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Uncertain Programming Models for Assignment Problem with Restriction of Qualification

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Abstract

In this paper, we consider two types of balance and unbalance assignment problems with restriction of qualification in uncertain environment. Within the framework of uncertainty theory and uncertain programming, we construct three models for uncertain assignment problem with restriction of qualification, i.e., expected minimum balance assignment model, α-minimum balance assignment model, and α-minimum unbalance assignment model. We then prove that the three models can be converted into assignment problem with restriction in deterministic forms, which can be solved via a judging matrix. Finally, some numerical examples are given to illustrate the proposed models.

Keywords: assignment problem, uncertain programming, judging matrix, balance, unbalance, expected minimum model, α-minimum model

1 Introduction

An assignment problem is one of the decision making problems that has been actively studied by several researchers. The assignment problem is a problem where assignees are being assigned to perform tasks. The assignees might employees who need to be given work assignments. A single-objective assignment problem is an assignment problem that must optimize one objective function, such as optimize of cost, time, or quality, etc. Various types of single-objective assignment problems in crisp environment are explained in Hiller and Libermann [11] and Taha [31]. In this research, we consider two types of assignment problems, i.e., balance and unbalance assignment problems with restriction of qualification. Balance assignment problem means that the number of people is equal to the number of job, while unbalance assignment problem means that the number of people is less (greater) than the number of job. Especially, we concern on balance and unbalance assignment problems with restriction of qualification, i.e., a worker has qualification for a job only if the cost of assigning the worker to the job is no more than a limit cost given, as proposed in Huang and Zhang [13] also Kumar and Gupta [15].

In real life problem, some parameters such as cost, time or quality, cannot be known exactly. Since the lack of history data, incomplete information, there are indeterminate phenomena in the data of cost, time, or quality, etc. Hence, we need a tool to handle the indeterminacy phenomena in some parameters of assignment problem. On the one hand, some researchers handled indeterminate phenomena in assignment problem by using probability theory. Walkup [32] is the pioneer who investigated probabilistic assignment problem in 1979. Grundel et al. [10] investigated an asymptotic properties of random multidimensional assignment problems. By taking the restriction and complementary relation between expected value and variance into consideration, Li et al. [18] proposed a random assignment model based on the synthesis effect. Nikolova and Stier-Moses [26] considered the traffic assignment problem on networks with stochastic travel times. Recently, Pour et al. [27] investigated a generalized personnel assignment problem under stochastic demand.

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On the other hand, some researchers handled the vague phenomena in assignment problem by using fuzzy set theory which is introduced by Zadeh [38] in 1965. Chen [2] initiated an assignment problem under fuzzy environment. After that, many researchers have done some types of assignment problems under fuzzy environment. Sakawa et al. [29] introduced interactive fuzzy linear programming for assignment problem. Huang and Zhang [13] gave a solution method for fuzzy assignment problem with restriction of qualification by using a judging matrix. Mukherjee and Basu [25] gave a model for fuzzy assignment problem with restriction on person cost depending on efficiency and restriction on job cost where both the costs are regarded as intuitionistic fuzzy numbers and they used a heuristic method for showing the existence of the solution of their model. Singh [30] gave a new method for solving dual hesitant fuzzy assignment problems with restrictions based on similarity measure. Recently, Kar et al. [14] proposed a multi objective multi-index generalized assignment problem using fuzzy programming technique. Further, Mehlawat et al. [24] proposed a new possibilistic optimization model for multiple criteria assignment problem.

In this research, we handle the uncertain phenomena in assignment problem by using uncertainty theory proposed by Liu [19]. Especially, we use the theory of uncertain programming which initiated by Liu [20]. Many researchers have solved decision making problems in uncertain environment [3, 7]. For example, Chen et al. [6] initiated diversified model for portfolio selection in uncertain environment. Gao [8] proposed uncertain models for single facility location problems on networks. Yang et al. [36] established a multi-period uncertain workforce planning model, where the labor demands and operation costs were assumed to be independent uncertain variables. Zhang and Peng [39] investigated uncertain programming model for optimal uncertain assignment problem. Rosyida et al. [28] have initiated uncertain coloring problem in an uncertain graph. Besides this, uncertainty theory has been also widely applied to various fields, such as transportation system [5, 9], production planning problem [16, 17], supply chain network optimization [23, 33, 34, 35], and contract theory with application [4, 22].

This paper is motivated from the results of Huang [12], Huang and Zhang [13], also Mukherjee and Basu [25]. We investigate uncertain balance and unbalance assignment problems with a restriction of qualification in this research. The differences of the proposed work and the previous results can be summarized as follows: (1) Huang [12] presented a method for deterministic unbalance assignement problem with a restriction of qualification; (2) Huang and Zhang [13] proposed a fuzzy assignment problem with restriction of qualification; and Mukherjee and Basu [25] provided a fuzzy assignment problem with a restriction where the parameters are intuitionistic fuzzy numbers. Nonetheless, we should use another tool to deal with indeterminate phenomena in an assignment problem if it cannot be handled by probability theory or fuzzy theory. Liu [19] discussed when there is no sample or the size of a sample is too small, the probability theory is not suitable to handle indeterminate phenomena. Therefore, we need an evaluation from domain experts to give a belief degree that an event will occur. On the other hand, a possibility measure in fuzzy set theory is also not appropriate to undertake indeterminate phenomena. Hence, we investigate uncertain balance and unbalance assignment problems with restriction of qualification and solve the problems via a judging matrix. It is different to the work from Zhang and Peng [39] that proposed uncertain programming model for optimal uncertain assignment problem without a restriction and they solved via Kuhn-Munkres algorithm. Different to the work in [39], we provide uncertain balance and unbalance assignment problems with a restriction and we solve via a judging matrix.

In this paper, we construct two models for balance assignment problem with a restriction of qualification in uncertain network, i.e., an expected minimum uncertain balance assignment model and an α-minimum uncertain balance assignment model. Further, we construct an α-minimum uncertain unbalance assignment model with a restriction of qualification. By using the advantages of uncertainty theory, the three models can be converted into deterministic forms that can be solved via a judging matrix. Some numerical examples are given to illustrate the three models.

This paper is structured as follows: some basic concepts of uncertainty theory, uncertain programming, and graph theory are reviewed in Section 2. In Section 3, we present an assignment problem with a restriction in deterministic environment. The two models for balance assignment problem with a restriction in uncertain network and the deterministic transformations are given in Section 4. In Section 5, the model for unbalance assignment problem with a restriction in uncertain network and the deterministic transformation are given. Finally, some numerical examples to illustrate the proposed models are given in Section 6. In Section 7, the conclusions are presented.

2 Preliminaries

2.1 Uncertainty Theory

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is said to be an event. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ is called an uncertain measure if it satisfies the following three axioms [19]:
Axiom I (**Normality**) $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ.
Axiom II (**Duality**) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ for any event Λ.
Axiom III (**Subadditivity**) For every countable sequence of events $\Lambda_1,\Lambda_2,\ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

In uncertainty theory, the triplet $(\Gamma,\mathcal{L},\mathcal{M})$ is called an uncertainty space. In order to provide the operational law, Liu [19] gave the following product axiom.

Axiom IV (**Product**) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\ldots$. The product of uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\ldots$, respectively.

Further, an uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers. To describe an uncertain variable ξ in practice, Liu [19] suggested the concept of uncertainty distribution as follows

$$\Phi(x)=\mathcal{M}\left\{\xi\le x\right\},\ \ \forall x\in\Re.$$

Furthermore, to tell the size of an uncertain variable, the expected value of an uncertain variable ξ is defined as the following form

$$E[\xi]=\int_0^{+\infty}\mathcal{M}\{\xi\ge x\}\mathrm{d}x-\int_{-\infty}^{0}\mathcal{M}\{\xi\le x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite. Assume that uncertain variable ξ has a finite expected value e, the variance of ξ is defined by Liu [19] as

$$V[\xi]=E[(\xi-e)^2].$$

The uncertain variables ξ_1, ξ_2,..., ξ_n are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i\in B_i)\right\}=\bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i\in B_i\}$$

for any Borel sets B_1, B_2,..., B_n of real numbers.

When the uncertain variables $\xi_1,\xi_2,\ldots,\xi_n$ are represented by uncertainty distributions, the operational law for strictly monotone function is given as follows.

Theorem 1 ([19]) *Assume that $\xi_1,\xi_2,\ldots,\xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If the function $f(x_1,x_2,\cdots,x_n)$ is strictly increasing with respect to $x_1,x_2,\ldots,x_m$, and strictly decreasing with respect to x_{m+1}, x_{m+2}, ..., x_n, then the uncertain variable $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ has an uncertainty distribution*

$$\Psi(x)=\sup_{f(x_1,x_2,\cdots,x_n)=x}\left(\min_{1\le i\le m}\Phi_i(x_i)\wedge\min_{m+1\le i\le n}(1-\Phi_i(x_i))\right)$$

whose inverse function is

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

2.2 Uncertain Programming

Initialized by Liu [20], uncertain programming is a type of mathematical programming involving uncertain variables. Assume that $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is an uncertain vector, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is an objective function, and $g_j(\boldsymbol{x}, \boldsymbol{\xi})$ are constraint functions, $j = 1, 2, \ldots, p$. A decision with minimum expected objective value subject to some chance constraints can be obtained via the uncertain programming model:

$$\begin{cases} \min\limits_{\boldsymbol{x}} & E[f(\boldsymbol{x}, \boldsymbol{\xi})] \\ s.t. & \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, j = 1, 2, \ldots, p. \end{cases} \tag{1}$$

Definition 2 ([20]) *A vector $\boldsymbol{x}$ is called a feasible solution to the uncertain programming model (1) if*

$$\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, j = 1, 2, \ldots, p.$$

Definition 3 ([20]) *A feasible solution $\boldsymbol{x}^*$ is called an optimal solution to the uncertain programming model (1) if $E[f(\boldsymbol{x}^*, \boldsymbol{\xi})] \le E[f(\boldsymbol{x}, \boldsymbol{\xi})]$ for any feasible solution $\boldsymbol{x}$.*

Based on the uncertainty theory, a useful theorem was given by Liu [20] to transform the uncertain programming into deterministic programming.

Theorem 4 ([20]) *Assume $f(\boldsymbol{x}, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$ and $g_j(\boldsymbol{x}, \xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$ for $j = 1, 2, \ldots, p$. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, then the uncertain programming model*

$$\begin{cases} \min\limits_{\boldsymbol{x}} & E[f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)] \\ s.t. & \mathcal{M}\{g_j(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n) \le 0\} \ge \alpha_j, j = 1, 2, \ldots, p \end{cases}$$

is equivalent to the crisp mathematical programming

$$\begin{cases} \min\limits_{\boldsymbol{x}} & \int_0^1 f(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) d\alpha \\ s.t. & g_j(\boldsymbol{x}, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \Phi_{k+1}^{-1}(1-\alpha_j), \cdots, \Phi_n^{-1}(1-\alpha_j)) \le 0, j = 1, 2, \ldots, p. \end{cases}$$

2.3 Graph Theory

In this subsection, the definition of graph and some basic terminology of graph theory cited from Bondy and Murty [1] are introduced. In this paper, every graph is assumed to be undirected, simple, and finite graph.

Definition 5 ([1]) *A graph G is an order triple $(V(G), E(G), \psi_G)$ consisting of a nonempty vertex set $V(G)$, an edge set $E(G)$, and an incidence function ψ_G that associates with each edge an unordered pair of vertices of G.*

The number of vertices in a graph G is often called the order of G. For a graph of order n, it can be usually described by the following adjacency matrix

$$\mathrm{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

where

$$x_{ij} = \begin{cases} 1, & \text{if there exists an edge between vertices } v_i \text{ and } v_j \\ 0, & \text{otherwise.} \end{cases}$$

A matching in $G = (V, E)$ is a subset M of the edge set $E(G)$ such that each pair of edges in M have different endpoint. A vertex x in G is said to be M-saturated if there is an edge of M which is incident with the vertex x. A matching M is said to be perfect matching if every vertex in G is M-saturated.

A perfect matching can be applied to real world decision making problems, such as assignment problem. In an assignment problem, there are n workers, $x_1, x_2, \ldots, x_n$, which are available for n jobs, $y_1, y_2, \ldots, y_n$. Each worker being qualified for one or more for these jobs. The personnel assignment problem is a problem to assign all workers to jobs for which they are qualified. For this problem, we can construct bipartite graph $G = (X, Y)$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. A vertex x_i is assigned to vertex y_j if and only if the worker x_i is qualified to job y_j and then the personnel assignment problem becomes a problem of determining a perfect matching in G.

3 Assignment Problem with Restriction of Qualification in Deterministic Environment

Assume that there are n workers available for n jobs. Each worker is assigned to one job, and each job must be done by exactly one worker. Each weight w_{ij} represents the cost for the job y_j is assigned to the worker x_i. Hence, the cost are represented as cost matrix W, where w_{ij} stands for the cost of assigning worker x_i to job y_j. The variable x_{ij} indicates whether the job y_j is assigned to the worker x_i or not. Hence, x_{ij} is a 0-1 decision variable expressed as follows:

$$x_{ij} = \begin{cases} 1, & \text{if the job } y_j \text{ is assigned to the worker } x_i \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In real life problem, there is a restriction of qualification, i.e., the cost w_{ij} is no more than c_j that the worker i has the qualification for the job j. The problem is how to determine the assignment of workers to the jobs such that satisfy the restriction each cost w_{ij} is no more than c_j and the assignment give the minimum total cost of completing all of the jobs. The restriction in the constraint is given as follows:

$$w_{ij}x_{ij} \leq c_j, i, j = 1, 2, \ldots, n. \tag{3}$$

Generally speaking, an optimal assignment problem with restriction of qualification in deterministic network is defined as follows:

$$\begin{cases} \min & \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}x_{ij} \\ s.t. & \sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots, n \\ & w_{ij}x_{ij} \leq c_j, \ \ i, j = 1, 2, \ldots, n \\ & x_{ij} \in \{0, 1\}, \ \ i, j = 1, 2, \ldots, n. \end{cases} \tag{4}$$

In the year 2013, Kumar and Gupta [15] gave a method to determine a solution for the model (4) via a judging matrix. Since there is additional constraint which will induce the problem have no solution, they investigated the existence of the solution by the judging matrix defined as follows.

Definition 6 ([15]) *Let $A = (a_{ij})_{n\times n}$, where*

$$a_{ij} = \begin{cases} 1, & \textit{if the weight } w_{ij} \leq c_j \textit{ corresponding to the constraint (3)} \\ 0, & \textit{otherwise.} \end{cases}$$

The matrix A is called the judging matrix.

Definition 7 ([15]) *Let $A = (a_{ij})_{n\times n}$ be a judging matrix. An element 1 at different row and different column in the judging matrix A is called an independent element 1.*

Further, the existence of solution of the model (4) is given in the following theorem.

Theorem 8 ([15]) *The model (4) has a solution if and only if the number of the independent element 1 of judging matrix A is n.*

We summarize the procedure for solving the assignment problem with restriction (4) as follows.

Step 1. Determine the judging matrix $A = (a_{ij})_{n\times n}$.

Step 2. Find the number of independent element 1 in the judging matrix A. According to Theorem 8, if the number of element 1 is $k = n$, then the model (4) has a solution and hence has an optimal solution. Otherwise if $k < n$, then the model (4) does not have a solution.

Step 3. Establish the solution matrix $B = (b_{ij})_{n\times n}$, where

$$b_{ij} = \begin{cases} c_{ij}, & \text{if } a_{ij} = 1 \\ M, & \text{if } a_{ij} = 0, \end{cases}$$

and M is an arbitrary large number [15].

Step 4. Substitute the solution matrix B in classical assignment problem and use the method in classical assignment to solve the model (4).

In real life problems, the weight w_{ij} can represent the cost and profit made by assigning the worker x_i to job y_j. In fact, the parameters cannot be known exactly because the lack of history data, insufficient information, etc. In this situation, the cost data can only be obtained from the domain experts and we should deal it by using uncertainty theory [21]. Based on the above discussions, we investigate assignment problem with restriction in uncertain network in the next section.

4 Balance Assignment Problem with Restriction of Qualification in Uncertain Environment

Let $W = (w_{ij})_{n\times n}$ be a weight matrix where the weight w_{ij} on each pair (x_i, y_j) represents the effectiveness (cost, profit, etc) made by assigning the worker x_i to the job y_j. The weight $W(M)$ represents the total effectiveness of assigning the workers in $X = \{x_1, x_2, \ldots, x_n\}$ to the jobs in $Y = \{y_1, y_2, \ldots, y_n\}$. The optimal assignment problem is an assignment problem which optimize the total effectiveness.

Based on uncertainty theory, we regard the weight w_{ij} as uncertain variables ξ_{ij} for all $i, j = 1, 2, \ldots, n$, and the uncertain weight matrix is represented by $\xi = (\xi_{ij})_{n\times n}$. Following the results in Gao and Kar [9] and Yang et al. [37], we assume that all of ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively.

We consider that only the cost ξ_{ij} is no more than c_j, the worker i has the qualification for the job j. In other words, the assignment (x_i, y_j) is possible only if the belief degree that the weight $\xi_{ij} \le c_j$ is bigger than α, where $\alpha \in (0, 1)$ is a predetermined confidence level. Therefore, there is an additional constraint as follows:

$$\mathcal{M}\{\xi_{ij} x_{ij} \le c_j\} \ge \alpha. \tag{5}$$

4.1 Expected Minimum Model for Uncertain Balance Assignment Problem with Restriction of Qualification

If the decision makers expect to find an assignment which satisfies the restriction in (5) and give the minimum total cost, then they can use expected minimum model. The concept of expected minimum assignment problem with restriction is introduced.

Definition 9 *Consider an assignment problem of n workers in $X = \{x_1, x_2, \ldots, x_n\}$ to n jobs in $Y = \{y_1, y_2, \ldots, y_n\}$. Each pair (x_i, y_j) has a weight w_{ij} that represents the effectiveness of cost or profit, etc. made by assigning the worker x_i to the job y_j. Let $\mathbb{M}$ be a class of all pair $M = \{(x_i, y_j)\}, i, j = 1, 2, \ldots, n$ that satisfy the restriction in (5). The assignment $M^* \in \mathbb{M}$ is called an expected minimum assignment problem with restriction if*

$$E[W(M^*)] \le E[W(M)]$$

holds for all assignment $M \in \mathbb{M}$.

Given a confidence level $\alpha \in (0,1)$, it follows from Definition 9 that the expected minimum assignment problem with restriction can be formulated as follows:

$$\begin{cases} \min & E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\xi_{ij}\right] \\ s.t. & \sum_{i=1}^{n} x_{ij} = 1,\ j=1,2,\ldots,n \\ & \sum_{j=1}^{n} x_{ij} = 1,\ i=1,2,\ldots,n \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha,\ i,j=1,2,\ldots,n \\ & x_{ij} \in \{0,1\},\ i,j=1,2,\ldots,n. \end{cases} \tag{6}$$

The expected minimum assignment model (6) can be converted into an equivalent deterministic model, as presented in Theorem 10.

Theorem 10 *Assume that the uncertain variables ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i,j = 1,2,\ldots,n$, respectively. The uncertain assignment model (6) is equivalent to the deterministic model:*

$$\begin{cases} \min & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\left(\int_0^1 \Phi_{ij}^{-1}(\alpha)d\alpha\right) \\ s.t. & \sum_{i=1}^{n} x_{ij} = 1, j=1,2,\ldots,n \\ & \sum_{j=1}^{n} x_{ij} = 1, i=1,2,\ldots,n \\ & \Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j, i,j=1,2,\ldots,n \\ & x_{ij} \in \{0,1\}, i,j=1,2,\ldots,n. \end{cases} \tag{7}$$

Proof. It follows from the linearity of expected value operator that

$$E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\xi_{ij}\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}E[\xi_{ij}]. \tag{8}$$

Based on the property of expected value operator, the objective (8) is equivalent to the objective in (7). The uncertain variables ξ_{ij} have regular uncertainty distributions Φ_{ij}. Hence, we can use the inverse of uncertainty distribution Φ_{ij}^{-1} to convert the constraint

$$\mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha$$

into deterministic constraint

$$\Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j.$$

Thus, the uncertain model (6) is equivalent to the deterministic model (7). The theorem is proved. □

4.2 α-Minimum Model for Uncertain Balance Assignment Problem with Restriction of Qualification

Given $\alpha \in (0,1)$. If the decision makers want to find an α-minimum assignment problem among all assignment in $\mathbb{M}$, then they can use α-minimum assignment model. The concept of α-minimum assignment model with a restriction on cost is presented below.

Definition 11 *Consider an assignment problem of n workers in $X = \{x_1, x_2, \ldots, x_n\}$ to n jobs in $Y = \{y_1, y_2, \ldots, y_n\}$. Let $\mathbb{M}$ be a class of all pair $M = \{(x_i, y_j)\}, i,j = 1,2,\ldots,n$ that satisfy the restriction in (5). The assignment $M^* \in \mathbb{M}$ is called an α-minimum assignment if*

$$\min\{\mathcal{W}|\mathcal{M}\{W(M^*) \le \mathcal{W}\} \ge \alpha\} \le \min\{\mathcal{W}|\mathcal{M}\{W(M) \le \mathcal{W}\} \ge \alpha\}$$

holds for all assignment $M \in \mathbb{M}$, where α is a predetermined confidence level.

Based on Definition 11, the model of α-minimum assignment problem with restriction given in (5) shown as follows:

$$\begin{cases} \min & \mathcal{W} \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\xi_{ij} \le \mathcal{W}\right\} \ge \alpha \\ & \sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} = 1, \ i = 1, 2, \ldots, n \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha, \ i, j = 1, 2, \ldots, n \\ & x_{ij} \in \{0, 1\}, \ i, j = 1, 2, \ldots, n. \end{cases} \tag{9}$$

Furthermore, the α-minimum assignment model (9) can be converted into an equivalent deterministic model, which is proved in Theorem 12.

Theorem 12 *Assume that the uncertain variables ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. The uncertain assignment model (9) is equivalent to the deterministic model:*

$$\begin{cases} \min & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha) \\ s.t. & \sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} = 1, \ i = 1, 2, \ldots, n \\ & \Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j, \ i, j = 1, 2, \ldots, n \\ & x_{ij} \in \{0, 1\}, \ i, j = 1, 2, \ldots, n. \end{cases} \tag{10}$$

Proof. It is obvious that the weight $W(M)$ is strictly increasing with respect to each component of w. Since the uncertain variables ξ_{ij} have regular uncertainty distributions Φ_{ij}, we can use the inverse of uncertainty distribution Φ_{ij} to convert the constraint

$$\mathcal{M}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\xi_{ij} \le \mathcal{W}\right\} \ge \alpha$$

into deterministic constraint

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha) \le \mathcal{W}.$$

Meanwhile, the constraint

$$\mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha$$

can be converted into deterministic constraint

$$\Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j.$$

Hence, the uncertain model (9) can be converted into the deterministic model:

$$\begin{cases} \min & \mathcal{W} \\ s.t. & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha) \le \mathcal{W} \\ & \sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,\ldots,n \\ & \sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,\ldots,n \\ & \Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j,\ i,j = 1,2,\ldots,n \\ & x_{ij} \in \{0,1\},\ i,j = 1,2,\ldots,n. \end{cases} \tag{11}$$

It is obvious that model (11) is equivalent to the α-minimum assignment model (10). The theorem is proved. □

We next investigate the analysis between the optimal objective of model (10) and the predetermined confidence level α, as stated in Lemma 13.

Lemma 13 *Given two confidence levels α_1 and α_2. Let M_1 and M_2 be the corresponding optimal objectives of model (10). If $\alpha_1 \ge \alpha_2$, then $M_1 \ge M_2$.*

Proof. Given $\alpha \in (0,1]$. Let us consider the α-pessimistic value of an uncertain variable ξ as follows:

$$\xi_{\inf}(\alpha) = \inf\{r|\mathcal{M}\{\xi \le r\} \ge \alpha\}.$$

Let D be the feasible domain of model (10). The α-pessimistic value of the uncertain variable ξ is an increasing function of α [19]. Hence, we obtain

$$M_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha_1) \ge M_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha_2).$$

Thus, the lemma is proved. □

5 Unbalance Assignment Problem with Restriction of Qualification in Uncertain Environment

Assume that there are m people available for n jobs, where $m < n$ such that only the cost w_{ij} is no more than c_j, the worker i has the qualification for the job j. We call it by unbalance assignment problem with restriction of qualification. Since there is indeterminacy phenomena in the cost matrix W, we could deal with the weight w_{ij} as independent uncertain variables ξ_{ij} with regular uncertainty distributions Φ_{ij}, $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, respectively, and the uncertain weight matrix is represented by $\xi = (\xi_{ij})_{m\times n}$. We next propose two α-minimum models for unbalance assignment problem with restriction of qualification in uncertain environment.

Model 1.

If each person is assigned to at most one job, then a first model of α-minimum uncertain unbalance assignment problem with a restriction is presented as follows:

$$\begin{cases} \min & \mathcal{W} \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\xi_{ij} \le \mathcal{W}\right\} \ge \alpha \\ & \sum_{i=1}^{m} x_{ij} = 1,\ j = 1,2,\ldots,n \\ & \sum_{j=1}^{n} x_{ij} \le 1,\ i = 1,2,\ldots,m \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha,\ i,j = 1,2,\ldots,n \\ & x_{ij} \in \{0,1\},\ i,j = 1,2,\ldots,n. \end{cases} \tag{12}$$

The constraint $\sum_{j=1}^{n} x_{ij} \le 1, i = 1, 2, \ldots, m$ means that each person can do at most one job.

It follows from Theorem 12 that the model (12) can be converted into the deterministic form:

$$\begin{cases} \min & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha) \\ s.t. & \sum_{i=1}^{m} x_{ij} = 1,\ j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} \le 1,\ i = 1, 2, \ldots, m \\ & \Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j,\ i, j = 1, 2, \ldots, n \\ & x_{ij} \in \{0, 1\},\ i, j = 1, 2, \ldots, n. \end{cases} \tag{13}$$

Model 2.
If each person could be assigned to at least one job, then we present a second model of α-minimum uncertain unbalance assignment problem with a restriction of qualification as follows:

$$\begin{cases} \min & \mathcal{W} \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\xi_{ij} \le \mathcal{W}\right\} \ge \alpha \\ & \sum_{i=1}^{m} x_{ij} = 1,\ j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} \ge 1,\ i = 1, 2, \ldots, m \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha,\ i, j = 1, 2, \ldots, n \\ & x_{ij} \in \{0, 1\},\ i, j = 1, 2, \ldots, n. \end{cases} \tag{14}$$

The constraint $\sum_{j=1}^{n} x_{ij} \ge 1, i = 1, 2, \ldots, m$ means that each person can do at least one job.

According to Theorem 12, the model (14) can be converted into the deterministic unbalance assignment problem as the following:

$$\begin{cases} \min & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\Phi_{ij}^{-1}(\alpha) \\ s.t. & \sum_{i=1}^{m} x_{ij} = 1,\ j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} \ge 1,\ i = 1, 2, \ldots, m \\ & \Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j,\ i, j = 1, 2, \ldots, n \\ & x_{ij} \in \{0, 1\},\ i, j = 1, 2, \ldots, n. \end{cases} \tag{15}$$

To determine the solution of the deterministic unbalance assignment problem (13) and (15), we use a proposition given in [12].

Proposition 14 *The unequal of workers and tasks assignment problem with restriction (13) has a solution if and only if the number of the independent element 1 of judging matrix $A = (a_{ij})_{m\times n}$ is $k = \min\{m, n\}$.*

The procedure to solve the deterministic unbalance assignment problems (13) and (15) can be summarized as follows:

Step 1. Determine the judging matrix $A = (a_{ij})_{m\times n}$.

Step 2. If the number k of the independent element 1 in the judging matrix A is $k < \min\{m, n\}$, then the problem has no solution and stop. Otherwise, if $k = \min\{m, n\}$, then go to step 3.

Step 3. Obtain a solution matrix $B = (b_{ij})_{l\times l}$, where $l = \max\{m, n\}$, and

$$b_{ij} = \begin{cases} c_{ij}, & \text{if } a_{ij} = 1 \\ M, & \text{if } a_{ij} = 0 \\ \max\limits_{1\le i\le m, 1\le j\le n} c_{ij}, & \text{if } j > n. \end{cases}$$

Step 4. Find the optimal solution of assignment problem with cost matrix B.

Step 5. Obtain the optimal solution of primal problem.

6 Numerical Examples

6.1 Numerical Example for Uncertain Balance Assignment Problem with a Restriction of Qualification

In this section we give an example to illustrate the proposed models. Assume that there are 4 workers available for 4 jobs. Each worker is assigned to one job, also each job must be exactly done by one worker. There is a time limit c_j for each job y_j. If the worker x_i is assigned to job y_j, then the cost is represented by c_{ij}. The problem is how to find a minimum cost assignment which satisfies the restriction: the worker x_i has the qualification for job y_j only if c_{ij} is no more than c_j for all $i, j = 1, 2, 3, 4$. In real life problems, the cost cannot be known exactly because insufficient information. In this situation, the cost data c_{ij} can be regarded as uncertain variable ξ_{ij} for $i, j = 1, 2, 3, 4$. We assume that the cost data c_{ij} are zigzag uncertain variables ξ_{ij} as shown in the matrix below.

$$\xi = \begin{pmatrix} \mathcal{Z}(1.5, 2.5, 4.5) & \mathcal{Z}(2.5, 3.5, 6.5) & \mathcal{Z}(2.5, 3.5, 6.5) & \mathcal{Z}(4.5, 5.5, 7.5) \\ \mathcal{Z}(2.5, 3.5, 6.5) & \mathcal{Z}(4.5, 5.5, 7.5) & \mathcal{Z}(4.5, 5.5, 7.5) & \mathcal{Z}(1.5, 3.5, 5.5) \\ \mathcal{Z}(1.5, 2.5, 4.5) & \mathcal{Z}(1.5, 2.5, 5.5) & \mathcal{Z}(1.5, 2.5, 5.5) & \mathcal{Z}(4.5, 5.5, 7.5) \\ \mathcal{Z}(4.5, 5.5, 7.5) & \mathcal{Z}(1.5, 2.5, 5.5) & \mathcal{Z}(4.5, 5.5, 7.5) & \mathcal{Z}(5.5, 6.5, 8.5) \end{pmatrix}$$

Given the confidence level $\alpha = 0.5$ and the time limit for each job is: $c_1 = 4.5, c_2 = 5, c_3 = 5, c_4 = 6$. Firstly, we formulate the problem by using the expected minimum assignment model (6) as follows.

$$\begin{cases} \min & E\left[\sum_{i=1}^{4}\sum_{j=1}^{4} x_{ij}\xi_{ij}\right] \\ s.t. & \sum_{i=1}^{4} x_{ij} = 1 \\ & \sum_{j=1}^{4} x_{ij} = 1 \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge \alpha \\ & x_{ij} \in \{0, 1\}, i, j = 1, 2, 3, 4. \end{cases} \tag{16}$$

Based on Theorem 10, the model (16) is equivalent to the deterministic model:

$$\begin{cases} \min & \sum_{i=1}^{4}\sum_{j=1}^{4} x_{ij}E[\xi_{ij}] \\ s.t. & \sum_{i=1}^{4} x_{ij} = 1 \\ & \sum_{j=1}^{4} x_{ij} = 1 \\ & \Phi_{ij}^{-1}(\alpha)x_{ij} \le c_j \\ & x_{ij} \in \{0, 1\}, i, j = 1, 2, 3, 4. \end{cases} \tag{17}$$

After that, the solution for the deterministic model (17) can be obtained by using the following steps:

Step 1. By using $\alpha = 0.5$, calculate $\Phi_{ij}^{-1}(0.5)$ for each element ξ_{ij}.

$$\Phi_{ij}^{-1}(0.5) = \begin{pmatrix} 2.5 & 3.5 & 3.5 & 5.5 \\ 3.5 & 5.5 & 5.5 & 3.5 \\ 2.5 & 2.5 & 2.5 & 5.5 \\ 5.5 & 2.5 & 5.5 & 6.5 \end{pmatrix}$$

Step 2. Establish the judging matrix $A = (a_{ij})_{n\times n}$.

$$\text{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Step 3. Determine the number of independent element 1.

$$\begin{pmatrix} 1^* & 1 & 1 & 1 \\ 1 & 0 & 0 & 1^* \\ 1 & 1 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \end{pmatrix}$$

The number of independent element 1 is $k = 4$ and then the problem has a solution.

Step 4. Calculate $E[\xi_{ij}]$ for each element ξ_{ij}.

$$E[\xi_{ij}] = \begin{pmatrix} 2.75 & 4 & 4 & 5.75 \\ 4 & 5.75 & 5.75 & 4 \\ 2.75 & 3 & 3 & 5.75 \\ 5.75 & 3 & 5.75 & 6.75 \end{pmatrix}$$

Step 5. Establish the solution matrix $B = (b_{ij})_{n\times n}$.

$$\text{B} = \begin{pmatrix} 2.75 & 4 & 4 & 5.75 \\ 4 & M & M & 4 \\ 2.75 & 3 & 3 & 5.75 \\ M & 3 & M & M \end{pmatrix}$$

Step 6. Obtain the optimal solution with the input is matrix B. By using LINGO software, we get the optimal solution:

$$X^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It follows from the result that the assignment $M^* = \{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$ gives the minimum expected value of the weight $W(M^*) = 12.75$ and satisfies the restriction $\mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge 0.5$ for $i, j = 1, 2, 3, 4$.

Secondly, given a confidence level $\alpha = 0.5$, we formulate the problem by using the α-minimum assignment model (9) as follows.

$$\begin{cases} \min & \mathcal{W} \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{4}\sum_{j=1}^{4} x_{ij}\xi_{ij} \le \mathcal{W}\right\} \ge 0.5 \\ & \sum_{i=1}^{4} x_{ij} = 1 \\ & \sum_{j=1}^{4} x_{ij} = 1 \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge 0.5 \\ & x_{ij} \in \{0,1\}, i,j = 1,2,3,4. \end{cases} \tag{18}$$

Based on Theorem 12, the model (18) is equivalent to the deterministic model:

$$\begin{cases} \min & \sum_{i=1}^{4}\sum_{j=1}^{4} x_{ij}\Phi_{ij}^{-1}(0.5) \\ s.t. & \sum_{i=1}^{4} x_{ij} = 1 \\ & \sum_{j=1}^{4} x_{ij} = 1 \\ & \Phi_{ij}^{-1}(0.5)x_{ij} \le c_j \\ & x_{ij} \in \{0,1\}, i,j = 1,2,3,4. \end{cases} \tag{19}$$

After converting into the deterministic model (19), the solution can be obtained by using the following steps:

Step 1. Calculate $\Phi_{ij}^{-1}(0.5)$.

$$\Phi_{ij}^{-1}(0.5) = \begin{pmatrix} 2.5 & 3.5 & 3.5 & 5.5 \\ 3.5 & 5.5 & 5.5 & 3.5 \\ 2.5 & 2.5 & 2.5 & 5.5 \\ 5.5 & 2.5 & 5.5 & 6.5 \end{pmatrix}$$

Step 2. Establish the judging matrix $A = (a_{ij})_{n\times n}$.

$$\text{A} = \begin{pmatrix} 1^* & 1 & 1 & 1 \\ 1 & 0 & 0 & 1^* \\ 1 & 1 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \end{pmatrix}$$

Step 3. Determine the number of independent element 1 in matrix A. From the solution of Model 17, we obtain the number of independent element 1 is $k = 4$. Hence, the problem has solution.

Step 4. Establish the solution matrix.

$$\text{B} = \begin{pmatrix} 2.5 & 3.5 & 3.5 & 5.5 \\ 3.5 & M & M & 3.5 \\ 2.5 & 2.5 & 2.5 & 5.5 \\ M & 2.5 & M & M \end{pmatrix}$$

Step 5. Obtain the optimal solution by using the input matrix $B = (b_{ij})_{n\times n}$:

$$X^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We can see that the assignment $M^* = \{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$ is a 0.5-minimum assignment with the minimum weight 11. This is the optimal solution which satisfies the restrictions: $\mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge 0.5$ for $i, j = 1, 2, 3, 4$. The optimal objective 11 for a decision level $\alpha = 0.5$ obtained by the α-minimum assignment model is less then the optimal objective obtained by the expected minimum model. This condition is also satisfied for all decision level $\alpha \in (0, 1)$.

6.2 Sensitivity Analysis of α for Solution of Balance Uncertain Assignment Model

In this part, we present the sensitivity analysis of optimal objectives with respect to the parameter α. The solution of α-minimum balance assignment problem with different values of α, which satisfy the restriction of qualification for different α are presented in Table 1.

Table 1: α-Minimum balance assignment with restriction of qualification

α	α-Minimum assignment	α-Weight
0.9	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	19
0.8	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	17
0.7	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	15
0.6	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	13
0.5	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	11
0.4	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	10
0.3	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	9
0.2	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	8
0.1	$\{(x_1, y_1), (x_2, y_4), (x_3, y_3), (x_4, y_2)\}$	7

We can see that the lower the value of α is, the lower the optimal objective is. After that, we present an α-minimum model for unbalance assignment problem in uncertain environment in the next numerical example.

6.3 Numerical Example for Uncertain Unbalance Assignment Problem with a Restriction of Qualification

We give numerical example to illustrate unbalance assignment problem in uncertain environment. There are 5 people available for 4 jobs and each worker is assigned to at most one job. The time cost needed to assign the person x_i to job y_j is represented as c_{ij}. The person x_i has the qualification for job y_j only if c_{ij} is no more than c_j for all $i = 1, 2, 3, 4, 5$, and $j = 1, 2, 3, 4$. The time limit for each job is $c_1 = 4, c_2 = 5, c_3 = 5, c_4 = 6$.

In fact, the cost c_{ij} cannot be known exactly and we could deal with uncertain variables ξ_{ij} for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4$. Assume that the cost data c_{ij} are zigzag uncertain variables ξ_{ij} as shown in the following matrix:

$$\xi = \begin{pmatrix} \mathcal{L}(3.2, 4.2) & \mathcal{L}(3.3, 7.3) & \mathcal{L}(5.2, 5.2) & \mathcal{L}(4.2, 5.2) \\ \mathcal{L}(4.3, 6.3) & \mathcal{L}(4.2, 6.2) & \mathcal{L}(6.2, 7.2) & \mathcal{L}(5.2, 8.2) \\ \mathcal{L}(2.2, 5.2) & \mathcal{L}(7.2, 8.2) & \mathcal{L}(3.3, 5.3) & \mathcal{L}(3.2, 4.2) \\ \mathcal{Z}(4.5, 5.5, 7.5) & \mathcal{Z}(1.5, 2.5, 5.5) & \mathcal{L}(4.2, 5.2) & \mathcal{L}(6.2, 9.2) \\ \mathcal{Z}(2.5, 3.5, 6.5) & \mathcal{Z}(1.5, 2.5, 4.5) & \mathcal{Z}(5.5, 6.5, 8.5) & \mathcal{Z}(4.5, 5.5, 7.5) \end{pmatrix}$$

The problem is how to find a minimum cost unbalance assignment in uncertain environment. In the first step, we formulate an α-minimum unbalance assignment model (9) by using a confidence level $\alpha = 0.8$:

$$\begin{cases} \min & \mathcal{W} \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{5}\sum_{j=1}^{4} x_{ij}\xi_{ij} \le \mathcal{W}\right\} \ge 0.8 \\ & \sum_{i=1}^{5} x_{ij} = 1, j = 1,2,3,4 \\ & \sum_{j=1}^{4} x_{ij} \le 1, i = 1,2,3,4,5 \\ & \mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge 0.8 \\ & x_{ij} \in \{0,1\}, i = 1,2,3,4,5; j = 1,2,3,4. \end{cases} \tag{20}$$

Based on Theorem 12, the model (18) is equivalent to the deterministic model:

$$\begin{cases} \min & \sum_{i=1}^{5}\sum_{j=1}^{4} x_{ij}\Phi_{ij}^{-1}(0.8) \\ s.t. & \sum_{i=1}^{5} x_{ij} = 1, j = 1,2,3,4 \\ & \sum_{j=1}^{4} x_{ij} \le 1, i = 1,2,3,4,5 \\ & \Phi_{ij}^{-1}(0.8)x_{ij} \le c_j \\ & x_{ij} \in \{0,1\}, i = 1,2,3,4,5; j = 1,2,3,4. \end{cases} \tag{21}$$

After that, the solution for the deterministic model (21) can be obtained by using the following steps:

Step 1. Calculate $\Phi_{ij}^{-1}(0.8)$ for each element ξ_{ij}.

$$\Phi_{ij}^{-1}(0.8) = \begin{pmatrix} 3.7 & 5.3 & 5.3 & 4.7 \\ 5.3 & 5.2 & 6.7 & 6.7 \\ 3.7 & 7.7 & 4.3 & 3.7 \\ 6.7 & 4.3 & 4.7 & 7.7 \\ 5.3 & 3.7 & 7.7 & 6.7 \end{pmatrix}$$

Step 2. Establish the judging matrix $A = (a_{ij})_{n\times n}$.

$$\text{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Step 3. Determine the number of independent element 1.

$$\begin{pmatrix} 1^* & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1^* \\ 0 & 1 & 1^* & 0 \\ 0 & 1^* & 0 & 0 \end{pmatrix}$$

The number of independent element 1 is $k = 4 = \min\{5, 4\}$ and then the problem has solution.

Step 4. Establish the solution matrix $B = (b_{ij})_{5\times 5}$.

$$\text{B} = \begin{pmatrix} 3.7 & M & M & 4.7 & 7.7 \\ M & M & M & M & 7.7 \\ 3.7 & M & 4.3 & 3.7 & 7.7 \\ M & 4.3 & 4.7 & M & 7.7 \\ M & 3.7 & M & M & 7.7 \end{pmatrix}$$

Step 5. Find the optimal solution with the input is matrix B as follows:

$$Y^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Step 6. Obtain the optimal solution of primal problem:

$$X^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It follows from the result that we assign the worker 1, 3, 4, 5 to the job 1, 4, 3, 2, respectively, and the worker 2 is not assigned to any job. This is the 0.8-minimum assignment which give the minimum cost 15.8 and satisfies the restriction of qualification: $\mathcal{M}\{\xi_{ij}x_{ij} \le c_j\} \ge 0.8$ for $i, j = 1, 2, 3, 4, 5$.

6.4 Sensitivity Analysis of α for Solution of Unbalance Uncertain Assignment Model

In this section, we give the sensitivity analysis of optimal objectives with respect to the parameter α for unbalance uncertain assignment model. What's more, the α-minimum assignment with different values of α for unbalance uncertain assignment problem with a restriction are presented in Table 2.

Table 2: α-Minimum unbalance assignment with restriction of qualification

α	α-Minimum assignment	α-Weight
0.8	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	15.8
0.7	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	14.2
0.6	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	12.6
0.5	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	11
0.4	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	10
0.3	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	9
0.2	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	8
0.1	$\{(x_1, y_1), (x_3, y_4), (x_4, y_3), (x_5, y_2)\}$	7

When the values of α are increasing, the optimal objectives are also increasing. Moreover, the optimal objective obtained by the α-minimum uncertain unbalance assignment model is less than or equal to the optimal objective obtained by the α-minimum uncertain balance assignment model for the decision level $0 < \alpha < 1$. However, this condition should be investigated in more general cases.

7 Conclusions

In this paper, we focus on an optimal assignment problem with a restriction of qualification. In real life problem, the cost data cannot be known exactly, because insufficient information, the lack of history data, etc. Uncertainty theory gives a new approach to deal with the indeterminate factors in assignment problem. Therefore, we investigate balance and unbalance assignment problem with restriction of qualification in an uncertain network. Within the framework of uncertain programming, we construct three models for assignment problem with restriction of qualification. In the first one, we provide an expected minimum balance assignment model with restriction. In the second one, we present an α-minimum balance assignment model with restriction for $\alpha \in (0, 1)$. In the third one, we give an α-minimum unbalance assignment model with restriction. By using the properties in the uncertainty theory, the three models can be converted into assignment problem with restriction in deterministic forms, which can be solved via the judging matrix. Furthermore, we get a fact when the values of α are increasing, the optimal objectives obtained by the α-minimum uncertain balance and unbalance models are also increasing.

For further research, we can investigate a property that state a connection between the optimal objectives obtained by the α-minimum uncertain unbalance model and the α-minimum uncertain balance assignment model. Furthermore, we can investigate a generalized assignment problem and a multi-objective generalized assignment problem in uncertain environment. Moreover, we can consider a multi-objective k-cardinality assignment problem in uncertain environment. Finally, uncertainty data in this paper is estimated by the domain expert. Estimating the uncertainty data with the help of of data mining tools will be an interesting future research points.

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