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Advances in Fuzzy Possibility Theory

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Abstract

Modelling under uncertainty has become an attractive topic for managerial decision making since uncertainty can lead to an increase in cost, risk, efficiency, and market competitiveness. It is acknowledged that fuzzy possibility theory may exhibit advantages of quantifying accurately uncertain information or imprecise data. To reduce the uncertain information embedded in the secondary possibility distribution of a type-2 fuzzy variable, a growing amount of research effort is dedicated to decrease greatly the computing complexity. This paper provides an overview of the latest advances and developments in the methods of reduction techniques and the applications for handling uncertainty occurring in the engineering and management problems. This paper seeks to: (i) offer a comprehensive review on the predominant type reduction methods, (ii) highlight the pertinent and widely used parametric credibilistic programming from the perspective of modeling and optimization methodology, as well as (iii) point out the findings and potential directions for future research.

Keywords: fuzzy possibility theory, type-2 fuzzy variable, reduction method, credibilistic programming

1 Introduction

Uncertainty appearing in the real decision making processes is an inherent characteristics of information. As an efficient approach, fuzzy set has a long tradition in capturing quantitatively imprecision or linguistic information. However, it is always difficult to assign an exact membership grade of each point for a fuzzy set. Type-2 fuzzy set, introduced by Zadeh, provides an extra degree of freedom to measure uncertainty over the ordinary fuzzy set because of its three dimensional nature. Since then, a large number of research works on the type-2 fuzzy theory have been reported by some researchers. Especially, an axiomatic approach, called fuzzy possibility theory, was deeply explored by Liu and Liu [28] to handle type-2 fuzziness. The authors presented some fundamental concepts with the intention of adopting a variable-based approach to dealing with type-2 fuzziness, which facilitate the use of modern mathematical tools to investigate fuzzy possibility theory.

Defuzzification maps fuzzy sets to crisp values. Type-2 defuzzification maps type-2 fuzzy sets to non-fuzzy values [34]. In order to facilitate type-2 defuzzification, type reduction which maps type-2 fuzzy sets to type-1 fuzzy sets is an undetachable module of type-2 fuzzy computation and optimization. Several type reduction methods for type-2 fuzzy sets have been presented in the literature. In fuzzy possibility theory, type reduction methods that collect the entire inherent vagueness of the data and form the output of type-2 fuzzy variables as the representative of the whole uncertainty also play an important role in various fuzzy disciplines. This paper attempts to provide a comprehensive summary on type reduction methods. They include the mean value (MV) reduction [32], critical value (CV) reduction [33], equivalent value (EV) reduction [37], value-at-risk (VaR) reduction [3, 4] and λ selection methods [9, 14, 26]. The above-mentioned reductions not only can decrease the computation complexity of type-2 fuzzy variables, but also can retain the important information represented by the original type-2 fuzzy variables.

Parametric credibilistic programming [23] is a new kind of optimization mean in the case of type-2 fuzziness. It is based on fuzzy possibility theory, type reduction methods, credibility measure theory [22] and

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fuzzy optimization techniques [24]. Qin et al. [33] employed parametric credibilitstic programming to data envelope analysis problem in 2011. Up to now, this optimization methodology has found extensive use in miscellaneous engineering and management problems, such as facility location and allocation, portfolio selection, emergency supplies prepositioning, and transportation problem. The present review also concentrates on these development of real applications. The purpose of this paper is to (i) conduct a representative review on the predominant type reduction methods in fuzzy possibility theory, (ii) highlight the pertinent and widely used parametric credibilistic programming, and (iii) summarize the findings and identify potential directions for future research.

The rest of the paper is organized as follows. Section 2 presents a brief description of the existing type reduction strategies in fuzzy possibility theory. Section 3 reviews parametric credibilistic programming and its applications. Conclusions will be presented in Section 4.

2 Reduction Methods in Fuzzy Possibility Theory

Let ξ be a type-2 fuzzy variable [28] with the secondary possibility distribution function $\tilde{\mu}_\xi(x)$, and type-2 possibility distribution function $\mu_\xi(x,u)$. The support [28] of ξ is defined as $\{(x,u)\in\Re^m\times[0,1]\mid\mu_\xi(x,u)>0\}$.

To reduce the type-2 fuzziness, one approach is to give a representing value for $\tilde{\mu}_\xi(x)$. Based on this idea, corresponding to different representing value, various methods are proposed to reduce the uncertain information embedded in the secondary possibility distribution.

Five reduction methods and their used representing values are listed in Table 1 that clearly shows the differences among these methods. Through CV reduction method, mean reduction method and EV reduction method, ξ are reduced to three different reduced fuzzy variables, respectively. The corresponding possibility distributions of the reduced variables have parameters θ_l and θ_r. There are parameters θ_l, θ_r and α in the possibility distributions of VaR reduced fuzzy variables and CVaR reduced fuzzy variables. The parameters θ_l and θ_r describe the degree of uncertainty that ξ takes its value, and determine the lower and upper boundaries of possibility distribution. The parameter α represents the possibility level in the support of ξ and determines the location of possibility distribution. When parameter α varies in the interval [0,1], the possibility distribution varies between the lower and upper boundaries. Therefore, by introducing the parameter α, the VaR reduction methods and CVaR reduction method generalize mean value reduction methods, CV reduction methods and EV reduction methods. Since $\mathrm{CVaR}_{\frac{\alpha}{2}}(\eta)=\mathrm{VaR}^L_\alpha(\eta)$ and $\mathrm{CVaR}_{1-\frac{\alpha}{2}}(\eta)=\mathrm{VaR}^U_\alpha(\eta)$, then the CVaR reduction method generalizes the VaR reduction methods.

Table 1: Reduction methods and their used representing values of $\tilde{\mu}_\xi(x)$

Methods	The representing values of $\eta=\tilde{\mu}_\xi(x)$
MV [32]	upper expectation $\mathcal{E}^*[\eta]$, lower expectation $\mathcal{E}_*[\eta]$, expectation $\mathcal{E}[\eta]$
CV [33]	optimistic critical value $\mathrm{CV}^*[\eta]$, pessimistic critical value $\mathrm{CV}_*[\eta]$, critical value $\mathrm{CV}[\eta]$
EV [37]	pessimistic equivalent value $\mathrm{EV}_*[\eta]$, optimistic equivalent value $\mathrm{EV}^*[\eta]$, equivalent value $\mathrm{EV}[\eta]$
VaR [3]	lower value-at-risk $\mathrm{VaR}^L_\alpha(\eta)$, upper value-at-risk $\mathrm{VaR}^U_\alpha(\eta)$
CVaR [4]	credibilistic value-at-risk $\mathrm{CVaR}_\alpha(\eta)$

If for any $x\in\Re$, $\tilde{\mu}_\xi(x)$ is a subinterval $[\mu_{\xi^L}(x;\theta_l),\mu_{\xi^U}(x;\theta_r)]$ of $[0,1]$ with parameters $\theta_l,\theta_r\in[0,1]$, then ξ is called a parametric interval-valued fuzzy variable [26]. It is a particular type-2 fuzzy variable. To characterize a parametric interval-valued fuzzy variable, its lower selection, upper selection, pessimistic cross selection and optimistic cross selection were presented.

According to Liu and Liu [26], the fuzzy variable ξ^L described by the possibility distribution $\mu_{\xi^L}(x;\theta_l)$ is called the lower selection of ξ, the fuzzy variable ξ^U characterized by the possibility distribution $\mu_{\xi^U}(x;\theta_r)$ is called the upper selection of ξ. For any $\lambda\in[0,1]$, a fuzzy variable ξ^λ is called a λ selection of ξ provided that ξ^λ is characterized by the parametric possibility distribution $\mu_{\xi^\lambda}(x;\theta)=(1-\lambda)\mu_{\xi^L}(x;\theta_l)+\lambda\mu_{\xi^U}(x;\theta_r),\theta=(\theta_l,\theta_r)$. In accordance with Chen and Shen [9], for any $\lambda\in[0,1]$, a fuzzy variable ξ_λ with the parametric possibility distribution $\mu_{\xi_\lambda}(x;\theta_l,\theta_r)=\lambda\mu_{\xi_P}(x;\theta_l,\theta_r)+(1-\lambda)\mu_{\xi_O}(x;\theta_l,\theta_r)$ is called a λ cross selection of ξ, where the fuzzy variables ξ_P and ξ_O are the pessimistic cross selection and optimistic cross selection of ξ, respectively. For a generalized parametric interval-valued trapezoidal fuzzy variable, i.e. $\mu_{\xi^L}(x;\theta_l)$ and $\mu_{\xi^U}(x;\theta_r)$ are generalized parametric possibility distributions, Guo and Liu [14] used two different parameters

λ_1 and λ_2 to determine the possibility distribution in the left span and right span of the trapezoidal fuzzy variable, respectively, and defined its dual parameter selection variable.

In the possibility distributions of λ selection variables, λ cross selection variables and dual parameter selection variables, there are parameters θ_l, θ_r and λ. The parameters θ_l and θ_r describe the degree of uncertainty that a type-2 fuzzy variable ξ takes its value, and determine the lower and upper boundaries of possibility distribution. The parameter λ determines the location of possibility distribution.

The above-mentioned methods reduce uncertain information embedded in the secondary possibility distribution, and retain the most important information in its parametric possibility distributions.

3 Parametric Credibilistic Programming

Parametric credibilistic programming is a main tool to model the optimization problem in the case of type-2 fuzziness. Parametric credibilistic programming has been employed in the different application fields, which include data envelope analysis [7, 32, 33, 41, 42], transportation problem [8, 10, 11, 17, 18, 19, 20, 21, 29, 31, 35], location problem [39, 40, 43], cellular manufacturing system design problem [1], portfolio selection [37], emergency supplies prepositioning problem [2, 6], project portfolio selection problem [27], inventory management [12, 13, 14], supply chain management [5, 16], energy management [36, 38], waste management [30], water pollution control problem [25], support vector machine model [15], and so on. The investigated references in this paper bear witness the actuality that parametric credibilistic programming has gained great attention in the last several years.

The reasons for the widespread use of parametric credibilistic programming are explained as follows. On the one hand, it takes advantage of explicitly characterizing uncertain information by type-2 fuzzy variables. On the other hand, it points out several feasible optimization criteria to develop a mathematical model after type reductions of type-2 fuzzy parameters which guarantee that the important information don't lost. Moreover, parametric credibilistic programming offers a more flexible decisions by controlling the parametric possibility distributions, which displays some superiorities over fixed possibility distributions.

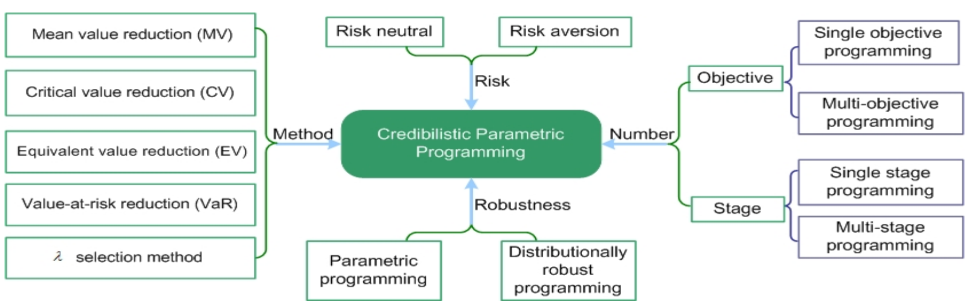

Figure 1: Classification criteria of parametric credibilistic programming

This paper assesses the state of the art in the area of parametric credibilistic programming models from different standards. Figure 1 presents the criteria used to dissect and organize this review. For instance, from the perspective of type reduction methods, we make the distinction among MV, CV, EV, VaR reduction and λ selection. From the perspectives of risk, we divide the models into risk neutral and risk aversion. From the perspective of number, we consider the amount of objective functions and modeling stage. In a second level of the classification, we offer a further categorization. For instance, we divide the models into single objective programming and multi-objective programming. Additionally, we distinguish between those papers which based on robust optimization methodology and those papers that ignore robustness. According to these classification criteria, the evaluation result is reported in Table 2.

In the light of the previously reviewed studies in Table 2, CV reduction method is taken preponderantly into consideration among different type reduction approaches. The reason lies the fact that the parametric possibility distributions of CV reduced fuzzy variables possess a better characteristics. That is, the possibility of demarcation points is not dependent on the parameters θ_l and θ_r. Through CV reduction method, the credibility constraints and VaR objective function can be transformed easily into their equivalent forms from a practical point of view. From the perspective of risk, the models with risk-averse objective is more than ones with risk-neutral objective. This phenomenon is beneficial from the risk meaning and tractability of value-at-risk criterion. Another finding that can be drawn from Table 2 is that there are only a few studies focusing on multiple objectives or multiple stages. Maybe it is a major challenge taking both dynamic and fuzzy properties of the uncertain data in account. Recently, Guo and Liu [13] have also given insights into new avenues for parametric credibilistic programming in a broad manner. The distributionally robust fuzzy optimization is still in the early stage, and some significant works remain to be explored in future.

Table 2: Classification for parametric credibilistic programming in applications

Reference	Type reduction					Risk		Objective		Stage		Robustness	
	MV	CV	EV	VaR	λ	Netural	Aversion	Single	Multiple	Single	Multiple	Yes	No
[32]	✓					✓		✓		✓			✓
[33]		✓					✓	✓		✓			✓
[41]		✓					✓		✓	✓			✓
[42]		✓				✓			✓		✓		✓
[7]					✓	✓		✓		✓			✓
[29]		✓				✓		✓		✓			✓
[18]		✓					✓	✓		✓			✓
[19]		✓					✓	✓		✓			✓
[8]		✓					✓		✓		✓		✓
[31]		✓					✓	✓		✓			✓
[10]		✓					✓	✓		✓			✓
[11]		✓					✓		✓	✓			✓
[17]	✓	✓					✓	✓		✓			✓
[35]		✓					✓		✓	✓			✓
[39]	✓						✓	✓		✓			✓
[40]	✓					✓			✓	✓			✓
[43]	✓						✓	✓			✓		✓
[37]			✓				✓	✓		✓			✓
[2]				✓		✓		✓		✓			✓
[6]				✓			✓	✓		✓			✓
[27]					✓		✓	✓		✓			✓
[14]					✓	✓		✓		✓			✓
[13]					✓	✓		✓		✓		✓	
[12]		✓				✓		✓		✓			✓
[5]				✓			✓	✓		✓			✓
[36]		✓				✓		✓		✓			✓
[38]		✓				✓		✓		✓			✓
[30]		✓				✓		✓		✓			✓
[25]		✓				✓		✓			✓		✓
[15]		✓				✓		✓		✓			✓

4 Conclusions

This paper offers a representative review on the recent use of fuzzy possibility theory for manipulating with uncertainties in the context of engineering and management application. As an axiomatic approach, fuzzy possibility theory would be beneficial to model real-life optimization problem under type-2 fuzziness framework. We summarize systematically the usual type reduction methods, that make type-2 fuzzy variables to type-1 fuzzy variables. Thus, the computation complexity of type-2 fuzzy variables can be decreased to a large extent. Most of all, the important information represented by the original type-2 fuzzy variables can be retained after the above-mentioned reductions. In addition, parametric possibility distributions of reduced fuzzy variables or selection fuzzy variables also provide a more flexible decisions in practical optimization problems, which is helpful to promote the development of parametric credibilistic optimization methods. A large number of papers coping with transportation problem, supply chain network design, and so on show the advantage of parametric credibilistic optimization methods, which give an informative view of the state of the art for researchers and practitioners. To conclude, notwithstanding the fact that fuzzy possibility theory has attracted much attention in the last ten years, we expect that this trend will continue.

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References

[1] Arghish, O., Tavakkoli-Moghaddam, R., Shahandeh-Nookabadi, A., and J. Rezaeian, An integrated cellular manufacturing system with type-2 fuzzy variables: three tuned meta-heuristic algorithms, *Journal of Intelligent & Fuzzy Systems*, vol.35, no.2, pp.2293–2308, 2018.

[2] Bai, X., Optimization for pre-positioning emergency supplies problem under fuzzy environment, *Systems Engineering – Theory & Practice*, vol.35, no.6, pp.1465–1473, 2015.

[3] Bai, X., and Y.K. Liu, Semideviations of reduced fuzzy variables: a possibility approach, *Fuzzy Optimization and Decision Making*, vol.13, pp.173–196, 2014.

[4] Bai, X., and Y.K. Liu, CVaR reduced fuzzy variables and their second order moments, *Iranian Journal of Fuzzy Systems*, vol.12, no.5, pp.45–75, 2015.

[5] Bai, X., and Y.K. Liu, Robust optimization of supply chain network design in fuzzy decision system, *Journal of Intelligent Manufacturing*, vol.27, no.6, pp.1131–1149, 2016.

[6] Bai, X., Gao, J., and Y.K. Liu, Prepositioning emergency supplies under uncertainty: a parametric optimization method, *Engineering Optimization*, vol.50, no.7, pp.1114–1133, 2018.

[7] Bai, X., Zhang, F., and Y.K. Liu, Modeling fuzzy data envelopment analysis under robust input and output data, *RAIRO-Operations Research*, vol.52, no.2, pp.619–643, 2018.

[8] Baidya, A., Bera, U.K., and M. Maiti, Multi-stage multi-objective solid transportation problem for disaster response operation with type-2 triangular fuzzy variables, *Hacettepe Journal of Mathematics and Statistics*, vol.45, no.5, pp.1485–1518, 2016.

[9] Chen, Y., and S. Shen, The cross selections of parametric interval-valued fuzzy variables, *Journal of Uncertain Systems*, vol.9, no.2, pp.156–160, 2015.

[10] Das, A., Bera, U.K., and M. Maiti, A breakable multi-item multi stage solid transportation problem under budget with Gaussian type-2 fuzzy parameters, *Applied Intelligence*, vol.45, no.3, pp.923–951, 2016.

[11] Das, A., Bera, U.K., and M. Maiti, Defuzzification and application of trapezoidal type-2 fuzzy variables to green solid transportation problem, *Soft Computing*, vol.22, no.7, pp.2275–2297, 2018.

[12] Debnath, B.K., Majumder, P., Bera, U.K., and M. Maiti, Inventory model with demands as type-2 fuzzy number: a fuzzy differential equation approach, *Iranian Journal of Fuzzy Systems*, vol.15, no.1, pp.1–24, 2018.

[13] Guo, Z., and Y.K. Liu, Modelling single-period inventory problem by distributionally robut fuzzy optimization method, *Journal of Intelligent & Fuzzy Systems*, vol.35, no.1, pp.1007–1019, 2018.

[14] Guo, Z., Liu, Y.K., and Y. Liu, Coordinating a three level supply chain under generalized parametric interval-valued distribution of uncertain demand, *Journal of Ambient Intelligence and Humanized Computing*, vol.8, no.5, pp.677–694, 2017.

[15] Ha, M., Yang, Y., and C. Wang, A new support vector machine based on type-2 fuzzy samples, *Soft Computing*, vol.17, pp.2065–2074, 2013.

[16] Jana, D.K., Pramanik, S., and M. Maiti, A parametric programming method on Gaussian type-2 fuzzy set and its application to a multilevel supply chain, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.24, no.3, pp.451–477, 2016.

[17] Jana, D.K., Pramanik, S., and M. Maiti, Mean and CV reduction methods on Gaussian type-2 fuzzy set and its application to a multilevel profit transportation problem in a two-stage supply chain network, *Neural Computing and Applications*, vol.28, no.9, pp.2703–2726, 2017.

[18] Kundu, P., Kar, S., and M. Maiti, Fixed charge transportation problem with type-2 fuzzy variables, *Information Sciences*, vol.255, pp.170–186, 2014.

[19] Kundu, P., Kar, S., and M. Maiti, Multi-item solid transportation problem with type-2 fuzzy parameters, *Applied Soft Computing*, vol.31, pp.61–80, 2015.

[20] Kundu, P., Kar, S., and M. Maiti, A fuzzy multi-criteria group decision making based on ranking interval type-2 fuzzy variables and an application to transportation mode selection problem, *Soft Computing*, vol.21, pp.3051–3062, 2017.

[21] Kundu, P., Majumder, S., Kar, S., and M. Maiti, A method to solve linear programming problem with interval type-2 fuzzy parameters, *Fuzzy Optimization and Decision Making*, vol.18, pp.103–130, 2019.

[22] Liu, Y.K., *Credibility Measure Theory: A Modern Methodology of Handling Subjective Uncertainty*, Science Press, Beijing, 2018.

[23] Liu, Y.K., Bai, X., and K. Yang, *Parametric Credibilistic Optimization Methods*, Science Press, Beijing, 2017.

[24] Liu, Y.K., Chen, Y., Liu, Y., and R. Qin, *Fuzzy Optimization Methods with Applications*, Science Press, Beijing, 2013.

[25] Liu, J., Li, Y., Huang, G., and L. Chen, A recourse-based type-2 fuzzy programming method for water pollution control under uncertainty, *Symmetry-Basel*, vol.9, no.11, ID 265, 2017.

[26] Liu, Y., and Y.K. Liu, The lambda selections of parametric interval-valued fuzzy variables and their numerical characteristics, *Fuzzy Optimization and Decision Making*, vol.15, no.3, pp.255–279, 2016.

[27] Liu, Y., and Y.K. Liu, Distributionally robust fuzzy project portfolio optimization problem with interactive returns, *Applied Soft Computing*, vol.56, pp.655–668, 2017.

[28] Liu, Z., and Y.K. Liu, Type-2 fuzzy variables and their arithmetic, *Soft Computing*, vol.14, no.7, pp.729–747, 2010.

[29] Liu, P., Yang, L., Wang, L., and S. Li, A solid transportation problem with type-2 fuzzy variables, *Applied Soft Computing*, vol.24, pp.543–558, 2014.

[30] Ma, X., Ma, C., Wan, Z., and K. Wang, A fuzzy chance-constrained programming model with type 1 and type 2 fuzzy sets for solid waste management under uncertainty, *Engineering Optimization*, vol.49, no.6, pp.1040–1056, 2017.

[31] Pramanik, S., Jana, D.K., Mondal, S.K., and M. Maiti, A fixed-charge transportation problem in two-stage supply chain network in Gaussian type-2 fuzzy environments, *Information Sciences*, vol.325, pp.190–214, 2015.

[32] Qin, R., Liu Y.K., and Z.Q. Liu, Modeling fuzzy data envelopment analysis by parametric programming method, *Expert Systems with Applications*, vol.38, no.7, pp.8648–8663, 2011.

[33] Qin, R., Liu, Y.K., and Z.Q. Liu, Methods of critical value reduction for type-2 fuzzy variables and their applications, *Journal of Computational and Applied Mathematics*, vol.235, no.5, pp.1454–1481, 2011.

[34] Runkler, T.A., Chen, C., and R. John, Type reduction operators for interval type-2 defuzzification, *Information Sciences*, vol.467, pp.464–476, 2018.

[35] Sengupta, D., Das, A., and U.K. Bera, A gamma type-2 defuzzification method for solving a solid transportation problem considering carbon emission, *Applied Intelligence*, vol.48, no.11, pp.3995–4022, 2018.

[36] Suo, C., Li, Y.P., Wang, C.X., and L. Yu, A type-2 fuzzy chance-constrained programming method for planning Shanghai's energy system, *International Journal of Electrical Power & Energy Systems*, vol.90, pp.37–53, 2017.

[37] Wu, X., and Y.K. Liu, Optimizing fuzzy portfolio selection problems by parametric quadratic programming, *Fuzzy Optimization and Decision Making*, vol.11, no.4, pp.411–449, 2012.

[38] Xu, Y., Liu, F., Huang, G., and G. Cheng, An optimization model under interval and fuzzy uncertainties for a by-product gas system of an iron and steel plant, *Engineering Optimization*, vol.51, no.3, pp.447–464, 2019.

[39] Yang, K., Liu, Y.K., and G. Yang, Optimizing fuzzy p-hub center problem with generalized value-at-risk criterion, *Applied Mathematical Modelling*, vol.38, pp.3987–4005, 2014.

[40] Yang, K., Yang, L.X., and Z.Y. Gao, Hub-and-spoke network design problem under uncertainty considering financial and service issues: a two-phase approach, *Information Sciences*, vol.402, pp.15–34, 2017.

[41] Zhou, X., Pedrycz, W., Kuang, Y., and Z. Zhang, Type-2 fuzzy multi-objective DEA model: an application to sustainable supplier evaluation, *Applied Soft Computing*, vol.46, pp.424–440, 2016.

[42] Zhou, X., Xu, Z., Yao, L., Tu, Y., Lev, B., and W. Pedrycz, A novel data envelopment analysis model for evaluating industrial production and environmental management system, *Journal of Cleaner Production*, vol.170, pp.773–788, 2018.

[43] Zhou, X., Yu, N., Tu, Y., Pedrycz, W., and B. Lev, Bi-level plant selection and production allocation model under type-2 fuzzy demand, *Expert Systems with Applications*, vol.86, pp.87–98, 2017.