Journal of Uncertain Systems
Vol.13, No.1, pp.42-55, 2019
Online at: www.jus.org.uk

Bi-Normed Intuitionistic Fuzzy $\beta-$Ideals of $\beta-$Algebras

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Received 3 September 2018; Revised 23 October 2018

Abstract

This paper extends fuzzy beta-ideal into intuitionistic fuzzy beta-ideal of a beta-algebra. The notion of bi-normed intuitionistic fuzzy beta-ideal has been discussed by coupling triangular norm and triangular conorm. Further some realted results using cartesion product and level subsets are also studied.

Keywords: $\beta-$algebra, $\beta-$ideals, intuitionistic fuzzy $\beta-$ideal, bi-normed intuitionistic fuzzy $\beta-$ideal

1 Introduction

Schweizer and Sklar [17, 18] were first presented Triangular norms and triangular conorms with the improvement of $T-$norms in statistical metric spaces. Also they dealt about the concept of associative functions and statistical triangle inequalities. Menger [13] has analysed the idea of probabilistic metric spaces which leads to additional input into the decision making concepts and theories of corporative recreations.Specifically, in the framework of hypotheses of fuzzy sets, the $T-$norms have been broadly used for fuzzy operations, fuzzy logics and fuzzy connection conditions. In probabilistic metric spaces, $T-$norms are utilized by Hadzic [7] et al. to generalize triangle inequality of common metric spaces. Individual $T-$norms most of the times apply in further disciplines of mathematics, since the class contains numerous recognizable functions. In [11, 12], a precise study concerning the properties and the related parts of t-norms have been considered by Klement et al.

After Zadeh's [24] presentation of fuzzy sets, there have been various speculations of this crucial idea. In [25] likewise they presented the idea of interval valued fuzzy subsets where the estimations of the membership functions are intervals of numbers rather than the numbers. Shieh [21] introduced the notion of infinite fuzzy relations equations with continuous t-norms. The thought of intuitionstic fuzzy set was characterized by Atanassov [2] and he portrayed the arrangement of all intuitionistic fuzzy sets in which two modal logic operators ($\Diamond$ and $\Box$) are also focused.

The fuzzy sets have been connected in algebraic structures begins from Rosenfeld [16]. Triangular normed fuzzy subalgebras of $BCK-$algebras was developed by Young Bae Jun et al. [9]. Kim [10] proposed intuitionistic (T,S) normed fuzzy subalgebras of $BCK-$algebras and Tapan senapati et al. [19, 20] started the concept of intuitionistic fuzzy bi-normed $KU-$subalgebra of $KU-$algebra and $KU-$ideal of a $KU-$algebra . Dutta et al. [6] introduced the concept of direct product of general doubt IF ideals of $BCK/BCI-$algebras with respect to triangular Bi-norm. Intuitionistic $(S,T)-$ fuzzy Lie ideals of lie algebras was extended by Muhammed Akram [1]. Hedayati [8] proposed the idea of interval valued intuitionstic fuzzy subsemi module of a semimodule as for t-norm T and s-norm S. Specifically, by the assistance of the harmoniousness relations on semimodules, build another interval valued intuitionistic (S,T)-fuzzy subsemi modules on semi module of quotient.

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$\beta-$algebra is another algebraic structre consturucted with two opeartions.The thought of $\beta-$algebras was started by Neggers [15]. Later various studies have been carried out on $\beta-$algebra using fuzzy. In [3, 4, 5] the authors talked about the idea of fuzzy $\beta-$subalgebras, fuzzy $\beta-$ideals of $\beta-$algebras and T fuzzy $\beta-$subalgebras of $\beta-$algebra. Sujatha et al. [22, 23] depicted the idea of intuitionistic fuzzy $\beta-$subalgebras of $\beta-$algebras and product on intuitionstic fuzzy $\beta-$subalgebras of $\beta-$algebras. Muralikrishna et al. [14] proposed the notion of $(S,T)-$ normed intuitionistic fuzzy $\beta-$subalgebras.

With all these inspiration, this paper intenns to study about the bi-normed intuitionistic fuzzy $\beta-$ideal. The paper is sorted out as pursue: Section 1 shows the introduction and section 2 gives some basic definitions and properties of $\beta-$algebra, fuzzy $\beta-$ideal, T-norm, $S-$norm and so on. Section 3 deals the concept and operations of bi-normed intuitionistic fuzzy $\beta-$ideals of $\beta-$algebra and discussed their properties. Then, the characterization based on the modal operators $\Box A$ and $\Diamond A$ have been applied. Following that the behaviour of this structure under homomorphisms is investigated. Section 4, focuses the Cartesion product of bi-normed intuitionistic fuzzy $\beta-$ideals. section 5, is dedicated for the level subset of bi-normed intuitionistic fuzzy $\beta-$ideals and section 6 gives the conclusion.

2 Preliminaries

This section reveals the necessary definitions required for the work.

Definition 1 ([15]) *A $\beta-$ algebra is non-empty set X with a constant 0 and two binary operations $+$ and $-$ are satisfying the following axioms:*
(i) $x-0=x$,
(ii) $(0-x)+x=0$,
(iii) $(x-y)-z=x-(z+y)\ \ \forall\ x,y,z\in X$.

Example 2 The set $X=\{0,1,2,3,4,5\}$ is a β-algebra with constant 0 and two binary operation $+$ and $-$ are defined on X by the following Cayley's table.

Table 1: β-algebra

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	3	1
3	3	4	5	0	1	2
4	4	3	1	2	5	0
5	5	2	3	1	0	4

$-$	0	1	2	3	4	5
0	0	1	2	3	5	4
1	1	0	4	5	3	2
2	2	5	0	4	1	3
3	3	4	5	0	2	1
4	4	3	1	2	0	5
5	5	2	3	1	4	0

Example 3 Consider Set of all integers Z. $(Z,+,-,0)$ is an infinite $\beta-$algebra where 0 , $+$ and $-$ have usual meanings.

Definition 4 ([3]) *A non-empty subset I of a $\beta-$algebra $(X,+,-,0)$ is called a $\beta-$ideal of X, if*
(i) $0\in I$,
(ii) $x+y\in I$,
(iii) $x-y$ *&* $y\in I$ *then* $x\in I\quad \forall x,y\in X$.

Example 5 Consider the $\beta-$algebra $(X,+,-,0)$ in the following Cayley's table.

Table 2: β-algebra for β-ideal

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$-$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The subset $I_1=\{0,1\}$ is a $\beta-$ideal of X.

Definition 6 *In a* $\beta-$*algebra* $(X, +, -, 0)$*, a partial ordering* $\leq$ *can be defined as,* $\forall x, y \in X, x \leq y \Leftrightarrow x-y = 0$.

Definition 7 ([24]) *A fuzzy set in* X *is represented by* $A = \{\langle x, \mu_A(x)\rangle/x \in X\}$ *defined as a function* $\mu : X \to [0, 1]$*. i.e, for each element in the universal set* X*,* $\mu_A(x)$ *is called the membership value of* $x \in X$.

Definition 8 ([2]) *An Intuitionistic fuzzy set in a nonempty set X is defined by* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in X\}$ *where* $\mu_A : X \to [0, 1]$ *is a membership function of* A *and* $\nu_A : X \to [0, 1]$ *is a non membership function of* A *satisfying* $0 \leq \mu_A(x) + \nu_A(x) \leq 1 \ \ \forall\ x \in X$.

Definition 9 ([5]) *Let* μ *be a fuzzy set in a* $\beta-$*algebra. Then* μ *is called a fuzzy* $\beta-$*ideal of X, if*
(i) $\mu(0) \geq \mu(x)$,
(ii) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$,
(iii) $\mu(x) \geq min\{\mu(x - y), \mu(y)\} \quad \forall x, y \in X$.

Example 10 Let $X = \{0, 1, 2, 3\}$ be a β-algebra with constant 0 and two binary operation $+$ and $-$ are defined on X by the following Cayley's table.

Table 3: β-algebra for fuzzy β-ideal

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$-$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then the following fuzzy set A in X defined as

$$\mu_A(x) = \begin{cases} 0.8 : & x = 0 \\ 0.6 : & x = 1 \\ 0.4 : & x = 2, 3 \end{cases}$$

Then A is clearly fuzzy $\beta-$ideal of X.

Definition 11 ([5]) *Let* μ *be a fuzzy set in a* $\beta-$*algebra. Then* μ *is called an anti fuzzy* $\beta-$*ideal of X, if*
(i) $\mu(0) \leq \mu(x)$,
(ii) $\mu(x + y) \leq max\{\mu(x), \mu(y)\}$,
(iii) $\mu(x) \leq max\{\mu(x - y), \mu(y)\} \quad \forall x, y \in X$.

Definition 12 ([11, 12]) *A function* $T : [0, 1] \times [0, 1] \to [0, 1]$ *is called a triangular norm (*$T-$*norm), if*
(i) $T(x, 1) = x$,
(ii) $T(x, y) = T(y, x)$,
(iii) $T(T(x, y), z) = T(x, T(y, z))$,
(iv) $T(x, y) \leq T(x, z) \ \ if \ \ y \leq z \ \ \forall x, y, z \in [0, 1]$.

The minimum ([7, 10, 12]) $T_M(x, y) = min(x, y)$, the product $T_P(x, y) = x.y$ and the Lukasiewicz $T-$norm $T_L(x, y) = max(x + y - 1, 0)\ \forall\ x, y \in [0, 1]$ are some of the $T-$norms.

Definition 13 ([11]) *A function* $S : [0, 1] \times [0, 1] \to [0, 1]$ *is called a traingular conorm (*$T-$*conorm), if*
(i) $S(x, 0) = x$,
(ii) $S(x, y) = S(y, x)$,
(iii) $S(S(x, y), z) = S(x, S(y, z))$,
(iv) $S(x, y) \leq S(x, z) \ \ if \ \ y \leq z \ \ \forall x, y, z \in [0, 1]$.

The maximum $S_M(x, y) = max(x, y)$, the probabilistic sum $S_P(x, y) = x + y - x.y$ and the Lukasiewicz $T-$conorm $S_L(x, y) = min(x + y, 1)\ \forall\ x, y \in [0, 1]$ are some of the $T-$norms.

3 Bi-Normed Intuitionistic Fuzzy $\beta-$Ideals

This section starts with the definitions of $T-$fuzzy and $S-$fuzzy $\beta-$ideals of a $\beta-$algebra. Further the notion of intutionistic fuzzy $\beta-$ideals on $\beta-$algebras were introduced. Then the concept has been extended to bi-normed intutionistic fuzzy $\beta-$ideals of $\beta-$algebras using the $(T,S)-$norm(both $T-$norm and $T-$conorm are coupled together) and studied the related results. Here after X is a $\beta-$algebra unless otherwise specified.

Definition 14 *Let $A = \{x, \mu_A(x) : x \in X\}$ be an fuzzy set in a $\beta-$algebra of X. A is called an $T-$fuzzy $\beta-$ideal [T-norm fuzzy $\beta-$ideal] of X, if*
(i) $\mu_A(0) \geq \mu_A(x)$,
(ii) $\mu_A(x+y) \geq T\{\mu_A(x), \mu_A(y)\}$,
(iii) $\mu_A(x) \geq T\{\mu_A(x-y), \mu_A(y)\} \quad \forall x, y \in X$.

Definition 15 *Let $A = \{x, \mu_A(x) : x \in X\}$ be an fuzzy set in a $\beta-$algebra of X. A is called an $S-$fuzzy $\beta-$ideal [T-conorm fuzzy $\beta-$ideal] of X, if*
(i) $\mu_A(0) \leq \mu_A(x)$,
(ii) $\mu_A(x+y) \leq S\{\mu_A(x), \mu_A(y)\}$,
(iii) $\mu_A(x) \leq S\{\mu_A(x-y), \mu_A(y)\} \quad \forall x, y \in X$.

Example 16 Let $X = \{0, 1, 2, 3\}$ be a $\beta-$algebra with constant 0 and binary operations $+$ and $-$ are defined on X as in the following cayley's table.

Table 4: β-algebra for $T-$fuzzy and $S-$fuzzy β-ideal

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Let $T_M, S_M : [0,1] \times [0,1] \to [0,1]$ be functions defined by $T_M(x,y) = max(x+y-1, 0)$ and $S_M(x,y) = min(x+y, 1)$ $\forall x, y \in [0,1]$. Here T_M is a $T-$norm and S_M is a $S-$norm. Define an fuzzy set A in X by

$$\mu_A(x) = \begin{cases} 0.7 : & x = 0 \\ 0.5 : & x = 1 \\ 0.4 : & x = 2, 3. \end{cases}$$

A is clearly $T-$fuzzy $\beta-$ideal and $S-$ fuzzy $\beta-$ideal of X.

Definition 17 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ be an intutionistic fuzzy set in a $\beta-$algebra X. A is called intuitionistic fuzzy $\beta-$ ideal of X, if*
(i) $\mu_A(0) \geq \mu_A(x) \quad \& \quad \nu_A(0) \leq \nu_A(x)$,
(ii) $\mu_A(x+y) \geq min\{\mu_A(x), \mu_A(y)\} \quad \& \quad \nu_A(x+y) \leq max\{\nu_A(x), \nu_A(y)\}$,
(iii) $\mu_A(x) \geq min\{\mu_A(x-y), \mu_A(y)\} \quad \& \quad \nu_A(x) \leq max\{\nu_A(x-y), \nu_A(y)\} \quad \forall x, y \in X$.

Example 18 Let $X = \{0, 1, 2, 3\}$ be a $\beta-$algebra with a constant 0 and the binary operations $+$ and $-$ are defined on X which is presented in the following cayley's table.

Table 5: β-algebra for IF β-ideal

+	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

−	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

The intuitionistic fuzzy set A in X defined as

$$\mu_A = \begin{cases} 0.8: & x = 0,1 \\ 0.5: & x = 2,3, \end{cases} \qquad \nu_A = \begin{cases} 0.1: & x = 0,1 \\ 0.3: & x = 2,3, \end{cases}$$

is an intuitionistic fuzzy $\beta-$ideal of X.

Theorem 19 *Let A be an intuitionistic fuzzy $\beta-$ ideal of a $\beta-$algebra X. If $x \leq y$ then $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y) \;\; \forall x, y \in X$.*
Proof: *For $x, y \in X, x \leq y \Rightarrow x - y = 0$,now*

$$\begin{aligned} \mu_A(x) &\geq min\{\mu_A(x-y), \mu_A(y)\} \\ &= min\{\mu_A(0), \mu_A(y)\} \\ &= \mu_A(y), \end{aligned}$$

and

$$\begin{aligned} \nu_A(x) &\leq max\{\nu_A(x-y), \nu_A(y)\} \\ &= max\{\nu_A(0), \nu_A(y)\} \\ &= \nu_A(y). \end{aligned}$$

Theorem 20 *Let A be a subset of X. Define an intuitionistic fuzzy set $\chi_A = \{x, \mu_{\chi_A}(x), \nu_{\chi_A}(x) : x \in X\}$ such that*

$$\mu_{\chi_A}(x) = \begin{cases} t_0 & if \; x \in \; A \\ t_1 & if \; x \notin \; A \end{cases} \qquad and \quad \nu_{\chi_A}(x) = \begin{cases} s_0 & if \; x \in \; A \\ s_1 & if \; x \notin \; A \end{cases}$$

where t_0, t_1, s_0 and $s_1 \in [0,1]$ with $t_0 > t_1 > s_1 > s_0$. χ_A is an intuitionistic fuzzy $\beta-$ideal of X if and only if A is a $\beta-$ideal of X.
Proof: *Suppose χ_A is an intuitionistic fuzzy set on X.*
(i) $\mu_{\chi_A}(0) \geq \mu_{\chi_A}(x) \;\; \forall x \in X$.

$$\therefore \; \mu_{\chi_A}(0) = t_0 \;\; or \;\; t_1 \; with \;\; t_0 > t_1. \tag{1}$$

If $\mu_{\chi_A}(0) = t_0$, then $0 \in A$.

$$\Rightarrow \mu_{\chi_A}(0) = t_1. \tag{2}$$

(1) and (2)$\Rightarrow t_1 \geq \mu_{\chi_A}(x) = t_0$, Which is a contradiction. Hence $\mu_{\chi_A}(0) = t_0$ gives $\;0 \in A$.
(ii) For $x, y \in A \;\Rightarrow \mu_{\chi_A}(x) = t_0 = \mu_{\chi_A}(y)$. Now,

$$\mu_{\chi_A}(x+y) \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = min\{t_0, t_0\} = t_0.$$

Therefore $\mu_{\chi_A}(x+y) = t_0 \;\Rightarrow x + y \in A$.
(iii) For any $x, y \in X$, if $x - y$ & $y \in A \;\Rightarrow \mu_{\chi_A}(x-y) = t_0 = \mu_{\chi_A}(y)$. Now

$$\mu_{\chi_A}(x) \geq min\{\mu_{\chi_A}(x-y), \mu_{\chi_A}(y)\} = min\{t_0, t_0\} = t_0 \Rightarrow x \in A.$$

Similarly, it can be proved for non membership function. Hence A is a $\beta-$ideal of X.

Conversely, assume A is a $\beta-$ideal of X.

$$\therefore 0 \in A \Rightarrow \mu_{\chi_A}(0) = t_0.$$

Also $Im(\mu_{\chi_A}) = \{t_0, t_1\}$ and $t_0 > t_1 \Rightarrow \mu_A(0) \geq \mu_{\chi_A}(x) \;\; \forall x \in X$. For $x, y \in A, x + y \in A$.

$$\Rightarrow \mu_{\chi_A}(x) = \mu_{\chi_A}(y) = \mu_{\chi_A}(x+y) = t_0 \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\}.$$

Hence $\mu_{\chi_A}(x+y) \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\}$.

For $x, y \in X$, *if* $x - y$ *and* $y \in A \Rightarrow x \in A$, *then*

$$\mu_{\chi_A}(x) = t_0 = min\{t_0, t_0\} = min\{\mu_{\chi_A}(x-y), \mu_{\chi_A}(y)\}.$$

For some $x \in X$, *if* $x - y \in A$ *and* $y \notin A \Rightarrow\ x \in A$, *then*

$$\mu_{\chi_A}(x) = t_1 = min\{t_0, t_1\} \geq min\{\mu_{\chi_A}(x-y), \mu_{\chi_A}(y)\}.$$

Similarly, it can be proved for non membership function.
$\therefore \chi_A$ *is an intuitionistic fuzzy* $\beta-$*ideal of* X.

Theorem 21 *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ *be an intuitionistic fuzzy set in a* $\beta-$*algebra.* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ *is an intuitionistic fuzzy* $\beta-$*ideal if and only if* $\mu_A(x)$ *is fuzzy* $\beta-$*ideal and* $\nu_A(x)$ *is anti fuzzy* $\beta-$*ideal of* X.

Definition 22 *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ *be an intutionistic fuzzy set in a* $\beta-$*algebra* X. *A is called a Bi-normed intuitionistic fuzzy* $\beta-$ *ideal of* $\beta-$*algebra* X, *if*
(i) $\mu_A(0) \geq \mu_A(x) \quad \& \quad \nu_A(0) \leq \nu_A(x)$,
(ii) $\mu_A(x+y) \geq T\{\mu_A(x), \mu_A(y)\} \quad \& \quad \nu_A(x+y) \leq S\{\nu_A(x), \nu_A(y)\}$,
(iii) $\mu_A(x) \geq T\{\mu_A(x-y), \mu_A(y)\} \quad \& \quad \nu_A(x) \leq S\{\nu_A(x-y), \nu_A(y)\} \quad \forall x, y \in X$.

Example 23 Consider the $\beta-$algebra $X = \{0, 1, 2, 3\}$ with a constant 0 and binary operations $+$ and $-$ are defined on X as in the following cayley's table.

Table 6: β-algebra for Binormed IF β-ideal

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

$-$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

Let $T_M, S_M : [0,1] \times [0,1] \to [0,1]$ be the functions defined by

$$T_M(x, y) = max(x + y - 1, 0)$$

and

$$S_M(x, y) = min(x + y, 1)\ \forall x, y \in [0,1].$$

Here T_M is a $T-$norm and S_M is a $S-$norm.

Define an intuitionistic fuzzy set $A = \{x, \mu_A(x), \nu_A(x) : x \in X\}$ in X by

$$\mu_A = \begin{cases} 0.6 : & x = 0, 1 \\ 0.5 : & otherwise \end{cases} \qquad \nu_A = \begin{cases} 0.2 : & x = 0, 1 \\ 0.4 : & otherwise. \end{cases}$$

Clearly, A is a bi-normed intuitionistic fuzzy $\beta-$ideal with respect to the norms defined.

Example 24 Consider the $\beta-$algebra $X = \{Z, +, -, 0\}$. The intuitionistic fuzzy set $A = \{x, \mu_A(x), \nu_A(x) : x \in X\}$ in X defined by

$$\mu_A(x) = \begin{cases} 0.7 : & x = 0 \\ 0.6 : & x < 0 \\ 0.5 : & x > 0 \end{cases} \qquad \nu_A(x) = \begin{cases} 0.2 : & x = 0 \\ 0.3 : & x < 0 \\ 0.4 : & x > 0 \end{cases}$$

is a Bi-normed intuitionistic fuzzy $\beta-$ideal of X.

Theorem 25 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ be a Bi-normed intuitionistic fuzzy $\beta-$ ideal of a $\beta-$algebra X. If $x \leq y$ then $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$ $\forall x, y \in X$.*
Proof: *For $x, y \in X, x \leq y \Rightarrow x - y = 0$. Then*

$$\begin{aligned}\mu_A(x) &\geq T\{\mu_A(x-y), \mu_A(y)\}\\ &= T\{\mu_A(0), \mu_A(y)\}\\ \mu_A(x) &= \mu_A(y)\end{aligned}$$

and

$$\begin{aligned}\nu_A(x) &\leq S\{\nu_A(x-y), \nu_A(y)\}\\ &= S\{\nu_A(0), \nu_A(y)\}\\ &= \nu_A(y).\end{aligned}$$

Theorem 26 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ be an Bi-normed intuitionstic fuzzy $\beta-$ideal of a $\beta-$algebra X. If $x + y \leq z$ then, $\mu_A(x) \geq T\{\mu_A(z), \mu_A(y)\}$ & $\nu_A(x) \leq S\{\nu_A(z), \nu_A(y)\}$ $\forall x, y, z \in X$.*
Proof: *For $x, y, z \in X$,*

$$\begin{aligned}\mu_A(x) &\geq T\{\mu_A(x-y), \mu_A(y)\}\\ &= T\{T\{\mu_A((x-y)-z), \mu_A(z)\}, \mu_A(y)\}\\ &= T\{T\{\mu_A((x-(z+y)), \mu_A(z)\}, \mu_A(y)\}\\ &= T\{T\{\mu_A(0), \mu_A(z)\}, \mu_A(y)\}\\ &= T\{\mu_A(z), \mu_A(y)\},\\ \nu_A(x) &\leq S\{\nu_A(x-y), \nu_A(y)\}\\ &= S\{S\{\nu_A((x-y)-z), \nu_A(z)\}, \nu_A(y)\}\\ &= S\{S\{\nu_A((x-(z+y)), \nu_A(z)\}, \nu_A(y)\}\\ &= S\{S\{\nu_A(0), \nu_A(z)\}, \nu_A(y)\}\\ &= S\{\nu_A(z), \nu_A(y)\}.\end{aligned}$$

Theorem 27 *Let A and B be Bi-normed intuitionistic fuzzy $\beta-$ideals of X. Then $A \cap B$ is also a Bi-normed intuitionistic fuzzy $\beta-$ideal of X. It can be generalised to the intersection of a family of Bi-normed intuitionistic fuzzy $\beta-$ideal of X.*

Theorem 28 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ be an intuitionistic fuzzy set in a $\beta-$algebra. A is a Bi-normed intuitionistic fuzzy $\beta-$ideal if and only if $\mu_A(x)$ is $T-$norm fuzzy $\beta-$ideal and $\nu_A(x)$ is a $S-$norm fuzzy $\beta-$ideal of X.*

Remark 29 Atanassov [2] characterized the set of all intuitionistic fuzzy sets with two operators which will transform each intuitionistic fuzzy set into fuzzy set. They are identical to the operators necessity and possibility defined in some modal logics. For every intuitionistic fuzzy set A, the modal operators $\Box A$ and $\Diamond A$ are defined as
$\Box A = \{\langle x, \mu_A(x)\rangle / x \in E\} = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle / x \in E\}$,
$\Diamond A = \{\langle x, 1 - \nu_A(x)\rangle / x \in E\} = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle / x \in E\}$.

The following deals the model operators applied on Binormed intuitionistic fuzzy $\beta-$ideal.

Theorem 30 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of X. $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is a Bi-normed intuitionistic fuzzy $\beta-$ideal of X if and only if $\Box A = \{X, \mu_A, (\mu_A)^c\}$ is a Bi-normed intuitionistic fuzzy $\beta-$ideal of X.*
Proof: *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of X.*

To complete the proof it is enough to prove the following claims.
Claim (i) $(\mu_A^c)(0) \leq (\mu_A^c)(x)$
For,

$$(\mu_A)(0) \geq (\mu_A)(x)$$

$$\begin{aligned}\Leftrightarrow 1-(\mu_A)(0) &\leq 1-(\mu_A)(x)\\ \Leftrightarrow (\mu_A^c)(0) &\leq (\mu_A^c)(x).\end{aligned}$$

Claim (ii) $\mu_A^c(x+y) \leq S\{\mu_A^c(x), \mu_A^c(y)\}$
For,

$$\begin{aligned}(\mu_A)(x+y) &\geq T\{(\mu_A)(x), (\mu_A)(y)\}\\ \Leftrightarrow 1-(\mu_A)(x+y) &\geq 1-T\{(\mu_A)(x), (\mu_A)(y)\}\\ \Leftrightarrow \mu_A^c(x+y) &\leq S\{(1-\mu_A(x)), (1-\mu_A(y))\}\\ \Leftrightarrow \mu_A^c(x+y) &\leq S\{\mu_A^c(x), \mu_A^c(y)\}.\end{aligned}$$

Claim (iii) $\mu_A^c(x) \leq S\{\mu_A^c(x-y), \mu_A^c(y)\}$
For,

$$\begin{aligned}(\mu_A)(x) &\geq T\{(\mu_A)(x-y), (\mu_A)(y)\}\\ \Leftrightarrow 1-(\mu_A)(x) &\geq 1-T\{(\mu_A)(x-y), (\mu_A)(y)\}\\ \Leftrightarrow \mu_A^c(x) &\leq S\{(1-\mu_A(x-y)), (1-\mu_A(y))\}\\ \Leftrightarrow \mu_A^c(x) &\leq S\{\mu_A^c(x-y), \mu_A^c(y)\}.\end{aligned}$$

Theorem 31 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set of X. $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is an Bi-normed intuitionistic fuzzy $\beta-$ideal of X if and only if $\Diamond A = (x, \nu_A^c, \nu_A)$ is a Bi-normed intuitionistic fuzzy $\beta-$ideal of X.*
Proof: *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an Bi-normed intuitionistic fuzzy set of X.*

To complete the proof it is enough to prove the following claims.
Claim(i) $(\nu_A^c)(0) \geq (\nu_A^c)(x)$
For,

$$\begin{aligned}(\nu_A)(0) &\leq (\nu_A)(x)\\ \Leftrightarrow 1-(\nu_A)(0) &\leq 1-(\nu_A)(x).\\ \Leftrightarrow (\nu_A^c)(0) &\geq (\nu_A^c)(x).\end{aligned}$$

Claim (ii) $\nu_A^c(x+y) \geq T\{\nu_A^c(x), \nu_A^c(y)\}$
For,

$$\begin{aligned}(\nu_A)(x+y) &\leq S\{(\nu_A)(x), (\nu_A)(y)\}\\ \Leftrightarrow 1-(\nu_A)(x+y) &\leq 1-S\{(\nu_A)(x), (\nu_A)(y)\}\\ \Leftrightarrow \nu_A^c(x+y) &\geq T\{(1-\nu_A(x)), (1-\nu_A(y))\}\\ \Leftrightarrow \nu_A^c(x+y) &\geq T\{\nu_A^c(x), \nu_A^c(y)\}.\end{aligned}$$

Claim (iii) $\nu_A^c(x) \geq T\{\nu_A^c(x-y), \nu_A^c(y)\}$
For,

$$\begin{aligned}(\nu_A)(x) &\leq S\{(\nu_A)(x-y), (\nu_A)(y)\}\\ \Leftrightarrow 1-(\nu_A)(x) &\leq 1-S\{(\nu_A)(x-y), (\nu_A)(y)\}\\ \Leftrightarrow \nu_A^c(x) &\geq T\{(1-\nu_A(x-y)), (1-\nu_A(y))\}\\ \Leftrightarrow \nu_A^c(x) &\geq T\{\nu_A^c(x-y), \nu_A^c(y)\}.\end{aligned}$$

Theorem 32 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an Bi-normed intuitionistic fuzzy $\beta-$ideal of X. If χ_A={ $x \in X$ / $\mu_A(x) = \mu_A(0)$ & $\nu_A(x) = \nu_A(0)$} then χ_A is a $\beta-$ideal of X.*
Proof: *Since $\mu_A(y) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0) \Rightarrow 0 \in \chi_A$. Let $x-y, y \in \chi_A$. Hence,*

$$\begin{aligned}\mu_A(x) &\geq T\{\mu_A(x-y), \mu_A(y)\}\\ &= T\{\mu_A(0), \mu_A(y)\}\end{aligned}$$

$$= T\{\mu_A(0), \mu_A(0)\}$$
$$= \mu_A(0).$$

But $\mu_A(0) \geq \mu_A(x) \Rightarrow \mu_A(x) = \mu_A(0)$.

Similarly it can be observed that $\mu_A(x) = \mu_A(0)$, *i.e* $x - y, y \in \chi_A \Rightarrow x \in \chi_A$.

$\therefore \chi_A$ *is an* $\beta-$*ideal of* X.

Theorem 33 *Let* $f : X \to X$ *be an endomorphism on* X *and* $A = \{x, \mu_A(x), \nu_A(x) : x \in X\}$ *be a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X. *Then* $A_f = \{f(x), \mu_f(x), \nu_f(x) : x \in X\}$ *where* $\mu_f : X \to [0, 1]$ *and* $\nu_f : X \to [0, 1]$ *are defined by* $\mu_f(x) = \mu(f(x))$ *and* $\nu_f(x) = \nu(f(x))$ $\forall x \in X$, *is a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X.

Proof: *Let* A *be a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X. *For* $x \in X$,

$$\mu_f(0) = \mu(f(0)) = \mu(0) \leq \mu(x),$$
$$\nu_f(0) = \nu(f(0)) = \nu(0) \geq \nu(x) \quad \forall x \in X.$$

Then

$$\begin{aligned}\mu_f(x + y) &= \mu(f(x + y)) \\ &\geq \mu(f(x) + f(y)) \\ &= T\{\mu(f(x)), \mu(f(y))\} \\ &= T\{\mu_f(x), \mu_f(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_f(x + y) &= \nu(f(x + y)) \\ &\leq \nu(f(x) + f(y)) \\ &= S\{\nu(f(x)), \nu(f(y))\} \\ &= S\{\nu_f(x), \nu_f(y)\}\end{aligned}$$

Also,

$$\begin{aligned}\mu_f(x) &= \mu(f(x)) \\ &\geq T\{\mu(f(x) - f(y)), \mu(f(y))\} \\ &= T\{\mu(f(x - y)), \mu(f(y))\} \\ &= T\{\mu_f(x - y), \mu_f(y)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\nu_f(x) &= \nu(f(x)) \\ &\leq S\{\nu(f(x) - f(y)), \nu(f(y))\} \\ &= S\{\nu(f(x - y)), \nu(f(y))\} \\ &= S\{\nu_f(x - y), \nu_f(y)\}.\end{aligned}$$

Hence A_f *is a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X.

Theorem 34 *Let* $f : X \to Y$ *be an onto homomorphism of* $\beta-$*algebras* X *and* Y. *If* A *is a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* Y, *then the preimage of* $f^{-1}(A)$ *is a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X.

Proof: *Let* A *be a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* Y. *For* $x \in X$,

$$f^{-1}(\mu_A(0)) = \mu_A(f(0)) = \mu_A(0) \geq \mu_A(x) \text{ and } f^{-1}(\nu_A(0)) = \nu_A(f(0)) = \nu_A(0) \leq \nu_A(x).$$

For $x, y \in X$,

$$\begin{aligned} f^{-1}(\mu_A)(x+y) &= \mu_A(f(x+y)) \\ &= \mu_A(f(x)+f(y)) \\ &\geq T\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= T\{f^{-1}(\mu_A(x)), f^{-1}(\mu_A(y))\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\mu_A)(x) &= \mu_A(f(x)) \\ &\geq T\{\mu_A(f(x)-f(y)), \mu_A(f(y))\} \\ &= T\{\mu_A(f(x-y)), \mu_A(f(y))\} \\ &= T\{f^{-1}((\mu_A)(x-y)), f^{-1}(\mu_A(y))\}. \end{aligned}$$

Similarly,

$$\begin{aligned} f^{-1}(\nu_A)(x+y) &= \nu_A(f(x+y)) \\ &= \nu_A(f(x)+f(y)) \\ &\leq S\{\nu_A(f(x)), \nu_A(f(y))\} \\ &= S\{f^{-1}(\nu_A(x)), f^{-1}(\nu_A(y))\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\nu_A)(x) &= \nu_A(f(x)) \\ &\leq S\{\nu_A(f(x)-f(y)), \nu_A(f(y))\} \\ &= S\{\nu_A(f(x-y)), \nu_A(f(y))\} \\ &= S\{f^{-1}((\nu_A)(x-y)), f^{-1}(\nu_A(y))\}. \end{aligned}$$

Hence $f^{-1}(A)$ *is a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X.

4 Product of Bi-Normed Intuitionistic Fuzzy $\beta-$Ideals

This section presents the notion of the cartesion product of two Bi-normed intuitionistic fuzzy $\beta-$ideals of $X$and Y.

Definition 35 *Consider the intuitionistic fuzzy subset* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *of* X *and* $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle : y \in Y\}$ *of* Y. *The Bi-normed Cartesian product of* A *and* B *is defined as*

$$A \times B = \{\mu_{A\times B}(x, y) \ \&\ \nu_{A\times B}(x, y) : x, y \in X \times Y\},$$

where $\mu_{A\times B}(x, y) = T\{\mu_A(x), \mu_B(y)\}$ *and* $\nu_{A\times B}(x, y) = S\{\nu_A(x), \nu_B(y)\}$.

Theorem 36 *If* A *and* B *be two Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* X *and* Y *respectively, then* $A \times B$ *is also a Bi-normed intuitionistic fuzzy* $\beta-$*ideal of* $X \times Y$.
Proof: *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *and* $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle : y \in Y\}$ *be the Bi-normed intuitionistic fuzzy ideals in* X *and* Y *respectively. Take* $(x, y) \in X \times Y$,

$$\begin{aligned} \mu_{A\times B}(0,0) &\geq T\{\mu_{A\times B}(0), \mu_{A\times B}(0)\} \\ &= T\{\mu_{A\times B}(x), \mu_{A\times B}(y)\} \\ &= \mu_{A\times B}(x, y), \\ \nu_{A\times B}(0,0) &\leq S\{\nu_{A\times B}(0), \nu_{A\times B}(0)\} \\ &= S\{\nu_{A\times B}(x), \nu_{A\times B}(y)\} \\ &= \nu_{A\times B}(x, y). \end{aligned}$$

Now take $a, b \in X \times Y$, where $a = (x_1, y_1)$ & $b = (x_2, y_2)$. Then, $\mu_{A\times B}(a + b) \geq T\{\mu_{A\times B}(a), \mu_{A\times B}(b)\}$ and also $\nu_{A\times B}(a + b) \leq S\{\nu_{A\times B}(a), \nu_{A\times B}(b)\}$. Now

$$\begin{aligned}
\mu_{A\times B}(a) &= \mu_{A\times B}(x_1, y_1) \\
&= T\{\mu_{A\times B}(x_1), \mu_{A\times B}(y_1)\} \\
&\geq T\{T\{\mu_A(x_1 - x_2), \mu_A(x_2)\}, T\{\mu_B(y_1 - y_2), \mu_B(y_2)\}\} \\
&\geq T\{T\{\mu_A(x_1 - x_2), \mu_B(y_1 - y_2)\}, T\{, \mu_A(x_2), \mu_B(y_2)\}\} \\
&= T\{\mu_{A\times B}((x_1, y_1) - (x_2, y_2)), \mu_{A\times B}(x_2, y_2)\} \\
&= T\{\mu_{A\times B}(a - b), \mu_{A\times B}(b)\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu_{A\times B}(a) &= \nu_{A\times B}(x_1, y_1) \\
&= S\{\nu_{A\times B}(x_1), \nu_{A\times B}(y_1)\} \\
&\leq S\{S\{\nu_A(x_1 - x_2), \nu_A(x_2)\}, S\{\nu_B(y_1 - y_2), \nu_B(y_2)\}\} \\
&\leq S\{S\{\nu_A(x_1 - x_2), \nu_B(y_1 - y_2)\}, S\{, \nu_A(x_2), \nu_B(y_2)\}\} \\
&= S\{\nu_{A\times B}((x_1, y_1) - (x_2, y_2)), \nu_{A\times B}(x_2, y_2)\} \\
&= S\{\nu_{A\times B}(a - b), \nu_{A\times B}(b)\}.
\end{aligned}$$

$A \times B$ is a Bi-normed intuitionistic fuzzy $\beta-$ideal of $X \times Y$.

Corollary 37 *If $A_1, A_2, ..., A_n$ be the Bi-normed intuitionistic fuzzy $\beta-$ideals of $X_1, X_2, ..., X_n$ respectively, then $\prod_{i=1}^{n} A_i$ is also a Bi-normed intuitionistic fuzzy $\beta-$ideal of $\prod_{i=1}^{n} X_i$.*

Lemma 38 *Let A and B be the two intuitionistic fuzzy sets of X and Y respectively. If $A \times B$ is a Bi-normed intuitionistic fuzzy $\beta-$ ideal of $X \times Y$ then $\mu_A(0) \geq \mu_A(x), \nu_A(0) \leq \nu_A(x)$ and $\mu_B(0) \geq \mu_B(y), \nu_B(0) \leq \nu_B(y)$ $\forall$ $x \in X$ and $y \in Y$.*
Proof: *Let A and B be the two intuitionistic fuzzy sets of X and Y respectively and $A \times B$ be a Bi-normed intuitionistic fuzzy $\beta-$ ideal of $X \times Y$.*

Suppose $\mu_A(0) \leq \mu_A(x), \nu_A(0) \geq \nu_A(y)$ and $\mu_B(0) \leq \mu_B(y), \nu_B(0) \geq \nu_B(x)$ for some $x \in X$ and $y \in Y$. Then

$$\begin{aligned}
\mu_{A\times B}(x, y) &\geq T\{\mu_A(x), \mu_B(y)\} \\
&= T\{\mu_B(0), \mu_A(0)\} \\
&= \mu_{A\times B}(0, 0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu_{A\times B}(x, y) &\leq S\{\nu_A(x), \nu_B(y)\} \\
&= S\{\nu_B(0), \nu_A(0)\} \\
&= \nu_{A\times B}(0, 0).
\end{aligned}$$

which is a contradiction, proving the result.

Theorem 39 *If A and B are two intuitionistic fuzzy sets of X and Y such that $A \times B$ is also a Bi-normed intuitionistic fuzzy $\beta-$ ideal of $X \times Y$, then either A is a Bi-normed intuitionistic fuzzy $\beta-$ideal of X or B is a Bi-normed intuitionistic fuzzy $\beta-$ideal of Y.*
Proof: *Let A and B be the two intuitionistic fuzzy sets of X and Y such that $A \times B$ is also a Bi-normed intuitionistic fuzzy $\beta-$ ideal of $X \times Y$. Hence*

$$\mu_B(0) \geq \mu_B(y), \nu_B(0) \leq \nu_B(y) \quad \forall y \in Y \tag{3}$$

by the Lemma 4.4. Then $\mu_{A\times B}(0, y) = T\{\mu_A(0), \mu_B(y)\}$ and $\nu_{A\times B}(0, y) = S\{\nu_A(0), \nu_B(y)\}$.

Since $A \times B$ is a Bi-normed intuitionistic fuzzy $\beta-$ideals of $X \times Y$, for (x_1, y_1) & $(x_2, y_2) \in X \times Y$, Then,

$$\mu_{A\times B}((x_1,y_1)+(x_2,y_2)) \geq T\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\}$$
and
$$\nu_{A\times B}(((x_1,y_1)+(x_2,y_2))) \leq S\{\nu_{A\times B}(x_1,y_1), \nu_{A\times B}(x_2,y_2)\},$$

$$\begin{array}{ll} \Rightarrow & \mu_{A\times B}((x_1+x_2),(y_1+y_2)) \geq T\{\mu_{A\times B}((x_1,y_1), \mu_{A\times B}(x_2,y_2)\} \\ \textit{and} & \\ & \nu_{A\times B}((x_1+x_2),(y_1+y_2)) \leq S\{\nu_{A\times B}((x_1,y_1), \nu_{A\times B}(x_2,y_2)\}. \end{array} \tag{4}$$

By putting $x_1 = x_2 = 0$ in (4),

$$\mu_{A\times B}(0,(y_1+y_2)) \geq T\{\mu_{A\times B}(0,y_1), \mu_{A\times B}(0,y_2)\}$$
and
$$\nu_{A\times B}(0,(y_1+y_2)) \leq S\{\nu_{A\times B}(0,y_1), \nu_{A\times B}(0,y_2)\}.$$

Hence

$$\mu_B(y_1+y_2) \geq T\{\mu_B(y_1), \mu_B(y_2)\} \text{ and } \nu_B(y_1+y_2) \leq S\{\nu_B(y_1), \nu_B(y_2)\} \quad \forall\, y_1, y_2 \in Y. \tag{5}$$

Also

$$\mu_{A\times B}(x_1,y_1) \ \geq T\{\mu_{A\times B}((x_1,y_1)-(x_2,y_2)), \mu_{A\times B}(x_2,y_2)\},$$

and

$$\nu_{A\times B}(x_1,y_1) \ \leq S\{\nu_{A\times B}((x_1,y_1)-(x_2,y_2)), \nu_{A\times B}(x_2,y_2)\} \ \ \forall (x_1,y_1),(x_2,y_2) \in X \times Y.$$

Hence

$$\mu_{A\times B}(x_1,y_1) \ \geq T\{\mu_{A\times B}((x_1-x_2),(y_1-y_2)), \mu_{A\times B}(x_2,y_2)\}$$

and

$$\nu_{A\times B}(x_1,y_1) \ \leq S\{\nu_{A\times B}((x_1-x_2),(y_1-y_2)), \nu_{A\times B}(x_2,y_2)\} \ \ \forall (x_1,y_1),(x_2,y_2) \in X \times Y.$$

Put $x_1 = x_2 = 0$ in the above equations,

$$\mu_B(y_1) \geq T\{\{\mu_B(y_1-y_2), \mu_B(y_2)\} and \nu_B(y_1) \leq S\{\{\nu_B(y_1-y_2), \nu_B(y_2)\} \ \ \forall\, y_1, y_2 \in Y. \tag{6}$$

Hence, from (3), (5) and (6), B is a Bi-normed intuitionistic fuzzy $\beta-$ideal of Y.

5 Level Subset of Bi-Normed Intuitionistic Fuzzy $\beta-$Ideals

This section discusses the level subset of Bi-normed intuitionistic fuzzy $\beta-$ideals.

Definition 40 *Let A be an intuitionistic fuzzy set of X. For $s, t \in [0, 1]$, the set $A_{s,t} = \{x, \mu_A(x) \geq s, \nu_A(x) \leq t : x \in X\}$ is called a intuitionistic level subset of X.*

Theorem 41 *If $A = \{\mu_A(x), \nu_A(x) : x \in X$ be a Bi-normed intuitionistic fuzzy $\beta-$ideal of X, then the set $A_{s,t} = \{x \in X : \mu_A(x) \geq s, \nu_A(x) \leq t\}$ is an $\beta-$ideal of X, $\forall s, t \in [0, 1]$.*

Proof: *If A is a Bi-normed intuitionistic fuzzy $\beta-$ideal of X. Now $\mu_A(0) \geq \mu_A(x)\ \forall x \in X \Rightarrow \mu_A(0) \geq s$ for any $s \in [0, 1] \Rightarrow 0 \in \mu_s$.*

For $x, y \in \mu_s$, $\Rightarrow$ $\mu_A(x) \geq s$, and $\mu_A(y) \geq s$. Now $\mu_A(x + y) \geq T\{\mu_A(x), \mu_A(y)\} \geq s$, hence $x + y \in \mu_s$.

Let $x, y \in X$ be such that $x - y, y \in \mu_{A_s} \Rightarrow \mu_A(x - y) \geq s$& $\mu_A(y) \geq s$. Now $\mu_A(x) \geq T\{\mu_A(x - y), \mu_A(y)\} \geq T\{s, s\} = s$. Hence $x \in \mu_{A_s}$. Therefore μ_s is a $\beta-$ideal of X.

Similarly, it can be proved that ν_t is also a $\beta-$ideal of X. Hence $A_{s,t}$ is a $\beta-$ideal of X.

This $\beta-$ideal is called intuitionistic level $\beta-$ideal of X.

The converse of the above theorem is also true as seen from the following.

Theorem 42 *If $A = \{\mu_A(x), \nu_A(x) : x \in X$ be an intuitionistic fuzzy set in X such that $A_{s,t}$ is $\beta-$ideal of X for every $s, t \in [0, 1]$, then A is Bi- normed intuitionistic fuzzy $\beta-$ideal of X.*

Combining the above two results, it can be observed that

Theorem 43 *Any $\beta-$ideal of X can be realized as a intuitionistic level $\beta-$ideal for some Bi-normed intuitionistic fuzzy $\beta-$ideal of X.*

Theorem 44 *Let A be a Bi-normed intuitionistic fuzzy $\beta-$ideal of X, $s \in [0,1]$ then,*
(i) If $s = 1$ then upper-level set $U(\mu_A, s)$ is either empty or $\beta-$ideal of X.
(ii) If $t = 0$ then lower-level set $L(\nu_A, t)$ is either empty or $\beta-$ideal of X.
(iii) If $T = min$ then upper-level set $U(\mu_A, s)$ is either empty or $\beta-$ideal of X.
(iv) If $S = max$ then lower-level set $L(\nu_A, t)$ is either empty or $\beta-$ideal of X.

Definition 45 *For the intuitionistic fuzzy subsets*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

of X and

$$B = \{\langle y, \mu_B(y), \nu_A(y)\rangle : y \in Y\}$$

of Y with $A \times B$ as the Bi-normed Cartesian product of A and B, the level subset of Bi-normed Cartesian product $A \times B$ of $X \times Y$ is defined as

$$(A \times B)_{s,t} = \{x, \mu_{A\times B}(x, y) \geq s, \nu_{A\times B}(x, y) \leq t : x \in X \times Y\}$$

where $\mu_{A\times B}(x, y) = T\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A\times B}(x, y) = S\{\nu_A(x), \nu_B(y)\}$.

Theorem 46 *If $A_{s,t}$ and $B_{s,t}$ are the two intuitionstic level $\beta-$ideal of the Bi-normed intuitionistic fuzzy $\beta-$ideals A and B of X and Y respectively, then $(A \times B)_{s,t}$ is an intuitionistic level $\beta-$ideal of $A \times B$ of $X \times Y$.*

Proof: *Let $A_{s,t}$ and $B_{s,t}$ be the two intuitionistic level $\beta-$ideals of the Bi-normed intuitionistic fuzzy $\beta-$ideals A and B of X and Y respectively.*

Take $x, y \in X \times Y$. Now $\mu_{A\times B}(0, 0) \geq T\{\mu_A(0), \mu_B(0)\} \geq T\{s, s\} \geq s$ for any $s \in [0, 1]$. Therefore $(0, 0) \in (A \times B)_{s,t}$.

Take $x = (x_1, x_2), y = (y_1, y_2) \in X \times Y$, then

$$\begin{aligned}
\mu_{A\times B}(x + y) &= \mu_{A\times B}((x_1, x_2) + y = (y_1, y_2) \\
&= \mu_{A\times B}((x_1 + y_1), (x_2 + y_2)) \\
&= T\{\mu_A(x_1 + y_1), \mu_B(x_2 + y_2)\} \\
&\geq T\{T\{\mu_A(x_1), \mu_A(y_1)\}, T\{\mu_B(x_2), \mu_B(y_2)\}\} \\
&= T\{T\{\mu_A(x_1), \mu_B(x_2)\}, T\{\mu_A(y_1), \mu_B(y_2)\}\} \\
&= T\{\mu_{A\times B}(x_1, x_2), \mu_{A\times B}(y_1, y_2)\} \\
&= T\{\mu_{A\times B}(x), \mu_{A\times B}(y)\} \\
&= T\{s, s\} \\
&\geq s.
\end{aligned}$$

Also for $x - y, y \in (A \times B)_{s,t}, \mu_{A\times B}(x) \geq T\{\mu_{A\times B}(x - y), \mu_{A\times B}(y)\} \geq T\{s, s\} = s \Rightarrow x \in (A \times B)_{s,t}$.

Similarly, the same arguments can be observed for $\nu_{A\times B}$.

Hence $(A \times B)_{s,t}$ is an intuitionistic level $\beta-$ideal of $A \times B$ of $X \times Y$.

6 Conclusion

This paper introduces bi-normed intuitionistic fuzzy $\beta-$ideal of a $\beta-$algebra. In deapth, the study analysed the bi-normed intuitionistic fuzzy $\beta-$ideal using modal operators, homomorphic image, cartesion product and level subset of an intuitionistic fuzzy set. One can extend these ideas to the other substructures like $H-$ideals and filters of a $\beta-$algebra.

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