b_0) + f_{\vee}(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_0, b_0)$ , there exists values  $a_+$ ,  $b_+$ ,  $a_-$ , and  $b_-$  for which  $a_+ + a_0 + a_- \leq 1$ ,  $b_+ + b_- + b_0 \leq 1$ , and

$$f_{\&}(a_+, b_+) + f_0(a_0, b_0) + f_{\vee}(a_-, b_-) = 1.$$

**Proposition 3.1.** For each “and”-operation  $f_{\&}(a, b)$  and each compatible “or”-operation  $f_{\vee}(a, b)$ , the corresponding picture “and”-operation has the form

$$f_0(a, b) = 1 - \max_{a_+ \leq 1-a, b_+ \leq 1-b} (f_{\&}(a_+, b_+) + f_{\vee}(1-a-a_+, 1-b-b_+)).$$

**Proposition 3.2.** For  $f_{\&}(a, b) = \min(a, b)$  and  $f_{\vee}(a, b) = \max(a, b)$ , the corresponding picture “and”-operation is  $f_0(a, b) = \min(a, b)$ .

**Proposition 3.3.** For  $f_{\&}(a, b) = a \cdot b$  and  $f_{\vee}(a, b) = a + b - a \cdot b$ , the corresponding picture “and”-operation is  $f_0(a, b) = a \cdot b$ .

**Definition 3.3.** Let  $f_{\vee}(a, b)$  be an “or”-operation, and let  $f_{\&}(a, b)$  be a compatible “and”-operation. We say that  $f_0(a, b)$  is a picture “or”-operation corresponding to  $f_{\vee}(a, b)$  and  $f_{\&}(a, b)$  if the following two properties are satisfied:

- first, if  $a_+ + a_0 + a_- \leq 1$  and  $b_+ + b_- + b_0 \leq 1$ , then

$$f_{\vee}(a_+, b_+) + f_0(a_0, b_0) + f_{\&}(a_-, b_-) \leq 1;$$

- second, for each pair  $(a_0, b_0)$ , there exists values  $a_+$ ,  $b_+$ ,  $a_-$ , and  $b_-$  for which  $a_+ + a_0 + a_- \leq 1$ ,  $b_+ + b_- + b_0 \leq 1$ , and

$$f_{\vee}(a_+, b_+) + f_0(a_0, b_0) + f_{\&}(a_-, b_-) = 1.$$

**Proposition 3.4.** For each “and”-operation  $f_{\&}(a, b)$  and each compatible “or”-operation  $f_{\vee}(a, b)$ , the corresponding picture “or”-operation has the form

$$f_0(a, b) = 1 - \max_{a_+ \leq 1-a, b_+ \leq 1-b} (f_{\vee}(a_+, b_+) + f_{\&}(1-a-a_+, 1-b-b_+)).$$

The Propositions 3.1 and 3.4 leads to a somewhat unexpected conclusion:

**Corollary 3.1.** For each pair of compatible “and”- and “or”-operations, the corresponding picture “or”-operation is the same as the corresponding picture “and”-operation.

## 4 Possible Extensions: Idea

**Fuzzy logic  $\rightarrow$  intuitionistic logic  $\rightarrow$  picture logic: a brief reminder.** We started with the usual fuzzy logic, in which, for each statement  $A$  with degree of confidence  $a_+$ , the degree of confidence in its negation  $\neg A$  is equal to  $a_- = 1 - a_+$ , i.e., for which  $a_+ + a_- = 1$ .

We then took into account that we may be uncertainty about both  $A$  and  $\neg A$ , i.e., we may have  $a_+ + a_- \leq 1$ . This is the case of intuitionistic fuzzy logic. In intuitionistic fuzzy logic, we can consider the remaining degree

$$a_0 \stackrel{\text{def}}{=} 1 - (a_+ + a_-)$$

that describes our uncertainty. In this case, we have  $a_+ + a_0 + a_- = 1$ .

The next natural step was to take into account that not all the difference  $1 - (a_+ + a_-)$  may be due to uncertainty, and that we may have situations in which, for the corresponding uncertainty degree  $a_0$ , we have  $a_0 < 1 - (a_+ + a_-)$ . In this case, in general, we have  $a_0 \leq 1 - (a_+ + a_-)$ . This is the case of picture fuzzy logic. In picture fuzzy logic, we can consider the remaining degree  $a_r \stackrel{\text{def}}{=} (1 - (a_+ + a_-)) - a_0$ . In this case, we have  $a_+ + a_0 + a_- + a_r = 1$ .

**Natural next steps.** It seems reasonable to apply the same idea again, and consider the cases in which  $a_+ + a_0 + a_- + a_r \leq 1$ . For this new extension of fuzzy logic, we can use the same idea as above to extend operations on  $a_+$ ,  $a_-$ , and  $a_0$  components to the operations of the remainders  $a_r$ . For example, for “and”-operations, we require that every time we have  $a_+ + a_0 + a_- + a_r \leq 1$  and  $b_+ + b_0 + b_- + b_r \leq 1$ , we should have

$$f_{\vee}(a_+, b_+) + f_0(a_0, b_0) + f_{\&}(a_-, b_-) + f_r(a_r, b_r) \leq 1,$$

and for each pair  $(a_r, b_r)$ , we should have at least one case when the above inequality becomes equality.

From this requirement, we can also extract an explicit formula for  $f_r(a, b)$ : for example, for  $f_{\&}(a, b) = \min(a, b)$ ,  $f_{\vee}(a, b) = \max(a, b)$ , and  $f_0(a, b) = \min(a, b)$ , we get  $f_r(a, b) = \min(a, b)$ .

For such extended logic, we can define a new remainder

$$a_n \stackrel{\text{def}}{=} 1 - (a_+ + a_0 + a_- + a_r)$$

for which  $a_+ + a_0 + a_- + a_r = 1$ . We can now again apply the same idea and consider cases for which  $a_+ + a_0 + a_- + a_r \leq 1$ , etc. Our approach enables us to define “and”- and “or”-operations for all such extensions.

## 5 Proofs

**Proof of Proposition 2.1.** The first condition implies that

$$f(a_-, b_-) \leq 1 - f_{\&}(a_+, b_+)$$

for all  $a_+$  and  $b_+$ , and  $b_-$  for which  $a_+ \leq 1 - a_-$  and  $b_+ \leq 1 - b_-$ . This means that  $f(a_-, b_-)$  should be smaller than or equal to the smallest of the values  $1 - f_{\&}(a_+, b_+)$  for such  $a_+$  and  $b_+$ , i.e., equivalently, that is smaller than or equal to 1 minus the largest possible value of  $f_{\&}(a_+, b_+)$ .

Since “and”-operation is non-strictly increasing in both variables, its largest value is attained when both  $a_+$  and  $b_+$  attain their largest values  $1 - a_-$  and  $1 - b_-$ . Thus, the first condition is equivalent to

$$f(a_-, b_-) \leq 1 - f_{\&}(1 - a_-, 1 - b_-).$$

The second part of the definition implies that we have equality:

$$f(a_-, b_-) = 1 - \max_{a_+ \leq 1 - a_-, b_+ \leq 1 - b_-} f_{\&}(a_+, b_+).$$

The “and”-operation is monotonic in both variables, so, for the given values of  $a_-$  and  $b_-$ , the largest value of  $f_{\&}(a_+, b_+)$  is attained when  $a_+$  and  $b_+$  attains the largest possible values  $a_+ = 1 - a_0 - a_+$  and  $b_+ = 1 - b_0 - b_+$ . Thus, we get the desired formula. The proposition is proven.

**Proof of Proposition 2.2** is similar.

**Proof of Proposition 3.1.** The first condition implies that

$$f_0(a_0, b_0) \leq 1 - (f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-))$$

for all  $a_+$ ,  $a_-$ ,  $b_+$ , and  $b_-$  for which  $a_+ + a_0 + a_- \leq 1$  and  $b_+ + b_0 + b_- \leq 1$ . Thus, we can conclude that

$$f_0(a_0, b_0) \leq 1 - \max_{a_+ + a_0 + a_- \leq 1, b_+ + b_0 + b_- \leq 1} (f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-)).$$

The second part of the definition implies that we have equality:

$$f_0(a_0, b_0) = 1 - \max_{a_+ + a_0 + a_- \leq 1, b_+ + b_0 + b_- \leq 1} (f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-)).$$

The “or”-operation is monotonic in both variables, so, for the given values of  $a_+$  and  $b_+$ , the largest value of  $f_{\vee}(a_-, b_-)$  (and thus, of the sum  $f_{\&}(a_+, b_+) + f_{\vee}(a_-, b_-)$ ) is attained when  $a_-$  and  $b_-$  attains the largest possible values  $a_- = 1 - a_0 - a_+$  and  $b_- = 1 - b_0 - b_+$ . Thus, we get the desired formula. The proposition is proven.

**Proof of Proposition 3.2.** For  $a_+ = b_+ = 0$ , we get

$$\begin{aligned} f_{\&}(a_+, b_+) + f_{\vee}(1 - a - a_+, 1 - b - b_-) &= f_{\&}(0, 0) + f_{\vee}(1 - a, 1 - b) \\ &= \min(0, 0) + \max(1 - a, 1 - b). \end{aligned}$$

For all other values  $a_+ \leq 1 - a$  and  $b_+ \leq 1 - b$ , we have

$$\begin{aligned} v &\stackrel{\text{def}}{=} f_{\&}(a_+, b_+) + f(1 - a - a_+, 1 - b - b_+) \\ &= \min(a_+, b_+) + \max(1 - a - a_+, 1 - b - b_+). \end{aligned}$$

If we add the same constant  $c$  to two numbers  $p$  and  $q$ , the same number will remain larger, so we get  $\max(p + c, q + c) = \max(p, q)$ . In particular, we get

$$v = \max(1 - a - a_+ + \min(a_+, b_+), 1 - b - b_+ + \min(a_+, b_+)).$$

In the first of the two minimized terms, we can use the fact that  $\min(a_+, b_+) \leq a_+$ , thus

$$1 - a - a_+ + \min(a_+, b_+) \leq 1 - a - a_+ + a_+ = 1 - a.$$

In the second term, we can similarly use the fact that  $\min(a_+, b_+) \leq b_+$ , thus

$$1 - b - b_+ + \min(a_+, b_+) \leq 1 - b - b_+ + b_+ = 1 - b.$$

Hence,  $v \leq \max(1 - a, 1 - b)$ . So, the maximum is indeed  $\max(1 - a, 1 - b)$ .

Thus,  $f_0(a, b) = 1 - \max(1 - a, 1 - b)$ . When the subtracted number is the largest, the difference is the smallest, so we get

$$f_0(a, b) = \min(1 - (1 - a), 1 - (a - b)) = \min(a, b).$$

The proposition is proven.

**Proof of Proposition 3.3.** For  $a_+ = b_+ = 0$ , we get

$$\begin{aligned} f_{\&}(a_+, b_+) + f_{\vee}(1 - a - a_+, 1 - b - b_+) &= f_{\&}(0, 0) + f_{\vee}(1 - a, 1 - b) \\ &= 0 \cdot 0 + 1 - a + 1 - b - (1 - a) \cdot (1 - b) \\ &= 1 - a + 1 - b - 1 + a + b - a \cdot b = 1 - a \cdot b. \end{aligned}$$

For all other values  $a_+ \leq 1 - a$  and  $b_+ \leq 1 - b$ , we have

$$\begin{aligned} &f_{\&}(a_+, b_+) + f(1 - a - a_+, 1 - b - b_+) \\ &= a_+ \cdot b_+ + 1 - a - a_+ + 1 - b - b_+ - (1 - a - a_+) \cdot (1 - b - b_+) \\ &= a_+ \cdot b_+ + 1 - a - a_+ + 1 - b - b_+ - 1 + a + a_+ + b + b_+ - a \cdot b - a \cdot b_+ - a_+ \cdot b - a_+ \cdot b_+ \\ &= 1 - a \cdot b - a \cdot b_+ - a_+ \cdot b \leq 1 - a \cdot b. \end{aligned}$$

Thus, the maximum is indeed  $1 - a \cdot b$ , and 1 minus this maximum is simply  $a \cdot b$ . The proposition is proven.

**Proof of Proposition 3.4** is similar to the proof of Proposition 3.1.

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