Journal of Uncertain Systems
Vol.13, No.1, pp.56-74, 2019
Online at: www.jus.org.uk

Hedging Contract for Value at Risk based Risk Management

Sovan Mitra*
University of Liverpool,Brownlow Hill,Liverpool,L69 7ZX,United Kingdom.

Received 5 December 2018; Revised 18 December 2018

Abstract

Risk management has become one of the core functions of all businesses over the past 30 years, it is a vital part of every business organisation. Whilst a number of risk management methods have been devised over the years to mitigate risk, many firms still suffer from being unable to manage losses. Value at Risk has become an industry standard for risk measurement and risk based decision making. However Value at Risk based decision making leads to problems in managing the risk identified using this method. In this paper we provide a method for hedging risk that is determined using the Value at Risk methodology.

Keywords: value at risk, risk management, hedging, uncertain systems, stochastic processes

1 Introduction

Risk management has grown over the past 30 years, and is a vital part of every business organisation. Essentially, risk management is the identification, mitigation or reduction of uncertain outputs. Risks can occur from a variety of sources, such as project risk, political risk, credit risk etc. to name a few risk factors. Given that such risks can have significant impact on firms, there has been substantial research to investigate methods to manage risk in general.

A key part of risk management is the identification and quantification of risk. In order for risk to be managed effectively, it must firstly be identified. This is a non-trivial task because it is widely reported that many unanticipated risks can cause significant damage to a company. The issue of risk identification is a significant area of research in itself, however the focus of our research is related to the other key part of risk management: risk quantification.

Risk quantification or risk measurement has grown into a significant area of research in itself, with applications across a range of industries. One of the most widely used risk measures in industry is VaR (Value at Risk), which has grown in popularity over the past 30 years. Whilst VaR has some significant disadvantages (such as VaR cannot take into account diversification and is not a coherent risk measure) it is still the most widely used risk measure; it is used by many industries. VaR has the advantage of being widely applicable to a range of purposes, not just investment analysis, it can be applied to many types of risk. Additionally, VaR can be easily applied to risk analysis and risk based decision making, as it clearly captures the risk facing a company.

Whilst VaR is a popular risk measure and has risk analysis advantages, the method of measuring risk using VaR causes problems in risk management (for example the VaR methodology tends to focus on losses, rather than gains, from any risky event). One of the more fundamental problems with VaR based risk analysis is that one needs to be able to devise a method of managing the risk. As mentioned previously, risk management has been studied extensively due to its importance in industry.

In the context of VaR, a possible risk management strategy would be risk elimination, that is the firm would exclude itself from such situations to remove exposure to such risks. Whilst this strategy is attractive in simplicity it is also not frequently practical, as many firms cannot exclude many situations and risks e.g. political risk, economic risk etc.. Another risk management strategy that is more frequently employed (and is more applicable to a range of situations) is hedging. This is essentially purchasing some service or product to transfer the risk incurred by the firm. Although the firm incurs a cost in purchasing the hedging contract, it is

*Corresponding author.
Email: Sovan.Mitra@liverpool.ac.uk (S. Mitra).

not exposed to extreme losses. This is also consistent with the VaR risk measurement methodology in reducing risk. A practical case in point is the supply chain risk management industry, where a VaR methodology may be applied along with a hedging contract.

We should note that whilst risk measures other than VaR exist, such as variance, spectral analysis, and partial moments (see for instance [22] for a review), such risk measures do not have such a direct physical representation in the real world. Hence alternative risk measures would not be as preferable to VaR in industries that are directly involved in real world operations. Furthermore, we should also note that the incentive to purchase a hedging contract (when a VaR method is applied) is not just applicable to the supply chain industry. Such hedging contracts would be useful in other risk areas, such as project management risks, economic risks and environmental risks.

In supply chain risk management, VaR has been used to measure the risk of firms that supply goods to buyers (see for example [6] and [12]). In a VaR methodology the risk represents the possibility that the supplier cannot sell its goods to a buyer, in other words the supplier has an excess supply of goods. Hence under a VaR risk management methodology the excess supply represents a risk or loss to the supplier. One way to hedge out such a risk is to transfer this risk by selling to alternative buyers. The supplier buys a contract so that it can sell excess supplies to alternative buyers at a pre-agreed price. The alternative buyer may be interested in such an arrangement because the supplier will typically agree to selling at a pre-agreed price that is typically discounted to standard market prices.

The model of the hedging contract is as follows. The supplier purchases a contract that enables it to sell its (excess) supply at a pre-agreed price to the alternative buyer. The supplier is not obligated to sell the goods at any particular time, however the buyer will pay for the goods once the supplier sells them. The buyer can then sell the goods onto its own customers. Although the buyer cannot force the time of sale of goods, the buyer can determine the timing of the sale of the hedging contract. We model the sale of the hedging contract as an optimal stopping problem. The method of optimal stopping has been used in many applications to improve resource allocation problems, for example see [2, 14, 16, 20].

In order for this hedging method to be viable the (alternative) buyers need to determine its buying price from the supplier. Additionally, the supplier's excess supply of goods will affect the price of the goods, since high supply reduces prices. In this paper we provide a solution to the buyer's purchasing problem when suppliers adopt a VaR risk methodology for risk management. Specifically, we provide a mathematical model of the hedging operation and optimise the buyer's purchase decision as an optimal stopping problem. We follow [21] in our proofs.

This paper makes a number of contributions. Firstly, we provide a mathematical model of the hedging contract; we model the operations, the hedging contract and the decision to use and sell the contract. Secondly, we model the buyer's decision as an optimal stopping problem and solve this to provide the optimal stopping criteria. Thirdly, we provide closed form solutions to payoffs in our model. Fourth, we derive the limiting and long term behaviour of our hedging contract, which provides insight on the long term impact of this process.

This paper is organised as follows. In the next section we introduce our preliminaries, the supplier model for excess supply, the associated pricing model and the discounting process. In the next section we model the buyer's problem in terms of an optimal stopping problem. In the next section we provide our solution to the optimal stopping problem in Theorem 1. In the next section we provide closed form solutions for our payoffs, specifically by proving Theorems 2 and 3. In the next section we examine the long term behaviour of hedging operation, by proving 4 lemmas which enable us to prove Theorem 4.

2 Preliminaries and Pricing Model

Let $S(t)$ denote the excess supply of goods held by a supplier at time t. Let us define the probability space for $S(t)$ that is $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_s)$, such that $\mathbb{P}_s(S(0) = s) = 1$. We assume that the filtration $\mathcal{F}_t$ for $t \geq 0$ is right continuous. Let the mapping $s \rightarrow \mathbb{E}_s[X]$ be measurable for some random variable X, where $\mathbb{E}_s$ denotes the expectation under probability measure $\mathbb{P}_s$.

Without loss of generality, we will assume that $(\Omega, \mathcal{F})$ equals the canonical space of continuous trajectories $\Omega = \mathcal{C}([0, \infty); \mathbb{R})$ with Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$, so that the shift operation $\Theta_t : \Omega \rightarrow \Omega$ is well-defined by $\Theta_t(\omega)(u) = \omega(t + u)$, for $\omega = (\omega(u))_{u \geq 0} \in \Omega$ and $u, t \geq 0$. We will make use of the following known properties: the Brownian motion $S(t)$ is a strong Markov process relative to $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_s)$ and for a given

Borel measurable subset $X \subseteq \mathbb{R}$ that the time

$$\Gamma_X = \inf\{t \geq 0 | S(t) \in X\},$$

is an $\mathcal{F}$-stopping time, and we use the convention $\inf \emptyset = \infty$.

The stochastic process $S = \{S(t)\}_{t\geq 0}$ is a one dimensional Brownian motion (also known as a Wiener process) on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_s)$. The $S(t)$ has continuous trajectories, that is $t \to S(t, \omega)$ is continuous in t for all $\omega \in \Omega$, the increments are independent, that is $S(t_2) - S(t_1)$ and $S(t_4) - S(t_3)$ are independent for $t_1 < t_2 \leq t_3 < t_4$, such that $\{t_1, t_2, t_3, t_4\} \in \mathbb{R}_{>0}$. Additionally,

$$S(t_2) - S(t_1) \sim \mathcal{N}(0, t_2 - t_1), \forall t_1 < t_2,$$

where $\mathcal{N}(\iota, \tilde{\sigma})$ denotes the Normal distribution with mean ι and variance $\tilde{\sigma}$.

The surplus model for $\{S(t) \in \mathbb{R}^+ | t \geq 0\}$ implies that an oversupply of goods exists with the supplier. The supplier will dispose of excess supply for a number of reasons: firstly the supplier will have limited space and will need to provide space for newly arriving inventory, and secondly demand in the goods may have fallen for a variety of reasons. The Brownian motion model for $S(t)$ provides a good model of surplus quantities because it enables us to model $S(t)$ in continuous time, whilst also taking into account the random (and unpredictable) nature of excess supply. This is not an uncommon model for modelling excess supply, see for example [1, 7, 16]. We note in passing that our model can also take into account $\{S(t) \in \mathbb{R}^- | t > 0\}$, that is negative values which indicate a supply shortage.

In addition to the quantity of surplus supply $S(t)$, we also require the price of each good. We model the price of goods as a function of the quantity of supply, that is price is modelled as $f(S(t))$. We model prices according to a set of scenarios as follows: for $\{f(S(t)) = 0 | S(t) \geq \mu_2\}$, where $\mu_2 \in \mathbb{R}_{>0}$ is a constant. In this scenario the excess supply is extremely high and the supplier simply needs to dispose of the goods, and so will sell it at zero price. From an economic perspective, an excessively high supply would push prices down along the supply curve, and so prices would be very close to 0.

In a contrary scenario, where there is a high shortage of goods, we have $\{f(S(t)) = \Lambda | S(t) \leq \mu_1\}$, where $\Lambda \in \mathbb{R}_{>0}$ and $\mu_1 \in \mathbb{R}_{<0}$ are constants. In scenarios of high shortage, the prices could theoretically increase as shortages increase (in accordance with demand and supply curves). However, we assume the price will be limited to $\Lambda \in \mathbb{R}_{>0}$, $\forall S(t) \leq \mu_1$, so that firms cannot charge excessively high prices to consumers. This is to reflect the pricing structure of many real world retail markets, where market prices are regulated to protect consumers from excessively high prices.

In between the previous two scenarios, the prices will change with quantity $S(t)$, where there is an inverse relation between $S(t)$ and $f(S(t))$. Consequently we have the model

$$f(s) \quad = \quad \Lambda.\mathbb{1}_{\{s \leq \mu_1\}} + (e + ls).\mathbb{1}_{\{\mu_1 < s \leq \mu_2\}},$$

where $S(t) = s$ is the current value of $S(t)$, $f(s)$ is the price of goods now, $\mu_1 = (\Lambda - e)/l, \mu_2 = -e/l$, e and l are constants, such that $\{\Lambda, e \in \mathbb{R}_{>0}\}$ and $\{l \in \mathbb{R}_{<0}\}$. The constant e determines the "equilibrium" price, that is the price that exists when there is neither an oversupply nor a shortage.

We now examine the hedging contract to sell surplus goods to buyers. The supplier purchases the contract V which enables the supplier to sell its surplus at a pre-agreed price K (which we refer to as the strike price). The supplier must purchase the contract *a priori* and has some forecasted estimate on the oversupply of goods in the future. In other words, the supplier must have some measure of risk on the oversupply. To be more precise, let us assume we have a real valued random variable $X \in \mathbb{R}$ within the measurable space $\{\Omega, \mathcal{F}\}$, where X follows a distribution of losses $\mathcal{G}$, then a risk measure $\tilde{\lambda}$ is defined by

$$\tilde{\lambda} : \mathcal{G} \mapsto \mathbb{R}.$$

We assume the standard risk measure assumptions of coherent risk measurement [3], that is translation invariance, subaddivity, monotonicity and positive homogeneity and are given (respectively) as

$$\begin{aligned}
\tilde{\lambda}(X + k) &= \tilde{\lambda}(X) + k, \text{ for } k \in \mathbb{R}, \\
\tilde{\lambda}(X_1 + X_2) &\leq \tilde{\lambda}(X_1) + \tilde{\lambda}(X_2), \\
\tilde{\lambda}(X_1) &\leq \tilde{\lambda}(X_2), \forall X_1 \leq X_2, \\
\tilde{\lambda}(kX) &= k\tilde{\lambda}(X), \forall k \in \mathbb{R}_{\geq 0}.
\end{aligned}$$

One popular risk measure is VaR (Value at Risk) which is defined as

$$VaR_{\hat{\varrho}}(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \hat{\alpha}\},$$

such that $\{\hat{\alpha} \in \mathbb{R}_{\geq 0} | 0 \leq \hat{\alpha} \leq 1\}$. Essentially, an individual specifies a confidence level (or risk level) $\hat{\alpha}$ and the associated threshold value is given by VaR. For our supplier, this risk measure implies that the supplier would like to use the hedging contract if s exceeds some threshold value s^*. We note the supplier would also trigger the contract in the event s exceeds a known amount s^*, since there will be operational constraints e.g. storage space.

The supplier determines the time to exercise the hedge. We can assume the supplier exercises the hedge when $s > s^*$, where s^* is a constant. The supplier will receive price K for the goods, and the buyer pays K to the supplier. We note that the buyer can sell the acquired goods later on, at any price it wishes to its own customers. We now examine the arbitrage constraints regarding our contract. Firstly, the buyer pays a price K for the goods, therefore

$$K \leq \Lambda,$$

that is the strike must be less than the maximum sale price of goods Λ, otherwise the buyer is guaranteed to make a loss with probability 1. This would be an arbitrage opportunity [5], such opportunities do not exist in realistic and well-functioning markets. An implication of this arbitrage constraint is that market regulation (which may enforce Λ) will limit K and so limit the sale of hedging contracts.

Secondly, the price V must be restricted by

$$V \leq K,$$

that is the contract price must be limited by the strike. If this inequality did not exist then the buyer can make a guaranteed profit if the supplier immediately exercises the contract. Finally, we must have

$$f(s^*) < K - V,$$

that is the net *cost* for the buyer should be greater than the price of the goods $f(s^*)$. Again, if this inequality does not exist then the buyer makes a profit by instantly selling goods at the market price $f(s^*)$.

We now examine the discounting factor. The discount rate $r(t)$ is required to obtain the present value of any future cashflows. For a risk averse investor the discount rate will equal the short rate $r'(t)$ plus the risk premium r_p , that is $r(t) := r'(t) + r_p$ [10]. We assume r_p is constant and so $r(t)$ can be modelled by a short rate model. The modelling of stochastic interest rate factors is extensive. If we assume a Markov diffusion process, under the risk neutral measure, then a stochastic interest rate factor is defined as

$$dr(t) = \varsigma_1(r(t), t)dt + \sigma(r(t), t)dB_0(t),$$

where $\sigma \in \mathbb{R}_{>0}, \varsigma_1 \in \mathbb{R}_{>0}, \varsigma_2 \in \mathbb{R}_{>0}$ are constants and $dB_0(t)$ is a Brownian motion. In particular, the interest rate diffusion can be modelled to be correlated with other assets, with associated Wiener processes $dB_1(t)$ and $dB_2(t)$. The correlation matrix $[dB_0(t), dB_1(t), dB_2(t)]$ is given by

$$\begin{matrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{01} & 1 & \rho_{12} \\ \rho_{02} & \rho_{12} & 1 \end{matrix}$$

where $\rho_{01}, \rho_{02}, \rho_{12}$ are constants, such that $\{\rho_{01}, \rho_{02}, \rho_{12} \in \tilde{\rho} \subset \mathbb{R} | \tilde{\rho}^2 \leq 1\}$. In fact the Brownian motions can be expressed as independent processes by

$$\begin{aligned} d\tilde{B}_0(t) &= dB_0(t), \\ d\tilde{B}_1(t) &= \rho_{01} dB_0(t) + \sqrt{1 - \rho_{01}^2} dB_1(t), \\ d\tilde{B}_2(t) &= \rho_{02} dB_0(t) + \frac{\rho_{12} - \rho_{01}\rho_{02}}{\sqrt{1 - \rho_{01}^2}} dB_1(t) + \sqrt{1 - \rho_{02}^2 - \frac{(\rho_{12} - \rho_{01}\rho_{02})^2}{1 - \rho_{01}^2}} dB_2(t). \end{aligned}$$

For our model a continuous affine process in one dimension would be sufficient, a review is available in [18]. A standard interest rate model for $r(t)$ is the Vasicek model [23], which has been used in derivatives pricing [11] and has been favoured in modelling [13]. The Vasicek model follows an Ornstein-Uhlenbeck process [4]. Let our discounting process $r(t)$ follow

$$dr(t) = (\varsigma_1 - \varsigma_2 r(t))dt + \sigma dB(t),$$

so that $dr(t) \in \mathbb{R}, \forall \varsigma_1, \varsigma_2, \sigma$.

To determine $r(t)$ we see that

$$\begin{aligned} d(exp(\varsigma_1 t)r(t)) &= exp(\varsigma_2 t)dr(t) + \varsigma_2 exp(\varsigma_2 t)r(t)dt \\ &= exp(\varsigma_2 t)(\varsigma_1 - \varsigma_2 r(t))dt + exp(\varsigma_2 t)\sigma dB(t) + \varsigma_2 exp(\varsigma_2 t)r(t)dt, \\ \Rightarrow exp(\varsigma_1 t)r(t) &= r(0) + \varsigma_1 \int_0^t exp(\varsigma_2 u)du + \sigma \int_0^t exp(\varsigma_2 u)dB(u) \\ \therefore r(t) &= r(0)exp(-\varsigma_2 t) + \left(\frac{\varsigma_1}{\varsigma_2}\right)(1 - exp(-\varsigma_2 t)) + \sigma exp(-\varsigma_2 t)\int_0^t exp(\varsigma_2 u)dB(u). \end{aligned}$$

Assuming that r(0) is a constant then we have a Gaussian distribution with mean

$$\begin{aligned} \mathbb{E}[r(t)] &= \mathbb{E}\left[r(0)exp(-\varsigma_2 t) + \left(\frac{\varsigma_1}{\varsigma_2}\right)(1 - exp(-\varsigma_2 t)) + \sigma exp(-\varsigma_2 t)\int_0^t exp(\varsigma_2 u)dB(u)\right] \\ &= \mathbb{E}\left[r(0)exp(-\varsigma_2 t) + \left(\frac{\varsigma_1}{\varsigma_2}\right)(1 - exp(-\varsigma_2 t))\right] + \mathbb{E}\left[\sigma exp(-\varsigma_2 t)\int_0^t exp(\varsigma_2 u)dB(u)\right] \\ &= \mathbb{E}\left[r(0)exp(-\varsigma_2 t) + \left(\frac{\varsigma_1}{\varsigma_2}\right)(1 - exp(-\varsigma_2 t))\right] \\ &= r(0)exp(-\varsigma_2 t) + \left(\frac{\varsigma_1}{\varsigma_2}\right)(1 - exp(-\varsigma_2 t)) \\ &\approx \left(\frac{\varsigma_1}{\varsigma_2}\right), \; for \; t \to \infty. \end{aligned}$$

We also have variance Var(r(t)) , denoted v^2, given by

$$\begin{aligned} v^2 &= \mathbb{E}[r(t)^2] - \mathbb{E}^2[r(t)] \\ &= \sigma^2 \left(\frac{1 - exp(-2\varsigma_1 t)}{2\varsigma_1}\right) \\ &\to \left(\frac{\sigma^2}{2\varsigma_1}\right), \; for \; t \to \infty. \end{aligned}$$

For a supplier hedging contract we expect a low volatility, and so $\varsigma_1 >> \sigma^2 \Rightarrow v^2 \to \delta, \forall t$, where δ is small. We now apply the Chebyshev inequality

$$\begin{aligned} \mathbb{P}(|r(t) - \mathbb{E}[r(t)]| \geq k) &\leq \frac{v^2}{k^2}, \\ \mathbb{P}(|r(t) - \mathbb{E}[r(t)]| \geq k) &\leq \frac{\delta^2}{k^2}, \end{aligned}$$

where $k \in \mathbb{R}_{>0}$ is a constant. Hence for small v the probability of values diverging from $\mathbb{E}[r(t)]$ will be small, hence

$$r(t) \approx \mathbb{E}[r(t)] \approx \left(\frac{\varsigma_1}{\varsigma_2}\right).$$

Thus the discount rate is effectively constant and we denote this by r for convenience.

3 Buyer Model as an Optimal Stopping Problem

The buyer's problem of purchasing goods from the supplier can be analysed in terms of an optimisation problem, specifically by dynamic programming and an optimal starting-stopping problem (see for instance [17] and [19]). Hence let us define θ that is a measurable function, such that $\theta : \mathbb{R} \to [0, \infty]$ then θ is r-excessive relative to $S(t)$ if θ is r-superaveraging, that is

$$exp(-rt)\mathbb{E}_s[\theta(S(t))] \leq \theta(s), \forall s \in \mathbb{R}, t \geq 0,$$

and that θ has the limit

$$\lim_{t \downarrow 0} exp(-rt)\mathbb{E}_s[\theta(S(t))] = \theta(s), \forall s \in \mathbb{R},$$

where $\{r \in \mathbb{R}^+\}$ is the discount rate. We now provide the following property, Proposition 1.

Proposition 1. *If $\theta : \mathbb{R} \to [0, \infty]$ is r-excessive relative to $S(t)$ then we can state that, firstly, the mapping*

$$t \to \theta(S(t)),$$

is right continuous on $[0, \infty)$ and has left hand limits on $(0, \infty]$, almost surely. Secondly, if $\theta(S(t))$ is integrable for all $t \geq 0$, then

$$exp(-rt)\theta(X(t)), \forall t \geq 0,$$

is a right continuous supermartingale. Thirdly, for all $s \in \mathbb{R}$ we have

$$\mathbb{E}_s[exp(-rT_2)\theta(S(T_2))] \leq \mathbb{E}_s[exp(-rT_1)\theta(S(T_1)], \forall T_2 \geq T_1,$$

for all stopping times T_1, T_2, almost surely.

The buyer's optimisation problem is undertaken with the probability measure $\mathbb{P}_s$; for the benefit of clarity this represents the real world or physical probability measure.

Assuming the supplier purchases the hedging contract, the supplier will exercise this contract if supply quantity exceeds his storage capacity, denoted by s*. This frequently occurs if the supplier cannot shift goods from its own sales. Hence the supplier exercises the hedging contract at time

$$\Gamma_{s*,\infty} := \inf\{t \geq 0 | S(t) \geq s^*\},$$

that is $\Gamma_{s*,\infty}$ denotes the minimum time t at which the range of $S(t)$ values exceed s^*, for $S(t) \in \{s^*, \infty\}$.

Let us, initially, assume the buyer has sufficient storage space so that it can sell the hedging contract to the supplier, and in the event the contract is triggered (by the supplier) the buyer can store the goods. The buyer will pay the strike price K for the goods, hence we have the following discounted expected payoff $g_2(K, s)$ for the buyer

$$g_2(K, s) = \mathbb{E}_s[-Kexp(-r\Gamma_{s^*,\infty})],$$

which can be expressed as

$$\begin{aligned} g_2(K, s) &= -K, \quad &for\ s > s^*, \\ &= -Kz(s), \quad &for\ s \leq s^*, \end{aligned}$$

where $z(s) = exp(\kappa(s - s^*))$, κ denotes $\kappa = \sqrt{2r}$ for convenience. The payoff under the second condition (for $s \leq s^*$) is obtained using the well known Laplace transform for the hitting time of $S(t)$ at a given point $x \in \mathbb{R}$, that is $\Gamma_x = \inf\{t \geq 0 | S(t) = x\}$, is given by

$$\mathbb{E}_s[-Kexp(-r\Gamma_x)] = exp(-\kappa|x - s|).$$

Whilst in our model the buyer cannot determine the time of exercise of the hedge (since the exercise is controlled by the supplier and the supplier's excess supply follows a random process), the buyer can however determine the time of offering the hedge and should *optimise* this time to maximise its income. Assuming the buyer behaves rationally, the optimisation problem can be modelled as an optimal stopping problem with stopping time $\tau \in \mathcal{T}$, where $\mathcal{T} \in \mathbb{R}^+$ denotes the set of all stopping times. If we take into account the income from the sale of the hedge V, where $V \in \mathbb{R}^+$, then our optimisation problem is

$$g_1(K, s) = \sup_{\tau_2 \in \mathcal{T}} \mathbb{E}_x[exp(-r\tau_2)(V + g_2(S(\tau_2)))],$$

where τ_2 is the optimal stopping time for our optimisation.

In order to enable optimal timing of the sale of the hedge, the buyer must have sufficient physical empty space (to store the additional goods from the supplier) at the point of offering such a hedge. This storage issue is an important aspect of the model because the lack of storage space is one of the main motives behind the supplier selling excess goods to the buyer. Hence we must take into account storage constraints with the buyer. Consequently, in order for the buyer to optimise its sale of the contract, we must also take into account the sale of its inventory to vacate space for the goods from the supplier. This sale time can also be modelled as an optimal stopping problem at optimal stopping time τ_1, such that the set of optimal times is given by

$$\mathcal{T}_2 = \{(\tau_1, \tau_2) \in \mathcal{T} \times \mathcal{T} | \tau_1 \leq \tau_2\}.$$

Given that the buyer sells at time τ_2, that is price $f(S(\tau_2))$, our optimisation problem is

$$\alpha_V(s) = \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} \mathbb{E}_s[exp(-r\tau_1)f(S(\tau_1)) + exp(-r\tau_2)(V + g_2(K, s))]. \tag{1}$$

4 Optimal Starting-Stopping Solution

To solve the optimal solution to equation (1) by the principle of recursion (and dynamic programming), the equation (1) would be solved as

$$U(s) := \sup_{\tau\in\mathcal{T}} \mathbb{E}_s[exp(-r\tau)(f(S(\tau)) + g_1(K, S(\tau)))].$$

Alternatively, our optimal starting-stopping problem is optimisation of the expression

$$J_s(\tau_1, \tau_2) = \mathbb{E}_s[\eta_1(S(\tau_1))exp(-r\tau_1) + \eta_2(S(\tau_2))exp(-r\tau_2)], \forall(\tau_1, \tau_2) \in \mathcal{T}_2,$$

where $\eta_1(.), \eta_2(.)$ are real-valued functions. In association with this optimisation, we define two optimal stopping problems

$$\begin{aligned}\hat{\beta}(s) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_s[exp(-r\tau)\eta_2(S(\tau))],\\ \hat{\alpha}(s) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_s[exp(-r\tau)\eta_1(S(\tau)) + \eta_1(S(\tau))],\end{aligned}$$

with stopping sets $\zeta_{\hat{\alpha}}$ and stopping time $\Gamma_{\hat{\alpha}}$ as

$$\zeta_{\hat{\alpha}} = \{s \in \mathbb{R} | \hat{\alpha}(s) = \eta_1(s) + \hat{\beta}(s)\}, \ \textit{with}\ \Gamma_{\hat{\alpha}} = \inf\{t \geq 0 | S(t) \in \zeta_{\hat{\alpha}}\},$$

similarly for

$$\zeta_{\hat{\beta}} = \{s \in \mathbb{R} | \hat{\beta}(s) = \eta_2(s)\}, \ \textit{with}\ \Gamma_{\hat{\beta}} = \inf\{t \geq 0 | S(t) \in \zeta_{\hat{\beta}}\}.$$

We are now ready to state Theorem 1 which provides a solution in terms of the optimal stopping problems previously discussed.

Theorem 1. *If the functions η_1 and η_2 are bounded and continuous, then we deduce that $\hat{\beta}(.)$ and $\hat{\alpha}(.)$ are continuous and bound functions. The function $\hat{\beta}(.)$ is the smallest r-excessive majorant of η_2 and $\Gamma_{\zeta_{\hat{\beta}}}$ is the optimal stopping time to $\hat{\beta}(s)$:*

$$\hat{\beta}(s) = \mathbb{E}_s[exp(-r\Gamma_{\zeta_{\hat{\beta}}})\eta_2(S(\Gamma_{\zeta_{\hat{\beta}}}))], \forall s \in \mathbb{R}.$$

Also, $\hat{\alpha}(s)$ is the smallest r-excessive majorant of $\eta(.) + \hat{\beta}(.)$ and $\Gamma_{\zeta_{\hat{\alpha}}}$ is the optimal stopping time to $\hat{\alpha}(s)$:

$$\hat{\alpha}(s) = \mathbb{E}_s[exp(-r\Gamma_{\zeta_{\hat{\alpha}}})(\eta_1(S(\Gamma_{\zeta_{\hat{\alpha}}})) + \hat{\beta}(S(\Gamma_{\zeta_{\hat{\alpha}}})))], \forall s \in \mathbb{R}.$$

Secondly, $\hat{\alpha}(.)$ satisfies $\hat{\alpha}(.) \geq \alpha(.)$, where

$$\hat{\alpha}(s) = J_s(\hat{\tau}_1, \hat{\tau}_2) = \alpha(s), \forall s \in \mathbb{R}, \ \textit{for}\ \hat{\tau}_1 = \Gamma_{\zeta_{\hat{\alpha}}}, \hat{\tau}_2 = \hat{\tau}_1 + \Gamma_{\zeta_{\hat{\beta}}} \circ \Theta_{\hat{\tau}_1}.$$

Furthermore, if we have $\eta_1(.) = f(.)$ and $\eta_2(.) = V + g_2(K,s)$ are continuous and bounded functions, then $g_1(K,s)$ and $U(s)$ are continuous, bounded functions and r-excessive, and $U(s)$ is a solution to the optimal starting-stopping problem in equation (1), that is

$$U(s) = \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} \mathbb{E}_s[exp(-r\tau_1)f(S(\tau_1)) + exp(-r\tau_2)(V + g_2(K,s))].$$

Proof. We apply the results of [9] where ψ_r, ϕ_r are defined in [9]: given that $\eta_2(.)$ is a bounded function then applying [9] gives

$$\begin{aligned}\lim_{s\to\infty} \sup \frac{\eta_2^+(s)}{\psi_r(s)} &= \lim_{s\to\infty} \sup \frac{\eta_2^+(s)}{exp(\kappa s)} = 0,\\ \lim_{s\to-\infty} \sup \frac{\eta_2^+(s)}{\phi_r(s)} &= \lim_{s\to-\infty} \sup \frac{\eta_2^+(s)}{exp(-\kappa s)} = 0.\end{aligned}$$

Using these equations, given that η_2 is continuous, and applying Propositions 5.11 and 5.13 in [9], we conclude that $\hat{\beta}(.)$ is continuous, the smallest r-excessive majorant of η_2 and $\Gamma_{\zeta_{\hat{\beta}}}$ is the optimal stopping time. Similarly, using analogous arguments, given that $\eta_1 + \hat{\beta}(.)$ is continuous and bounded, we can then conclude that $\hat{\alpha}$ is continuous, bounded, the smallest r-excessive majorant of $\eta_1 + \hat{\beta}(.)$ and that the optimal stopping time is $\Gamma_{\zeta_{\hat{\alpha}}}$.

To prove the second part of the Theorem, our proof follows the same way as in [15] and the reader is referred to [15] for more information. Given that we have already deduced that $\hat{\alpha}(.), \hat{\beta}(.)$ are continuous, bounded functions, with their respective majorants, then for any set of stopping times $(\tau_1, \tau_2) \in \mathcal{T}_2$ and using Proposition 1 then we have the following:
Since

$$J_s(\tau_1,\tau_2) = \mathbb{E}_s[exp(-r\tau_1)\eta_1(S(\tau_1)) + exp(-r\tau_2)\eta_2(S(\tau_2))],$$

and that

$$\begin{aligned}\hat{\alpha}(s) &\geq \mathbb{E}_s[exp(-r\tau_1)\hat{\alpha}(S(\tau_1))]\\ &\geq \mathbb{E}_s[exp(-r\tau_1)\eta_1(S(\tau_1)) + exp(-r\tau_1)\hat{\beta}(S(\tau_1))],\\ \Rightarrow \hat{\alpha}(s) &\geq \mathbb{E}_s[exp(-r\tau_1)\eta_1(S(\tau_1)) + exp(-r\tau_2)\eta_2(S(\tau_2))].\\ \therefore \hat{\alpha}(s) &\geq \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} J_s(\tau_1,\tau_2).\end{aligned}$$

Now if we have

$$\hat{\tau}_1 = \Gamma_{\zeta_{\hat{\alpha}}},\ \hat{\tau}_2 = \hat{\tau}_1 + \Gamma_{\zeta_{\hat{\beta}}} \circ \Theta_{\hat{\tau}_1},$$

and $\hat{\alpha}(s)$ and $\hat{\beta}(s)$ are the functions for the optimal stopping problem with respective stopping times $\hat{\tau}_1$ and $\hat{\tau}_2$, then we can apply the strong Markov property of $S(t)$ so that since

$$J_s(\hat{\tau}_1,\hat{\tau}_2) = \mathbb{E}_s[exp(-r\hat{\tau}_1)\eta_1(S(\hat{\tau}_1)) + \mathbb{E}_s[exp(-r\hat{\tau}_2)\eta_2(S(\hat{\tau}_2))|\mathcal{F}_{\hat{\tau}_1}]],$$

then

$$J_s(\hat{\tau}_1,\hat{\tau}_2) = \mathbb{E}_s[exp(-r\hat{\tau}_1)(\phi_1 S(\hat{\tau}_1) + \hat{\beta}(S(\hat{\tau}_1)))] = \hat{\alpha}(s).$$

Now a simple instance of the above proof, and if we have $\eta_1(.) = f(.)$ and $\eta_2(.) = V + g_2(K,s)$ are continuous and bounded functions, then $g_1(K,s)$ and $U(s)$ are continuous, bounded functions and r-excessive. Also $U(s)$ is a solution to the optimal starting-stopping problem in equation (1), so that we can write

$$U(s) = \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} \mathbb{E}_s[exp(-r\tau_1)f(S(\tau_1)) + exp(-r\tau_2)(V + g_2(K,s))].$$

Hence our proof is completed. ∎

5 Closed Form Solutions for Payoffs

In this section we prove Theorems 2 and 3, and so provide closed form solutions to $g_1(s, K)$ and $U(s)$. To achieve this we follow [21] and first introduce Proposition 2.

Proposition 2. (Proposition 5.12, [9]) *Let us denote*

$$H(s) := \frac{\psi_r(s)}{\phi_r(s)} = exp(2\kappa s),$$

where $H : \mathbb{R} \to (0, \infty)$, $s \mapsto \psi_r(s) := exp(\kappa s)$, and $s \mapsto \phi_r(s) := exp(-\kappa s)$ are the fundamental solutions of the differential equation

$$\frac{1}{2}.\frac{d^2}{ds^2} w(s) - rw(s) = 0.$$

Let us also define a continuous and bounded function $g : \mathbb{R} \mapsto \mathbb{R}$, and let $\lambda : [0, \infty) \mapsto [0, \infty)$ be the smallest non-negative concave majorant of

$$F(x) = \frac{g(H^{-1}(x))}{\phi_r(H^{-1}(x))}.\mathbb{1}_{\{x>0\}}. \tag{2}$$

The function $U(.)$ for the optimal stopping problem

$$U(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[exp(-r\tau)g(S(\tau))],$$

can be expressed as $U(s) = \phi_r(s)\lambda(H(s))$ with optimal stopping time τ^* as

$$\tau^* = \Gamma_{\zeta_U} := \inf\{t \geq 0 : S(t) \in \zeta_U\},$$

where

$$\zeta_U := \{s \in \mathbb{R} : U(s) = g(s)\} = \{s \in \mathbb{R} : \phi_r(s)\lambda(H(s)) = g(s)\}.$$

We now provide closed form solutions for $g_1(s, K)$ and $U(s)$.

Theorem 2. *The function $g_1(s, K)$ is given by*

$$g_1(s, K) = z(s)\left(\Upsilon_3.\mathbb{1}_{\{s>\mu_3\}} - K.\mathbb{1}_{\{s\leq\mu_3\}}\right) + V.\mathbb{1}_{\{s\leq\mu_3\}},$$

with stopping region $\zeta_{g_1(s,K)} = (-\infty, \mu_3]$, where

$$\mu_3 = \mu_4 + s^*, \mu_4 = \frac{ln(V/2K)}{\kappa}, \Upsilon_3 = \frac{V^2}{4K}.$$

Proof. To prove this Theorem we apply Proposition 2. Let $x^* = H(s^*) = exp(2\kappa s^*)$ then we have

$$F_1(x) = \frac{V + g_2\left(K, \dfrac{lnx}{2\kappa}\right)}{x^{-\frac{1}{2}}} = \hat{q}_1(x).\mathbb{1}_{\{x>x^*\}} + q_1(x).\mathbb{1}_{\{0<x\leq x^*\}},$$

where

$$\begin{aligned} q_1(x) &= \sqrt{x}\left(V - \sqrt{x}\frac{K}{\sqrt{x^*}}\right), \ for\ q_1(x) \in (0, \infty), \\ \hat{q}_1(x) &= \sqrt{x}(V - K), \ for\ \hat{q}_1(x) \in (0, \infty), \end{aligned}$$

and $q_1(x)$,$\hat{q}_1(x)$ are real-valued functions.

Now we find the smallest, positive, concave majorant of F_1. We observe that

$$\begin{aligned} q_1'(x) &= \frac{1}{2}\frac{V}{\sqrt{x}} - \frac{K}{\sqrt{x^*}} \Rightarrow q_1''(x) = -\frac{1}{4}Vx^{-\frac{3}{2}}, \\ \hat{q}_2'(x) &= \frac{1}{2}\left(\frac{V}{\sqrt{x}} - \frac{K}{\sqrt{x}}\right) \Rightarrow \hat{q}_2''(x) = -\frac{1}{4}Vx^{-\frac{3}{2}} + \frac{1}{4}Kx^{-\frac{3}{2}}. \end{aligned}$$

We recall that $0 < V < K$. From the previous equations we deduce that $\hat{q}_1(x)$ is a monotonically decreasing, negative, and convex function on $(0,\infty)$, whereas $q_1(x)$ is concave on $(0,\infty)$. We have the unique root of $q_1(x)$

$$\gamma_1 = \left(\frac{V\sqrt{x^*}}{K}\right)^2,$$

and the unique root of $\hat{q}_1(x)$ is $\gamma_2 = \gamma_1/4$, where $\gamma_2 < \gamma_1 < x^*$ because $V < K$. We can consequently conclude that q_1 on (i) $(0,\gamma_2]$: concave, increasing and positive (ii) (γ_2,γ_1): decreasing, concave, decreasing and positive (iii) (γ_2,∞): concave, decreasing and negative. Therefore it follows that F_1 on: (i) $(0,\gamma_2]$: concave, increasing and positive (ii) (γ_2,γ_1): decreasing, concave, decreasing and positive (iii) (γ_2,x^*): concave, decreasing and negative (iv) (x^*,∞): convex, decreasing and negative.

We therefore deduce that the smallest non-negative concave majorant of F_1 is

$$\lambda(x) = q_1(x.\mathbb{1}_{\{0\le x\le\gamma_2\}} + \gamma_2.\mathbb{1}_{\{x>\gamma_2\}}).$$

Now if we use Proposition 2 then we obtain:

$$g_1(s,K) = z(s)\left(\Upsilon_3.\mathbb{1}_{\{s>\mu_3\}} - K.\mathbb{1}_{\{s\le\mu_3\}}\right) + V.\mathbb{1}_{\{s\le\mu_3\}}$$

with stopping region $\zeta_{g_1(s,K)}$ given by

$$\zeta_{g_1(s,K)} = \left(-\infty, \frac{ln(V/2K)}{\kappa} + s^*\right] = (-\infty,\mu_3].$$

■

We note in passing that $V \le K$ is a necessary condition (as mentioned earlier) to prevent arbitrage profit taking, as we would expect in a well functioning and realistic market. If this condition is not obeyed then the solution would change (however this would admit arbitrage and so would be unrealistic). Another useful conclusion is that

$$0 < V \le K \Rightarrow \mu_4 < 0,$$

and Theorem 2 implies that if a supplier is expected to exercise his contract immediately then it is not optimal for the buyer to offer the contract.

We now provide a closed form solution to $U(s)$ in the following Theorem, which is characterised in terms of s_4, where $s_4 \in (-\infty, -(e+V)/l]$ is a solution to the equation

$$0 = \frac{1}{2}exp(-\kappa s)\left(e + V + \frac{l}{\kappa} + ls\right) - Kexp(-\kappa s^*).$$

Based on Theorem 2, it is beneficial to analyse $U(s)$ when $s^* \ge \mu_1 - \mu_3$ and $s^* < \mu_2 - \mu_3$, as these situations represent the typical scenarios facing the supplier. The other scenarios occur during extremely high excess supply or high shortage of goods, both of which would not normally happen.

Theorem 3. *The function $U(s)$ over $s^* \ge \mu_1 - \mu_4$ and $s^* < \mu_2 - \mu_4$, is given by:*

(a) for the condition $\mu_3 > s_4 > \mu_1$ then

$$\begin{aligned} U(s) &= \Lambda + V - Kz(s), && for\ -\infty < s \le \mu_1 \\ &= e + ls + V - Kz(s) && for\ \mu_1 < s < s_4 \\ &= exp(-\kappa(s-s_4))(e + ls_4 + V - Kexp(\kappa(s_4 - s^*))), && for\ s > s_4, \end{aligned}$$

with stopping region $(-\infty, s_4]$*;*
(b) for the condition $s_4 \leq \mu_1$ *then*

$$\begin{aligned} U(s) &= \Lambda + V - Kz(s), && \text{for } -\infty < s \leq \mu_1 \\ &= exp(-\kappa s)exp(\mu_1\kappa)\left(\Lambda + V - Kexp(\mu_1\kappa - \kappa s^*)\right), && \text{for } s > \mu_1, \end{aligned}$$

with stopping region $(-\infty, \mu_1]$*;*
(c) for the condition $s_4 \geq \mu_3$ *then*

$$\begin{aligned} U(s) &= \Lambda + V - Kz(s), && \text{for } -\infty < s < \mu_1 \\ &= e + ls + V - Kz(s), && \text{for } \mu_1 \leq s \leq \mu_3 \\ &= e + ls + z(s)\Upsilon_3, && \text{for } \mu_3 < s \leq -\mu_5 \\ &= -\frac{l}{\kappa}exp\left(-\frac{\kappa e}{l} - 1 - \kappa s\right) + z(s)\Upsilon_3, && \text{for } s > -\mu_5, \end{aligned}$$

where

$$-\mu_5 = -\left(\frac{e}{l} + \frac{1}{\kappa}\right),$$

with stopping region $(-\infty, -\mu_5]$*.*

Proof. We first define some auxillary functions: $x = H(s) = exp(2\kappa s)$, so that $x^* = H(s^*) = exp(2\kappa s^*)$. We also have

$$\begin{aligned} & Q(x) = \sqrt{x}\left(e + \Upsilon_1 + V - \sqrt{x}\frac{K}{\sqrt{x^*}}\right), \text{ where } \Upsilon_1 = \frac{l.ln(x)}{2\kappa} \\ \Rightarrow\ & Q'(x) = \frac{1}{2\sqrt{x}}\left(e + V + \frac{l}{\kappa} + \Upsilon_1 - \frac{K}{\sqrt{x^*}}\right) \Rightarrow Q''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\left(e + V + \Upsilon_1\right), \end{aligned}$$

with

$$\begin{aligned} & \hat{Q}(x) = \sqrt{x}\left(e + \Upsilon_1 + \frac{\sqrt{x^*}}{\sqrt{x}}\Upsilon_3\right) \\ \Rightarrow\ & \hat{Q}'(x) = \frac{1}{2\sqrt{x}}\left(e + \frac{l}{\kappa} + \Upsilon_1 - \frac{K}{\sqrt{x^*}}\right) \Rightarrow \hat{Q}''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\left(e + \Upsilon_1\right), \end{aligned}$$

with

$$\begin{aligned} & Q_1(x) = \sqrt{x}\left(\Lambda + V - \sqrt{x}\frac{K}{\sqrt{x^*}}\right) \\ \Rightarrow\ & Q_1'(x) = \frac{1}{2\sqrt{x}}(\Lambda + V) - \frac{K}{\sqrt{x^*}} \Rightarrow Q_1''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(\Lambda + V), \end{aligned}$$

with

$$\hat{Q}_1(x) = \Lambda\sqrt{x} + \sqrt{x^*}\frac{V}{4K} \Rightarrow \hat{Q}_1'(x) = \frac{1}{2\sqrt{x}}.\Lambda \Rightarrow \hat{Q}_1''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\Lambda,$$

with

$$Q_2(x) = \sqrt{x}V - x\frac{K}{x^*} \Rightarrow Q_2'(x) = \frac{V}{2\sqrt{x}} - \frac{K}{\sqrt{x^*}} \Rightarrow Q_2''(x) = -\frac{V}{4}x^{-\frac{3}{2}},$$

and

$$\hat{Q}_2(x) = (\sqrt{x^*})\Upsilon_3 \Rightarrow \hat{Q}_2'(x) = 0 \Rightarrow \hat{Q}_2''(x) = 0.$$

Hence we can deduce $\hat{Q}_1(.)$ is concave and increasing on $\mathbb{R}_{\geq 0}$. The function $Q''(.)$ is also continuous on $\mathbb{R}_{\geq 0}$, it has a unique root at

$$x_1 = exp\left(-\frac{2\kappa}{l}(e + V)\right),$$

it is negative on $(0, x_1)$, and positive on (x_1, ∞). Also, as $Q'(.)$ is continuous, monotone and decreasing on $(0, x_1)$, $\lim_{x \downarrow 0} Q'(x) = \infty$, $Q'(x_1) < 0$, and $Q'(.)$ has a unique root $x_4 \in (0, x_1)$. Moreover, as $Q'(.)$ is continuous, monotone and increasing on (x_1, ∞) with $\lim_{x \uparrow \infty} Q'(x) = -K/\sqrt{x^*} < 0$, we conclude that $Q'(.)$ is negative on (x_4, ∞) and has a minimum at x_1.

Next we deduce that $Q(.)$ is concave and increasing on $(0, x_4)$, concave and decreasing on (x_4, x_1), convex and decreasing on (x_1, ∞). Also, $\hat{Q}'(.)$ is continuous on $(0, \infty)$, has a unique root at

$$x_2 = exp((\frac{-2\kappa e}{l}) - 2),$$

$\hat{Q}'(.)$ is positive on $(0, x_2)$ and negative on (x_2, ∞). Moreover $\hat{Q}''(.)$ has a unique root

$$x_3 = exp(\frac{-2\kappa e}{l}) < x_1,$$

it is negative on $(0, x_3)$ and positive on (x_3, ∞), where $x_2 < x_3$.

Next we deduce that $\hat{Q}(.)$ is concave and increasing on $(0, x_2]$, concave and decreasing on $(x_2.x_3)$, convex and decreasing on (x_3, ∞), and

$$\hat{Q}(x_2) = \frac{-2l exp(\frac{-\kappa e}{l} - 1)}{2\kappa} + (\sqrt{x^*})\Upsilon_3 > \hat{Q}_2(x),\ for\ \{x \in \mathbb{R} | x \geq 0\}.$$

Also, $Q_1(.)$ is concave on $(0, \infty)$, $Q_1'(.)$ is continuous and monotone on $\mathbb{R}_{\geq 0}$. Moreover, as $\lim_{x \downarrow 0} Q_1'(x) = \infty$, and $\lim_{x \uparrow \infty} Q_1'(x) = -K/\sqrt{x^*} < 0$, $Q_1'(.)$ has a unique root $x_5 \in \mathbb{R}_{\geq 0}$ where

$$x_5 = \left(\frac{\sqrt{x^*}(\Lambda + V)}{2K} \right)^2.$$

We therefore can conclude that $Q_1(.)$ is increasing on $(0, x_5]$ and decreasing on (x_5, ∞).

The function $Q_2(.)$ is concave on $\mathbb{R}_{\geq 0}$, with $Q_2'(.)$ continuous and monotonically decreasing on $\mathbb{R}_{\geq 0}$. Furthermore, $Q_2'(.)$ has a unique root $\gamma_2 \in \mathbb{R}_{\geq 0}$, which is

$$\gamma_2 = \left(\frac{\sqrt{x^*} V}{2K} \right)^2 < x_5.$$

Therefore $Q_2(.)$ is increasing on $(0, \gamma_2]$ and decreasing on (γ_2, ∞). Let

$$\gamma_s = exp\left(2\kappa\mu_1\right),$$

and so

$$\begin{aligned} Q_1(\gamma_s) &= Q(\gamma_s) = \sqrt{\gamma_s}\left(\Lambda + V - K\frac{\sqrt{\gamma_s}}{x^*}\right), \\ Q_1'(\gamma_s) &= \frac{1}{2\sqrt{\gamma_s}}(\Lambda + V) - \frac{K}{\sqrt{x^*}}, \\ Q'(\gamma_s) &= \frac{\Lambda + \frac{l}{\kappa}}{2\sqrt{\gamma_s}}, \end{aligned}$$

therefore there is a continuous fit between Q_1, Q at γ_s but we fail to have a smooth fit because $Q(\gamma_s) < Q_1(\gamma_s)$. Now for $\hat{Q}_1$ and $\hat{Q}$ we have

$$\begin{aligned} \hat{Q}_1(\gamma_s) &= \hat{Q}(\gamma_s) = \Lambda\sqrt{\gamma_s} + (\sqrt{x^*})\Upsilon_3, \\ \hat{Q}_1'(\gamma_s) &= \frac{\Lambda}{2\sqrt{\gamma_s}}, \\ \hat{Q}'(\gamma_s) &= \frac{\Lambda + \frac{l}{\kappa}}{2\sqrt{\gamma_s}}, \end{aligned}$$

and shows an continuous fit at γ_s but not a smooth fit because $\hat{Q}_1(\gamma_s) > \hat{Q}(\gamma_s)$. At point x_3 we have

$$
\begin{aligned}
Q(x_3) &= Q_2(x_3) = exp\left(\frac{-\kappa e}{l}\right)V - exp\left(\frac{-2\kappa e}{l}\right)\frac{K}{\sqrt{x^*}},\\
Q'(x_3) &= \frac{1}{2}exp\left(\frac{\kappa e}{l}\right)\left(V + \frac{l}{\kappa}\right) - \frac{K}{\sqrt{x^*}},\\
Q'_2(x_3) &= \frac{1}{2}exp\left(\frac{\kappa e}{l}\right)V - \frac{K}{\sqrt{x^*}}.
\end{aligned}
$$

This is also a continuous fit but fails because $Q'(x_3) < Q'_2(x_3)$. However, at γ_2 there is

$$
\begin{aligned}
Q_2(\gamma_2) &= \hat{Q}_2(\gamma_2) = (\sqrt{x^*})\Upsilon_3,\\
Q'_2(\gamma_2) &= \hat{Q}'_2(\gamma_2) = 0,
\end{aligned}
$$

which proves a continuous and smooth fit between $\hat{Q}_2$ and Q_2 at γ_2. Also for Q and $\hat{Q}(.)$ we have a smooth and continuous fit at γ_2

$$
\begin{aligned}
Q(\gamma_2) &= \hat{Q}'(\gamma_2)e\sqrt{\gamma_2} + l\sqrt{\gamma_2}\frac{ln(\gamma_2)}{2\kappa} + (\sqrt{x^*})\Upsilon_3,\\
Q'(\gamma_2) &= \hat{Q}'(\gamma_2) = \frac{1}{2}\gamma^{-\frac{1}{2}}\left(e + \frac{l}{\kappa} + \frac{l.ln(\gamma_2)}{2\kappa}\right).
\end{aligned}
$$

Similarly, for $Q_1(.)$ and $\hat{Q}_1(.)$ there is a continuous and smooth fit at γ_2

$$
\begin{aligned}
Q_1(\gamma_2) &= \hat{Q}_1(\gamma_2) = \Lambda\left(\frac{\sqrt{x^*}V}{2K}\right) + (\sqrt{x^*})\Upsilon_3,\\
Q'_1(\gamma_2) &= \hat{Q}'_1(\gamma_2) = \frac{\Lambda K}{V\sqrt{x^*}}.
\end{aligned}
$$

Also at x_3 we have

$$
\begin{aligned}
\hat{Q}(x_3) &= \hat{Q}_2(x_3) = (\sqrt{x^*}).\Upsilon_3,\\
\hat{Q}'(x_3) &= exp\left(\frac{\kappa e}{l}\right)\frac{l}{2\kappa},\\
\hat{Q}'_2(x_3) &= 0,
\end{aligned}
$$

and so we have a continuous fit at x_3 but not a smooth fit because $\hat{Q}'(x_3) < \hat{Q}'_2(x_3)$.

With all the previous results we can solve the two stage optimal stopping problem. We have $0 < \gamma_s \leq \gamma_2 < x_3$ and $F_2(.)$ is given by

$$
\begin{aligned}
F_2(.) &= \frac{\{f + g_1\}(H^{-1}(x))}{x^{-\frac{1}{2}}}, &(3)\\
&= Q_1(x).\mathbb{1}_{\{0<x<\gamma_s\}} + Q(x).\mathbb{1}_{\{\gamma_s\leq x\leq\gamma_2\}} + \hat{Q}(x).\mathbb{1}_{\{\gamma_2<x\leq x_3\}} &(4)\\
&\quad + \hat{Q}_2(x).\mathbb{1}_{\{x_3<x\}}. &(5)
\end{aligned}
$$

With the previous derivations we are now in a position to prove the Theorem. Firstly, we characterise the small non-negative majorant λ of $F_2(.)$ as given in equations (3)-(5). As deduced previously, we have: $Q_1(.)$ is concave everywhere, increasing on $(0, x_5]$ where $x_5 > \gamma_s$; we have $Q'(\gamma_s) < Q'_1(\gamma_s)$, and $Q'(\gamma_2) > \hat{Q}'(\gamma_2)$;$\hat{Q}(.)$ is concave everywhere, increasing on $(0, x_2]$, decreasing on (x_2, ∞), with $0 < x_2 < x_3 < x_1$;$\hat{Q}(x_2) > \hat{Q}(x), \forall x \in \mathbb{R}_{\geq 0}$. Now from equations (3)-(5) we can deduce that the transformed stopping region is $\{\lambda(.) = F_2(.)\} = (0, \Pi]$ where $\Pi \leq x_2$.

We now determine $U(s)$ for the specific subregions by applying Proposition 2. For:
(a) in this region we have $\gamma_2 > x_4 > \gamma_s$, the stopping region is $(0, x_4]$, for $x \geq x_4$ the function $\lambda(.)$ is constant with value $Q(x_4)$:

$$
\lambda(x) = \sqrt{x}\left(\Lambda + V - K\frac{\sqrt{x}}{\sqrt{x^*}}\right), \qquad for\ 0 < x < \gamma_s
$$

$$
\begin{aligned}
&= \sqrt{x}\left(e+\Upsilon_1+V-K\frac{\sqrt{x}}{\sqrt{x^*}}\right),\ for\ \gamma_s<x\le x_4\\
&= \sqrt{x_4}\left(e+l.\frac{ln(x_4)}{2\kappa}+V-K\frac{\sqrt{x_4}}{x^*}\right),\ for\ x_4<x.
\end{aligned}
$$

Hence by applying Proposition 2 we have

$$
\begin{aligned}
U(s) &= \Lambda+V-Kz(s), && for\ -\infty<s\le\mu_1\\
&= e+ls+V-Kz(s), && for\ \mu_1<s<s_4\\
&= exp(-\kappa(s-s_4))(e+ls_4+V-Kexp(\kappa(s_4-s^*))), && for\ s>s_4,
\end{aligned}
$$

(b)in this region we have $x_4\le\gamma_s$, with stopping region $(0,\gamma_s]$, and $x\ge\gamma_s$ we have $\lambda(.)$ is constant with value $Q(\gamma_s)$:

$$
\begin{aligned}
\lambda(x) &= \sqrt{x}\left(\Lambda+V-K\frac{\sqrt{x}}{\sqrt{x^*}}\right),\ for\ 0<x\le\gamma_s\\
&= \sqrt{\gamma_s}\left(\Lambda+V-K\frac{\sqrt{x}}{\sqrt{x^*}}\right),\ for\ x>\gamma_s.
\end{aligned}
$$

Hence by applying Proposition 2 we have

$$
\begin{aligned}
U(s) &= \Lambda+V-Kz(s), && for\ -\infty<s\le\mu_1\\
&= exp(-\kappa s)exp(\mu_1\kappa)(\Lambda+V-Kexp(\mu_1\kappa-\kappa s^*)), && for\ s>\mu_1.
\end{aligned}
$$

(c) in this region we have $x_4\ge\gamma_2$, however this inequality can be re-expressed as $x_2\ge\gamma_2$; the proof is as follows. For $x_4\ge\gamma_2\Rightarrow\hat{Q}'(\gamma_2)\ge0$ because $\hat{Q}'(\gamma_2)=Q'(\gamma_2)$, and $Q'(.)$ is non-negative on $(0,x_4]$. Also as $\hat{Q}'(.)$ is non-negative on $(0,x_2]$, negative on (x_2,∞) and has unique root at x_2, then $\hat{Q}(\gamma_2)\ge0\Rightarrow x_2\ge\gamma_2$.

The stopping region given by $(0,x_2]$, and $x\ge x_2$ the function $\lambda(.)$ is constant with value $\hat{Q}(x_2)$

$$
\begin{aligned}
\lambda(x) &= \sqrt{x}\left(\Lambda+V-K\frac{\sqrt{x}}{\sqrt{x^*}}\right).\mathbb{1}_{\{0<x<\gamma_s\}}+\sqrt{x}\left(e+\Upsilon_1+V-K\frac{\sqrt{x}}{\sqrt{x^*}}\right).\mathbb{1}_{\{\gamma_s<x\le\gamma_2\}}\\
&+\sqrt{x}\left(e+\Upsilon_1+\frac{\sqrt{x}}{\sqrt{x^*}}.\Upsilon_3\right).\mathbb{1}_{\{\gamma_2<x\le x_2\}}-\frac{l}{\kappa}\sqrt{x_2}+\sqrt{x^*}.\Upsilon_3.\mathbb{1}_{\{x>x_2\}}.
\end{aligned}
$$

Hence by applying Proposition 2 we have

$$
\begin{aligned}
U(s) &= \Lambda+V-Kz(s), && for\ -\infty<s<\mu_1\\
&= e+ls+V-Kz(s), && for\ \mu_1\le s\le\mu_3\\
&= e+ls+exp(\kappa(s^*-s))\Upsilon_3, && for\ \mu_3<s\le-\mu_5\\
&= -\frac{l}{\kappa}exp\left(-\frac{\kappa e}{l}-1-\kappa s\right)+exp(\kappa(s^*-s))\Upsilon_3, && for\ s>-\mu_5,
\end{aligned}
$$

Hence this completes our proof. ■

6 Limiting Behaviour

Whilst we have provided equations to our payoffs on a single exchange (that is selling of goods to vacate space at the buyer and then selling the contract), it would be useful to understand the long term behaviour after n exchanges or operations, specifically for $n\to\infty$. In other words we wish to know the limiting behaviour of our operation. In order to analyse this situation we make the assumption that the buyer must await for the supplier to exercise the n^{th} contract, before the $(n+1)^{th}$ contract can be offered. Consequently, the following function can be defined recursively as

$$
\alpha^n=\sup_{(\tau_1,\tau_2)\in\mathcal{T}_2}\mathbb{E}_s[f(S(\tau_1))exp(-r\tau_1)+(V+g_2^n(K,S(\tau_2)))exp(-r\tau_2)], \tag{6}
$$

where

$$g_2^n(K, S(\tau_2)) = \mathbb{E}_s[exp(-r\Gamma_{[s^*,\infty)})(\alpha_V^{n-1}(S(\Gamma_{[s^*,\infty)}) - K)],$$

with $\alpha_V^0 := 0$, and

$$\beta^n(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[exp(-r\tau)(V + g_2^n(K, S(\tau_2)))]. \tag{7}$$

Hence we are particularly interested in the limiting behaviour of

$$\begin{aligned} \alpha^*(s) &= \lim_{n\to\infty} \alpha^n(s), \forall s \in \mathbb{R}, \\ \beta^*(s) &= \lim_{n\to\infty} \beta^n(s), \forall s \in \mathbb{R}. \end{aligned}$$

In order to understand the limiting behaviour of our optimisation problem, we follow [21] and present a number of proofs. First we provide a Lemma on the functions $\alpha^n(.)$, $\beta^n(.)$, $g_2^n(.)$ and their sequences (Lemma 1). We then provide a Lemma on the optimal stopping problem and Brownian motion (Lemma 2). Next we provide a Lemma (Lemma 3) on optimal stopping problems for $g_2^\xi(.)$ (to be defined later) and the stopping region for function g_1^ξ (to be defined later). We then use the previous Lemmas to prove the limiting function of $\alpha^*(.)$ (Lemma 4), and the previous Lemmas are used to prove Theorem 4 on $\alpha^*(.)$.

Lemma 1. *The functions $\alpha^n(.), \beta^n(.)$ and $g_2^n(.)$ are continuous and bounded functions for all $n \geq 1$. Also, the sequence of functions $\{\alpha^n(.)\}_{n\geq 0}, \{\beta^n(.)\}_{n\geq 1}$ and $\{g_2^n(.)\}_{n\geq 1}$ are non-decreasing.*

Proof. First, we apply Theorem 1 to show β^1 and α^1 are continuous and bounded. We observe that functions $\alpha^n(.), g_2^{n+1}(.)$ and $\beta^{n+1}(.)$ are continuous and bounded, we have $\alpha^0 = 0$ is bounded and continuous, which implies the function $g_2^1(.)$ is bounded and continuous by definition. By Theorem 1 this implies that β^1 and α^1 are continuous and bounded. Therefore for $n = 1$ we have α^{n-1} is continuous and bounded implying that α^n, β^n and g_2^n are continuous and bounded. If we then assume that for a given $n \geq 1$ that α^{n-1} is continuous and bounded, we can also assert similarly that α^n, β^n and g_2^n are continuous and bounded. By induction we therefore conclude that $n \geq 1$.

We now apply Theorem 1 to prove that the sequences $\{\alpha^n\}_{n\geq 0}$, $\{\beta^n\}_{n\geq 1}$ and $\{g_2^n\}_{n\geq 1}$ are non-decreasing. By Theorem 1 we have α^1 is non-negative, therefore $g_2^2 \geq g_2^1$. Now as α^1, α^2 are solutions to the equation (6) (respectively), then

$$g_2^2 \geq g_2^1 \Rightarrow \alpha^2 \geq \alpha^1.$$

Similarly, if $\alpha^n \geq \alpha^{n-1}$ for some $n \geq 1$, then $g_2^{n+1} \geq g_2^n$ and therefore $\alpha^{n+1} \geq \alpha^n$. By a similar approach we deduce that $\beta^{n+1} \geq \beta^n$ because β^n is the solution to equation (7). Hence by induction on $n \geq 1$ the proof is completed. ■

Using the previous Lemma we have $\{\alpha^n\}_{n\geq 0}$ and $\{\beta^n\}_{n\geq 1}$ are non-decreasing sequences of continuous and bounded functions. Now using Theorem 1 we can also deduce that α^n for equation (6) also satisfies

$$\alpha^n(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[exp(-r\tau)(f(S(\tau)) + \beta^n(S(\tau))], n \geq 1. \tag{8}$$

We now wish to determine the optimal stopping problem for Brownian motion.

Lemma 2: *For Brownian motion $S(t)$ with initial value $s \in \mathbb{R}$, on probability space $(\Omega, \mathcal{F}, \mathbb{P}_s)$, then*

$$U(s_1) \leq U(s_2), \text{ for } s_1 > s_2 \geq s^*,$$

where function $g(.) : \mathbb{R} \to \mathbb{R}$ is continuous, bounded and decreasing after s^*, that is $g(s_1) \leq g(s_2)$, and $U(s)$ is the optimal stopping problem

$$U(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[exp(-r\tau)g(S(\tau))].$$

Proof: Let us define the stopping region $\zeta_U = \{s \in \mathbb{R} : U(s) = g(s)\}$ for the optimal stopping problem, with $\Gamma_{\zeta_U} = \inf\{t \geq 0 : S(t) \in \zeta_U\}$. For $\zeta_U = \emptyset \Rightarrow U = 0$ and the claim holds. For $\zeta_U \neq \emptyset$ then we have at least one of the points, or none of points s_1, s_2 are in the stopping region. If we take the scenario that at least one point is in the stopping region, let $s_1 \in \zeta_U \Rightarrow U(s_1) = g(s_1)$ and $g(s_1) \leq g(s_2) \leq U(s_2)$. If $s_2 \in \zeta_U \Rightarrow S(\Gamma_{\zeta_U}) \geq s_2$, $\forall s > s_2 \Rightarrow g(S(\Gamma_{\zeta_U})) \leq g(s_2) = U(s_2), \forall s > s_2$, including s_1. We therefore have $U(s_1) \leq U(s_2)$ by using the non-negativity of $U(.)$ and optimal value of Γ_{ζ_U}.

We now examine the scenario for neither points $\{s_1, s_2\}$ in the stopping region. First we examine $\zeta_U \cap \{s_1, s_2\} \neq \emptyset$. Let $s_4 \in \zeta_U \Rightarrow U(s_4) = g(s_4)$, with $s_4 \in \zeta_U \cap \{s_1, s_2\}$, then $g(s_2) \geq g(s_4) \Rightarrow U(s_2) \geq U(s_4)$. Also as $S(\Gamma_{\zeta_U}) \geq s_4, \forall s > s_4 \Rightarrow g(S(\Gamma_{\zeta_U})) \leq g(s_4)$, and $g(s_4) = U(s_4)$. Hence $U(s_1) \leq U(s_4) \leq U(s_2)$. We now examine the scenario $\zeta_U \cap \{s_1, s_2\} = \emptyset$: if $\zeta_U \cap (\infty, s^*] = \emptyset \Rightarrow \zeta_U \subseteq [a, \infty)$ where $a \in \zeta_U \cap (s^*, \infty)$ because ζ_U is non-empty and closed. Given that $0 \leq U(a)$, $U(a) = g(a)$ and if we now use the optimality of Γ_{ζ_U} then we have $g(s^*) < U(s^*) \leq g(a)$, which contradicts the assumption that $g(.)$ is decreasing $[s^*, \infty)$, hence $\zeta_U \cap (-\infty, s^*] \neq \emptyset$. Now suppose first that $\zeta_U \subset (-\infty, s_2)$, which is a closed and non-empty set, then there must exist a point $l \in \zeta_U$, such that $\zeta_U \subseteq (-\infty, l]$. Therefore

$$\begin{aligned} U(s_1) &= exp(\kappa(l - s_1))U(l), \\ U(s_2) &= exp(\kappa(l - s_2))U(l) \Rightarrow U(s_1) \leq U(s_2). \end{aligned}$$

Now finally, suppose that $\Gamma_U \cap (-\infty, a_1] \cup [a_2, \infty)$, where $a_1 < s_2 < s_1 < a_2$ and $a_1, a_2 \in \zeta_U$. If we apply Lemma 4.3 from [9] we obtain $U(s) = f(a_1, a_2, s, \kappa)$ and by differentiating we obtain $U'(s)$ and the conditions to determine that $U(.)$ is decreasing on $[s^*, \infty)$. ■

We now introduce our Lemma on the functions $g_1^{\xi}(.)$ and $g_2^{\xi}(s, K)$.

Lemma 3: *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function, decreasing on $[x^*, \infty)$ and $\xi(s^*) > K$. Let us also define*

$$g_2^{\xi}(s, K) = \mathbb{E}_s[exp(-r\Gamma_{[s^*,\infty]}) - K)],$$

and also the optimal stopping problem

$$g_1^{\xi}(s) = \sup_{\tau_2} \mathbb{E}_s[exp(-r\tau_2)(V + g_2^{\xi}(s, K))],$$

then we have $(\infty, s^*] \subseteq \Gamma_{\zeta_{g_1^{\xi}}} = \{g_1^{\xi} = V + g_2^{\xi}\}$.

Proof: We first apply Proposition 2 to $V + g_2^{\xi}(s, K)$, and define $F : [0, \infty) \to \mathbb{R}$ as

$$F(x) = \sqrt{x}\left(V + \frac{(\xi(s^*) - K)\sqrt{x}}{exp(\kappa x^*)}\right).\mathbb{1}_{\{0<x\leq x^*\}} + \sqrt{x}\left(V + \frac{ln(x)}{2\kappa} - K\right).\mathbb{1}_{\{x>x^*\}}$$

where $x^* = F(s^*) = exp(2\kappa s^*)$. Also, let us define $\hat{q}_1(.)$ and $\hat{q}_2(.)$, both defined on $\mathbb{R}_{\geq 0}$, with

$$\hat{q}_1(x) = \sqrt{x}\left(V + \xi\left(\frac{ln(x)}{2\kappa}\right) - K\right),$$

and

$$\begin{aligned} \hat{q}_2(x) = \sqrt{x}\left(V + \frac{\xi(s^* - K)\sqrt{x}}{exp(\kappa s^*)}\right) \quad &\Rightarrow \quad \hat{q}_2'(x) = \frac{1}{2}Vx^{-\frac{1}{2}} + \frac{\xi(s^*) - K}{exp(\kappa s^*)} \\ &\Rightarrow \quad \hat{q}_2''(x) = -\frac{1}{4}Vx^{-\frac{3}{2}}, \end{aligned}$$

noting that $\hat{q}_2'(.) = F(.)$ over $[0, x^*]$, and $\hat{q}_1'(.) = F(.)$ over $[x^*, \infty]$. Now if $\xi(s^*) > K$ then the differentials of $\hat{q}_2$ imply that $\hat{q}_2$ is non-negative, concave and increasing function. Therefore the smallest non-negative concave majorant of $\hat{q}_2$ is itself, and $\hat{q}_2$ majorises F because $\hat{q}_2(x) \geq \hat{q}_1(x)$ and $\hat{q}_1(x) = F(x)$ over $[x^*, \infty)$:

$$\hat{q}_2(x) - \hat{q}_1(x) = \sqrt{x}\left(V + \frac{\xi(s^* - K)\sqrt{x}}{exp(\kappa s^*)}\right) - \sqrt{x}\left(V + \xi\left(\frac{ln(x)}{2\kappa}\right) - K\right)$$

$$
\begin{aligned}
&= \frac{x}{exp(\kappa s^*)}(\xi(s^*) - K) - \sqrt{x}\left(\xi\left(\frac{ln(x)}{2\kappa}\right) - K\right) \\
&\geq \sqrt{x}\left(\frac{\sqrt{x}}{exp(\kappa s^*)} - 1\right)(\xi(s^*) - K) \Rightarrow \hat{q}_2(x) \geq \hat{q}_1(x),
\end{aligned}
$$

where we utilise $s \to \xi(s)$ is decreasing on $[s^*, \infty)$, $\sqrt{x} > \sqrt{x^*}$ and $x^* = exp(\kappa s^*)$ over (x^*, ∞). Therefore the smallest non-negative concave majorant of F and λ therefore gives $\lambda \leq \hat{q}_2$. However, on $[0, x^*]$ we have $F = \hat{q}_2$ and so $\lambda = \hat{q}_2 = F, \forall[0, x^*]$, alternatively $(-\infty, s^*] \subseteq \Gamma_{\zeta_1^\xi}$. Hence the proof is completed. ■

We now introduce Lemma 4, which requires Lemmas 2 and 3. Lemma 4 is required so that we can prove Theorem 4.

Lemma 4. *The limiting function of $\alpha^* = \lim_{n\to\infty} \alpha^n$ is bounded.*

Proof. To prove this, we examine α^* value with respect to K. First for $\alpha^* \leq K$, we apply Lemma 2 and using an induction argument prove that $\alpha^n(s^*) \geq \alpha^n(s), \forall s \geq s^*, \forall n \geq 1$. This inequality true for $n = 0$, and if we assume it is also true for $n = m = 1$, then payoff $V + g_2^m$ is decreasing on $[s^*, \infty)$ and by Lemma 2 the same also holds for $\beta^{m+1}(s)$. Since $f(.)$ is also decreasing on $\mathbb{R}$, by Lemma 2 and the characterisation of α^{m+1} in equation (8) to claim that α^{m+1} is decreasing on $[s^*, \infty)$. Now since $\alpha^n(s^*) \leq K, \forall n \geq 0$ then $g_2^n(s) \leq 0$ on $(-\infty, s^*], \forall n \geq 1$. However, by the prior argument $g_2^n(s) = \alpha^{n-1}(s) - K \leq \alpha^n(s^*) - K \leq 0$ over $(s^*, \infty) \Rightarrow g_2^n(s) \leq 0, \forall s \in \mathbb{R}$. Now we recall that $V > 0$ is constant, $f(.) \leq \Lambda$, the characteristation of α^n in equation (6) to deduce that $\alpha^n \leq \Lambda + V, \forall n \geq 1 \Rightarrow \alpha^* \leq \Lambda + V$.

We now examine the case for $\alpha^*(s^*) > K$ and that there exists an index $n \geq 1$: for equations (7) and (8) we assign the stopping sets, respectively $\zeta_\beta^n = \{s \in \mathbb{R} | \beta^n(s) = V + g_2^n(s)\}$, $\zeta_\alpha^n = \{s \in \mathbb{R} | \alpha^n(s) = f(s) + \beta^n(s)\}$, with respective stopping times $\tau_1^n = \Gamma_{\zeta_{\alpha^n}}$, and $\tau_2^n = \tau_1^n + \Gamma_{\zeta_{\beta^n}} \circ \Theta_{\tau_1^n}$. We now apply Theorem 1 $\forall n \geq 1$, so that

$$\alpha^n(s) = \mathbb{E}_s[exp(-r\tau_1^n)f(S(\tau_1^n)) + exp(-r\tau_2^n(V + g_2^n(S(\tau_2^n)))].$$

Now if we define n_0 as the first $n \geq 1$ such that $\alpha^n > K$, then using previous arguments we have $\forall n \geq n_0$ the inequality $g_2^{n+1}(s) \leq g_2^{n+1}(s^*), \forall s \geq s^*$. Moreover, s^* is the global maximum for $g_2^{n+1}(s)$ and from equation (7) one can show $\forall n \geq n_0$ that $\beta^{n+1}(s^*) = V + g_s^{n+1}(s^*)$, and $\beta^{n+1}(s^*) \geq \beta^{n+1}(s)$. The remainder of the proof is found by following [21] and the proof is outlined here. Firstly by deducing that the stopping region is $\zeta_{\alpha^{n+1}} \cap (-\infty, s') \neq \emptyset, \forall n \geq n_0$ and $s' < s^*$ is a constant. We then deduce that for every $m \geq 1$ there exists a point $s \in \zeta_{\alpha^{m+n_0}}$, such that $\alpha^{m+n_0}(s^*) \leq \alpha^{m+n_0}(s)$. The next deduction is that for all $m \geq 1$ then we must have at least one point $s_{m+n_0} \in \zeta_{\alpha^{m+n_0}} \cap (-\infty, s']$, such that $\alpha^{m+n_0}(s^*) \leq \alpha^{m+n_0}(s_{m+n_0})$. Now by applying Lemma 3 we find that $\alpha^n \leq \alpha^{n+1} \leq V + \Lambda + \Upsilon_2, \forall n \geq n_0$, where $\Upsilon_2 < \infty$. If we take the limit for $n \to \infty$ then we have $\alpha^* \leq V + \Lambda + \Upsilon_2$. ■

Now that we have stated our lemmas, we are now ready to state our Theorem.

Theorem 4. *The functions α^* and β^* are lower semicontinuous functions and are bounded. The functions have the following properties: firstly, α^* is the smallest r-excessive majorant of $f + \beta^*$. Secondly, β^* is the smallest r-excessive majorant of $V + g_2^*$, where*

$$g_2^*(s) = (\alpha^*(s) - K).\mathbb{1}_{\{s>s^*\}} + (\alpha^*(s^*) - K).z(s).\mathbb{1}_{\{s\leq s^*\}}.$$

Thirdly, α^* and β^* are functions of the optimal stopping problems:

$$
\begin{aligned}
\alpha^*(s) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_s[exp(-r\tau)(f(S(\tau)) + \beta^*(S(\tau)))], \\
\beta^*(s) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_s[exp(-r\tau)(V + g_2^*(S(\tau)))].
\end{aligned}
$$

Finally, α^* is the function of the implicit optimal starting-stopping problem

$$\alpha^*(s) = \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} \mathbb{E}_s[exp(-r\tau_1)f(S(\tau_1)) + exp(-r\tau_2)(V + g_2^* S(\tau_2))].$$

Proof. We note that α^* and β^* are the supremum of non-decreasing sequences of bounded and continuous functions, hence we can deduce the functions exists and are lower semicontinuous. Moreover, Lemma 4 implies that α^* is bounded, and so g_2^* and β^* are bounded.

We now wish to prove that α^* is the smallest r-excessive majorant of $f + \beta^*$. From [8] we can assert that as the limiting function of an increasing sequence of r-excessive functions, then α^* is also r-excessive. Now given that α^n is the smallest r-excessive majorant of $f + \beta^n$ for $n \geq 1$ then $\alpha \geq f + \beta^n, \forall n \geq 1$. If we now take the limit in terms of n then we obtain $\alpha^* \geq f + \beta^*$, and α^* is the r-excessive majorant of $f + \beta^*$. To prove that α^* is the smallest, let $\bar{\alpha} : \mathbb{R} \to [0, \infty]$ be any r-excessive function dominating $f + \beta^*$, that is $\bar{\alpha} \geq f + \beta^* \geq f + \beta^n, \forall n \geq 1$. Given that we know that α^n is the smallest r-excessive majorant of $f + \beta^n, \forall n \geq 1$, then

$$\bar{\alpha} \geq \alpha^n, \forall n \geq 1 \Rightarrow \bar{\alpha} \geq \sup_n \alpha^n \Rightarrow \bar{\alpha} \geq \alpha^*.$$

Moreover, to prove β^* is the smallest r-excessive majorant of $V + g_2^*$, we use a similar argument to that applied to α^* previously.

To prove that α^* and β^* are functions of the optimal stopping problems, we first observe that β^* is the smallest r-excessive function majorising $V + g_2^*$, therefore it is the function of the optimal stopping problem

$$\beta^*(s) \quad = \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[exp(-r\tau)(V + g_2^*(S(\tau)))].$$

Now by applying Proposition 5.13 in [9] we can conclude β^* is continuous and bounded. If we now repeat previous arguments then we can conclude that β^* is the function of the optimal stopping problem

$$\alpha^*(s) \quad = \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}_s[exp(-r\tau)(f(S(\tau)) + \beta^*(S(\tau)))],$$

and so is continuous and bounded. If we now apply Theorem 1 then we can deduce that

$$\alpha^*(s) = \sup_{(\tau_1, \tau_2) \in \mathcal{T}_2} \mathbb{E}_s[exp(-r\tau_1) f(S(\tau_1)) + exp(-r\tau_2)(V + g_2^* S(\tau_2))].$$

Hence this completes our proof. ■

7 Conclusion

Risk management is a fundamental function of all businesses nowadays, and its importance has grown over the past 30 years. In particular, suppliers need to manage their goods and Value at Risk (VaR) is a popular risk measurement for risk analysis. Within a VaR risk methdology we have analysed a risk management strategy, through transferring risk with a hedging contract to alternative buyers. We provide a mathematical model of the hedging contract and operation, modelling it as an optimal stopping problem. We solve the problem to derive the optimal stopping criteria. We provide closed form solutions to the payoffs involved in the model, and we derive the limiting and long term behaviour of our operations. In terms of future work, we would like extend our model to take into account additional transaction costs (such as taxes) to determine the impact on payoffs. We would also like to investigate the impact of introducing switching clauses in our contract to improve hedging costs.

References

[1] Aïd, R., A review of optimal investment rules in electricity generation, *Quantitative Energy Finance*, pp.3–40, 2013.

[2] Angelis, T.D., Ferrari, G., Martyr, R., and J. Moriarty, Optimal entry to an irreversible investment plan with non convex costs, *Mathematics and Financial Economics*, vol.11, no.4, pp.423–454, 2017.

[3] Artzner, P., Delbaen, F., Eber, J., Heath, D., and H. Ku, Coherent multiperiod risk measurement, Manuscript, ETH Zurich, 2003.

[4] Bingham, N.H., and K. Rudiger, *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, Springer, 2004.

[5] Björk, T., *Arbitrage Theory in Continuous Time*, Oxford University Press, 2004.

[6] Chiu, C.-H., and T.-M. Choi, Optimal pricing and stocking decisions for newsvendor problem with value-at-risk consideration, *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, vol.40, no.5, pp.1116–1119, 2010.

[7] Cho, I.-K., and S.P. Meyn, Efficiency and marginal cost pricing in dynamic competitive markets with friction, *Theoretical Economics*, vol.5, no.2, pp.215–239, 2010.

[8] Chung, K.L., and J.B. Walsh, *Markov Processes, Brownian Motion, and Time Symmetry*, World Publishing Corporation, 2013.

[9] Dayanik, S., and I. Karatzas, On the optimal stopping problem for one-dimensional diffusions, *Stochastic Processes and their Applications*, vol.107, no.2, pp.173–212, 2003.

[10] Dixit, A.K., and R.S. Pindyck, *Investment under Uncertainty*, Princeton University Press, 1994.

[11] Hull, J., *Options, Futures and Other Derivatives*, Prentice Hall New Jersey, 2000.

[12] Lim, J.J., Zhang, A.N., and P.S. Tan, A practical supply chain risk management approach using var, *IEEE International Conference on Industrial Engineering and Engineering Management*, 2013.

[13] Mamon, R.S., Three ways to solve for bond prices in the vasicek model, *Journal of Applied Mathematics and Decision Sciences*, vol.8, no.1, pp.1–14, 2004.

[14] Martyr, R., Finite-horizon optimal multiple switching with signed switching costs, *Mathematics of Operations Research*, vol.41, no.4, pp.1432–1447, 2016.

[15] Menaldi, J.-L., Robin, M., and M. Sun, Optimal starting-stopping problems for markov-feller processes, *Stochastics: An International Journal of Probability and Stochastic Processes*, vol.56, nos.1-2, pp.17–32, 1996.

[16] Moriarty, J., and J. Palczewski, Real option valuation for reserve capacity, *European Journal of Operational Research*, vol.257, no.1, pp.251–260, 2017.

[17] Ross, S.M., *Introduction to Stochastic Dynamic Programming*, Acad. Press, 1995.

[18] Schachermayer, W., Filipovic, D., and D. Duffie, Affine processes and applications in finance, *The Annals of Applied Probability*, vol.13, no.3, pp.984–1053, 2003.

[19] Sethi, S.P., *Optimal Control Theory: Applications to Management Science and Economics*, Springer, 2018.

[20] Sun, M., Nested variational inequalities and related optimal starting-stopping problems, *Journal of Applied Probability*, vol.29, no.1, pp.104–115, 1992.

[21] Szabo, D.Z., and R. Martyr, Real option valuation of a decremental regulation service provided by electricity storage, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol.375, article 20160300, 2017.

[22] Szegö, G., Measures of risk, *European Journal of Operational Research*, vol.163, no.1, pp.5–19, 2005.

[23] Vasicek, O., An equilibrium characterization of the term structure, *Journal of Financial Economics*, vol.5, no.2, pp.177–188, 1977.