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Decision Making Criteria Hybridization for Finding Optimal Decisions' Subset Regarding Changes of the Decision Function

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Abstract

An open question of dealing with the changing decision function is treated. The changing is metastated, i.e. it is described as this function changes through a set of metastates. By that, the goal is to solve a decision making problem applying a lot of applicable criteria. Thus a hybridization rule for finding an optimal decisions' subset is formulated. It is intended for combining criteria of various natures. For this, preference of straightforward calculation of normalized expected utility is explained. The expected utility is calculated from the function of three variables by eliminating states and metastates. The elimination is realized with extremization, multiplication, integration. The integration over finite sets is substituted with summation. Hybridization of a great number of criteria must expectedly disregard controversies of some criteria and possible non-appropriateness of the expected utility calculation.

Keywords: decision making criteria, metastate, expected utility normalization, criteria hybridization

1 Introduction

A lot of criteria exist and, likely, will be developed, for solving decision making problems [3, 18, 24]. Operating over the decision function, which preferably is reduced to a matrix, a part of them may invoke and apply additional statistical data in a form of probabilistic measures. Probability-based criteria aiming at maximizing expected utility (minimizing expected losses) perform poorly when statistical observations are insufficient (because then probabilistic measures are unreliable) [25, 31, 33, 47]. The performance is poorer when the decision function changes through a set of metastates [5, 29, 37] (see Figure 1).

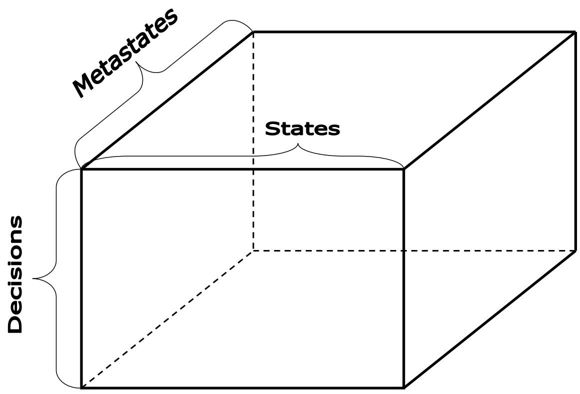

Figure 1: A three-dimensional sketch of the decision function depending on its third variable which is metastate

Metastates are generated due to uncertain evaluation of ordinary situations (couples of a decision and an ordinary state), or influence of the time course [7, 28]. For instance, entries of a common decision flat matrix in analyzing consumer preferences [10, 43, 46] change after market volatilities, inflation, competitions, etc. Similarly, risk

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decision matrices in developing occupational health and safety and environment policy are influenced by the changing standards and technologies [32, 40, 41]. This creates slices of the decision matrix, which are called metastates. On the other hand, continuous metastates can be used for converting ordinary situations which are evaluated as intervals [15, 23] into meta-situations whose evaluations then become real-valued points [21]. Then, nonetheless, efficient decisions must be made as over states and metastates, as well as under a lot of criteria. This factually addresses a ubiquitous problem of unification in decision making theory.

2 Background

A question of regarding multiple states of a decision making problem was considered in [15, 16, 23, 26, 37]. An issue of multiple criteria applicable to solve the problem was also considered in [2, 12, 18, 24, 37, 42]. An approach for reducing the multiple state decision making problem along with regarding multiple criteria by their hybridization was developed in [37]. Nevertheless, an algorithm of reducing a finite series of decision making problems to a single problem suggested in [37] relies on additional statistics. Without statistical data, the algorithm is consistent only if there is a nonempty intersection of the optimal decisions' subsets for the metastates.

Hybridization of multiple criteria for decision making lies in finding an appropriate combination of them [4, 12]. Such combination is a weighted sum of normalized expected utilities calculated for criteria [1, 4, 21, 22, 35, 36, 38, 39]. Greater weights correspond to more important criteria. Expected utilities for deficient or unreliable criteria are summed with lesser weights. As the expected utility is a function of the single variable (decision), then there is an open question of how to come to such a function from the function of three variables (Figure 2). Expectations shall be found over both states and metastates. So what should be eliminated first? Or does an order of the elimination not matter? These questions are to be answered unambiguously.

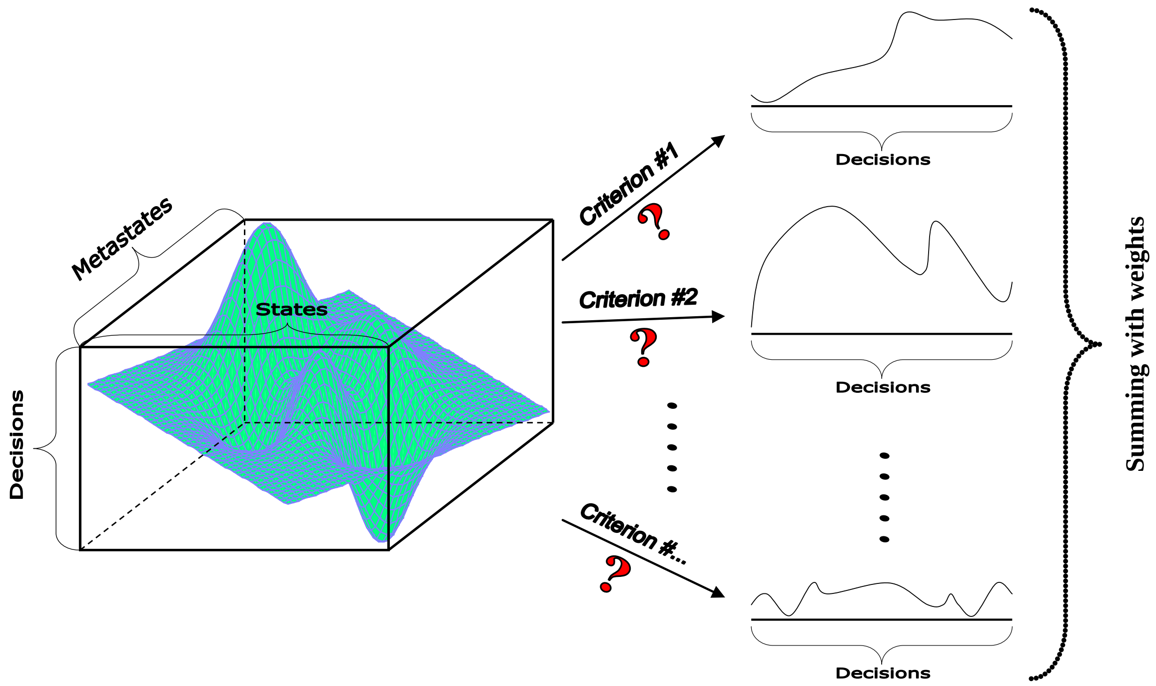

Figure 2: An open question of unambiguously mapping the function of three variables (Figure 1) into a function of the single variable (decision)

Moreover, finding the expected utility can be done in two ways:

1. Straightforwardly, without standardization (normalization) of the decision function [14, 17]. The expected utility is normalized subsequently (Figure 3).

2. With preliminarily standardizing the decision function, whereupon the expected utility, if necessary, is normalized (Figure 4). Despite this method seems one stage longer, it allows to compare aftermaths of criteria before normalization (see, e. g., [20, 34]). Comparisons of criteria can be used to correct their weights [2, 27, 42, 48]. For instance, if a criterion gives a more equable curve of the expected utility (like the third one in Figure 4), then its weight should be decreased [13, 30].

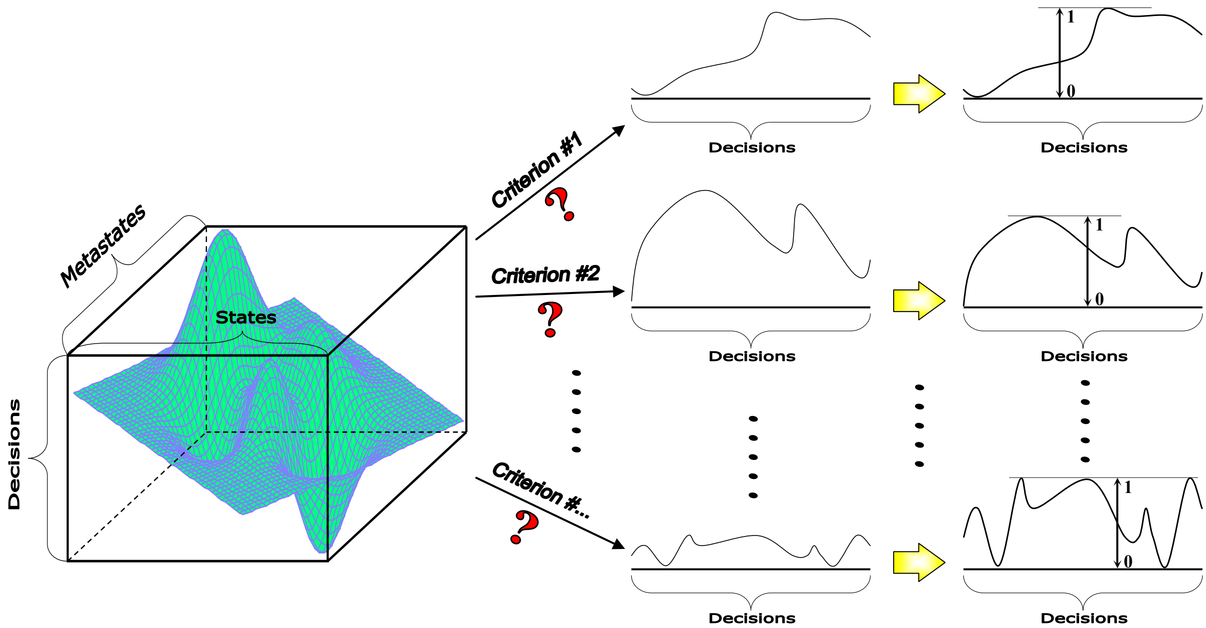

Figure 3: A sketch of straightforward calculation of the normalized expected utility

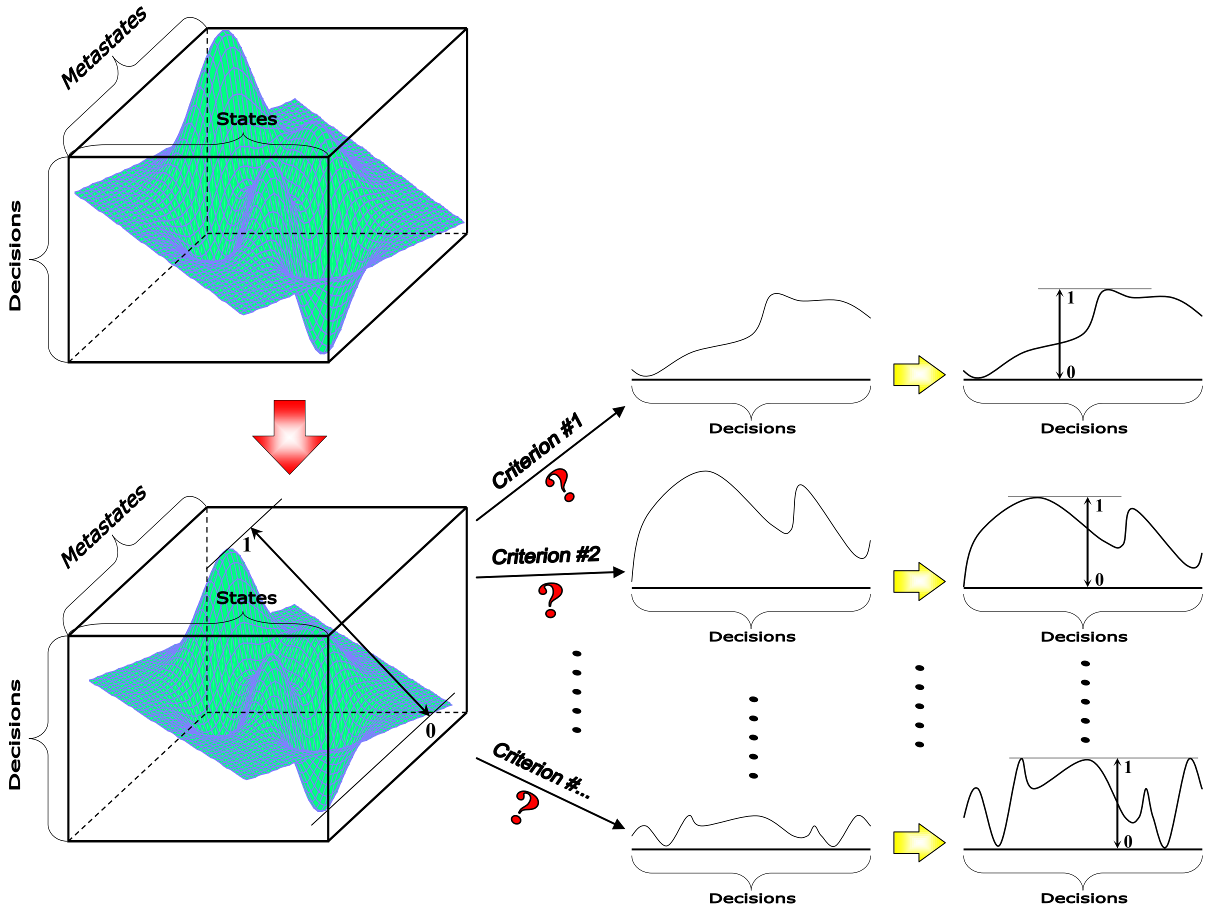

Figure 4: A sketch of calculating the normalized expected utility after the decision function is preliminarily standardized

We cannot compare expected utilities with calculating them straightforwardly. Calculation of the normalized expected utility, however, must be a common routine. Any options are unwanted. After all, absence of unambiguity is the main issue of the modern decision making theory [9, 27, 45]. Uniqueness of an optimal decision is not the only desired objective. Calculation of the normalized expected utility and hybridization of multiple criteria must also be unique [6, 19, 37]. The latter must help in arranging/sorting decisions unambiguously, which is even a wider problem than just searching for the mentioned optimal decision uniqueness.

3 Goal

Due to unclear ways of hybridizing criteria over decisions along with states and metastates, the goal is to formulate a hybridization rule for finding an optimal decisions' subset. This rule should allow regarding that an ordinary decision function may change. The changing is metastated (i.e., described as this function changes through a set of metastates).

For achieving the goal, the following tasks are going to be fulfilled:

1. Preliminary convention of the utility notion and denotations.
2. Substantiation of a common routine showing how the normalized expected utility is calculated.
3. Formulae for combining criteria.
4. Numerical experiments for validating the substantiated routine and criteria combination formulae.

4 Preliminary Convention

Although the conception of utility is the center of decision making theory, decision making problems operate with real utility rarely. Usually, utility represents income, benefit, win, gain, etc., but it is easier to deal with another measure of utility which is loss or risk. At least, loss functions are widespread objects with a purpose of minimization.

Thus, let us define the utility function as follows. Denote a set of decisions by X, a set of states by S, and a set of metastates by M. If these sets are finite then $X = \{x_i\}_{i=1}^{N}$, $S = \{s_j\}_{j=1}^{Q}$, $M = \{m_k\}_{k=1}^{K}$, by $N = |X|$, $N \in \mathbb{N} \setminus \{1\}$, $Q = |S|$, $Q \in \mathbb{N} \setminus \{1\}$, $K = |M|$, $K \in \mathbb{N} \setminus \{1\}$. An ordinary decision (utility) function, defined commonly on a set $X \times S$ at some $m \in M$, changes as m is changed.

In the situation

$$\{x, s, m\} \text{ by } x \in X, \quad s \in S, \quad m \in M, \tag{1}$$

which is a meta-situation, a real value $u(x, s, m)$ is an evaluation of utility. Therefore, the loss function $u(x, s, m)$ defined on the set $X \times S \times M$ is to be minimized with respect to decisions X. If function $u(x, s, m)$ represents benefit, then it is to be maximized with respect to decisions X.

Now, let us define a probabilistic measure. Denote by $p(x, s, m)$ a nonnegative function which is a probabilistic measure over ordinary states for each decision $x \in X$ and each metastate $m \in M$. The function $p(x, s, m)$ is defined on the set S with a Lebesgue measure $\mu_S(s)$, so its condition of unit normalization is [8, 42, 44]

$$\int_S p(x, s, m)\, d\mu_S(s) = 1 \quad \forall\, x \in X \text{ and } \forall\, m \in M. \tag{2}$$

Value $p(x, s, m)$ is proportional to a probability of that situation (1) happens.

On the other hand, metastates have their own probabilistic measure. This measure should not depend on decisions and ordinary states. Let a nonnegative function $w(m)$ be a probabilistic measure over metastates. Value $w(m)$ is proportional to a probability of that metastate m happens. The function $w(m)$ is defined on the set M with a Lebesgue measure $\mu_M(m)$, so its condition of unit normalization is

$$\int_M w(m)\, d\mu_M(m) = 1. \tag{3}$$

Theoretically, sets of states and metastates may appear such, that integrals (2) and (3) come non-summable. But, without losing generality, in practice, these sets can always be supplemented artificially so that integrals (2) and (3) might exist. Clearly, a transition to Riemann integrals is intuitively easily fulfilled, where it is possible.

5 Normalized Expected Utility

Expected utility by the minimax (minimaximax) principle is

$$a(x) = \max_{s \in S} \max_{m \in M} u(x, s, m). \tag{4}$$

Obviously, those maximization operators in (4) can be swapped:

$$a(x) = \max_{m \in M} \max_{s \in S} u(x, s, m). \tag{5}$$

Generally, operators of extremization over sets S and M are identical, so the swap can always be applied (if the order of extremization matters for calculation speed).

When the set S is finite, the product criterion (rule) for a positive function $u(x, s, m)$ gives

$$a(x) = \max_{m \in M} \prod_{s \in S} u(x, s, m). \tag{6}$$

When the set M is finite, the product rule gives

$$a(x) = \max_{s \in S} \prod_{m \in M} u(x, s, m). \tag{7}$$

If they both are finite, then

$$a(x) = \prod_{s \in S} \prod_{m \in M} u(x, s, m) = \prod_{m \in M} \prod_{s \in S} u(x, s, m). \tag{8}$$

If a probabilistic measure for calculating expected utility is invoked, then a one extremization operator remains. This is maximization for the loss function. Then expected utility is either

$$a(x) = \max_{m \in M} \int_S p(x, s, m) u(x, s, m) d\mu_S(s) \tag{9}$$

or

$$a(x) = \max_{s \in S} \int_M w(m) u(x, s, m) d\mu_M(m) \tag{10}$$

for knowing the measures over states or a measure over metastates, respectively. If all probabilistic measures are available, then

$$a(x) = \int_M w(m) \int_S p(x, s, m) u(x, s, m) d\mu_S(s) d\mu_M(m). \tag{11}$$

Therefore, the expected utility is obtained from the function of three variables (Figure 2) by eliminating both states and metastates. The elimination is realized with extremization, (or/and) multiplication, integration. The integration over finite sets is substituted with summation [8, 11, 35, 36, 38, 39, 42]. Multiplication, if any, always precedes extremization. The integration, if any, always precedes extremization as well.

Expected utilities by formulae (4)–(11) and similar ones to them have absolutely different units of measurement. This is why, for comparing them and subsequently combining them, the function $a(x)$ must be normalized—just like in the rightmost part of Figures 3 and 4. But suppose that the utility function is standardized preliminarily (Figure 4). For instance, a common routine for such standardization is

$$\tilde{u}(x, s, m) = \frac{u(x, s, m) - \min_{y \in X} \min_{t \in S} \min_{n \in M} u(y, t, n)}{\max_{y \in X} \max_{t \in S} \max_{n \in M} u(y, t, n) - \min_{y \in X} \min_{t \in S} \min_{n \in M} u(y, t, n)} \tag{12}$$

that gives $\tilde{u}(x, s, m) \in [0; 1]$. Hence, situations (1) evaluated with (12) by various criteria are comparable. However, if weights of criteria are known beforehand, then this standardization is useless. Besides, for applying the product rule, standardization (12) does not fit because it requires that $\tilde{u}(x, s, m) \in (0; 1]$ shall be. Furthermore, when probability-based criteria are applied (like Germeyer's or maximum probability criterion), standardization (12) is needless. That all implies the most common routine for calculating the normalized expected utility lies in the straightforward calculation shown in Figure 3.

6 Formulae for Combining Criteria

Suppose we have altogether H criteria to be hybridized. Formally, $H \in \mathbb{N} \setminus \{1\}$. Denote by $a_h(x)$ the expected utility by the h-th criterion, $h = \overline{1, H}$. Then the normalized expected utility by this criterion is [14, 29, 37, 38]

$$\tilde{a}_h(x) = \frac{a_h(x) - \min_{y \in X} a_h(y)}{\max_{y \in X} a_h(y) - \min_{y \in X} a_h(y)} \quad \forall\, x \in X \quad \text{at} \quad h = \overline{1, H}, \tag{13}$$

if only

$$\max_{y \in X} a_h(y) \neq \min_{y \in X} a_h(y). \tag{14}$$

Normalization (13) implies that, for every h-th criterion, $\exists\, x^{\langle 0 \rangle} \in X$ such that $\tilde{a}_h\left(x^{\langle 0 \rangle}\right) = 0$ and $\exists\, x^{\langle 1 \rangle} \in X$ such that $\tilde{a}_h\left(x^{\langle 1 \rangle}\right) = 1$, i.e. $\tilde{a}_h(x) \in [0; 1]$. If condition (14) is violated, i.e.

$$\max_{y \in X} a_h(y) = \min_{y \in X} a_h(y),$$

then the expected utility by the h-th criterion is constant and thus it does not make any sense. That criterion is excluded from hybridization.

Note that some of functions $\{a_h(x)\}_{h=1}^{H}$ may reflect benefits instead of losses, even working with a loss function $u(x, s, m)$. This is like when the minimum variance criterion is applied to a gain function. So let H_- be a subset of indices of criteria which correspond to losses, and H_+ be a subset of indices of criteria which correspond to gains. Here we have $H_- \cup H_+ = \left\{\overline{1, H}\right\}$. Then the criteria are combined as

$$X^* = \arg\min_{x \in X} \left(\sum_{h_- \in H_-} \lambda_{h_-} \tilde{a}_{h_-}(x) - \sum_{h_+ \in H_+} \lambda_{h_+} \tilde{a}_{h_+}(x) \right) \tag{15}$$

with the h-th criterion weight λ_h, where

$$\sum_{h=1}^{H} \lambda_h = \sum_{h_- \in H_-} \lambda_{h_-} + \sum_{h_+ \in H_+} \lambda_{h_+} = 1$$

by

$$\lambda_h \in (0; 1) \quad \forall\, h = \overline{1, H}. \tag{16}$$

Weights (16) determining importance of criteria are deduced empirically, with expert estimation procedures. If these weights are unknown or cannot be given, then the hybridization rule (15) is simplified to:

$$X^{*} = \arg\min_{x \in X} \left(\sum_{h_- \in H_-} \tilde{a}_{h_-}(x) - \sum_{h_+ \in H_+} \tilde{a}_{h_+}(x) \right). \tag{17}$$

Formula (17) is the easiest way for combining criteria. This way is the most used as it does not need any additional information, including weights (16).

7 Numerical Experiments for Validating the Substantiated Routine and Criteria Combination Formulae

Consider an example, where the loss function $u(x, s, m)$ is a $4 \times 3 \times 3$ decision matrix

$$\mathbf{U} = (u_{ijk})_{4\times3\times3} = \left((u_{ij1})_{4\times3}, (u_{ij2})_{4\times3}, (u_{ij3})_{4\times3} \right)$$

by $X = \{x_i\}_{i=1}^{4}$, $S = \{s_j\}_{j=1}^{3}$, $M = \{m_k\}_{k=1}^{3}$:

$$\mathbf{U} = \left(\begin{bmatrix} 4 & 2 & 2 \\ 3 & 1 & 4 \\ 5 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 5 & 1 \\ 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 5 & 2 \\ 2 & 3 & 5 \\ 4 & 4 & 2 \\ 3 & 5 & 2 \end{bmatrix} \right).$$

Probabilistic measures $\left\{ \{ p(x, s, m) \}_{m \in M} \right\}_{x \in X}$ over ordinary states constitute a $4 \times 3 \times 3$ stochastic matrix $\mathbf{P} = (p_{ijk})_{4\times3\times3}$:

$$\mathbf{P} = (p_{ijk})_{4\times3\times3} = \left((p_{ij1})_{4\times3}, (p_{ij2})_{4\times3}, (p_{ij3})_{4\times3} \right)$$
$$= \left(\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}, \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}, \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} \right).$$

The probabilistic measure $w(m)$ over those three metastates is just

$$\mathbf{W} = (w_k)_{1\times3} = [0.1 \quad 0.3 \quad 0.6].$$

Expected losses by minimaximax (4) or (5) are

$$\{a_1(x_i)\}_{i=1}^{4} = \{5, 5, 5, 5\}.$$

Clearly, this criterion of the ultimate pessimism is useless here as $X_1^{*} = X$. Expected losses by the product criterion in (8) are

$$\{a_2(x_i)\}_{i=1}^{4} = \{2880, 5760, 4800, 2880\}$$

which give two optimal decisions: $X_2^{*} = \{x_1, x_4\}$. Expected losses by (9) are

$$\left\{a_3(x_i)\right\}_{i=1}^{4} = \left\{\max_{k=\overline{1,3}} \sum_{j=1}^{3} p_{ijk} u_{ijk}\right\}_{i=1}^{4}$$
$$= \left\{\max_{k=\overline{1,3}}\{2.8, 2, 2.9\}, \max_{k=\overline{1,3}}\{2, 1.9, 3\}, \max_{k=\overline{1,3}}\{1.6, 1.8, 3.2\}, \max_{k=\overline{1,3}}\{2.3, 2, 3.2\}\right\}$$
$$= \{2.9, 3, 3.2, 3.2\}$$

which give $X_3^* = \{x_1\}$. An unexpected solution is obtained with another criterion by (10) exploiting probabilities of metastates:

$$\left\{a_4(x_i)\right\}_{i=1}^{4} = \left\{\max_{j=\overline{1,3}} \sum_{k=1}^{3} w_k u_{ijk}\right\}_{i=1}^{4}$$
$$= \left\{\max_{j=\overline{1,3}}\{2.5, 4.1, 1.7\}, \max_{j=\overline{1,3}}\{2.7, 2.2, 4.6\}, \max_{j=\overline{1,3}}\{3.8, 4, 1.7\}, \max_{j=\overline{1,3}}\{2.3, 4.4, 2.1\}\right\}$$
$$= \{4.1, 4.6, 4, 4.4\}$$

whereupon $X_4^* = \{x_3\}$. The fifth version of the optimal decisions' subset is obtained by calculating expected losses as (11), that reminds the full Bayes—Laplace criterion:

$$\left\{a_5(x_i)\right\}_{i=1}^{4} = \left\{\sum_{k=1}^{3} w_k \sum_{j=1}^{3} p_{ijk} u_{ijk}\right\}_{i=1}^{4}$$
$$= \{2.8 \cdot 0.1 + 2 \cdot 0.3 + 2.9 \cdot 0.6,\ 2 \cdot 0.1 + 1.9 \cdot 0.3 + 3 \cdot 0.6,$$
$$1.6 \cdot 0.1 + 1.8 \cdot 0.3 + 3.2 \cdot 0.6,\ 2.3 \cdot 0.1 + 2 \cdot 0.3 + 3.2 \cdot 0.6\}$$
$$= \{2.62, 2.57, 2.62, 2.75\}$$

whereupon $X_5^* = \{x_2\}$.

None of those five criteria gives an acceptable solution to the being considered decision making problem. Thus, hybridization is necessary. The first criterion is excluded as condition (14) is violated. After normalization of expected losses,

$$\left\{\tilde{a}_2(x_i)\right\}_{i=1}^{4} = \left\{0, 1, \frac{2}{3}, 0\right\}, \quad \left\{\tilde{a}_3(x_i)\right\}_{i=1}^{4} = \left\{0, \frac{1}{3}, 1, 1\right\},$$
$$\left\{\tilde{a}_4(x_i)\right\}_{i=1}^{4} = \left\{\frac{1}{6}, 1, 0, \frac{2}{3}\right\}, \quad \left\{\tilde{a}_5(x_i)\right\}_{i=1}^{4} = \left\{\frac{5}{18}, 0, \frac{5}{18}, 1\right\},$$

where $H_- = \left\{\overline{2, 5}\right\}$ and $H_+ = \varnothing$, we get an optimal decisions' subset by (17):

$$X^* = \arg\min_{x_i,\, i=\overline{1,4}} \left(\sum_{h_-=2}^{5} \tilde{a}_{h_-}(x_i)\right) = \arg\min_{x_i,\, i=\overline{1,4}} \left\{\frac{4}{9}, \frac{7}{3}, \frac{35}{18}, \frac{8}{3}\right\} = \{x_1\}. \tag{18}$$

Despite $X_2^* = \{x_1, x_4\}$ by the product criterion in (8), here decision x_4 is the worst due to (18). Besides, $X_4^* = \{x_3\}$ and $X_5^* = \{x_2\}$, although decisions x_3 and x_2 are almost as poor as decision x_4. So, this is an example of that standalone criteria may give inconsistent and contradictory results until they are combined into a single hybridized criterion. However, apart from the useless minimaximax, here the best decision $X^* = \{x_1\}$ is given also by the product criterion and expected losses by (9).

More general examples are obtained by numerical experiments formed on a base of pseudorandom numbers drawn from the standard uniform distribution on the open interval $(0;1)$. For instance, the decision matrix

$$\mathbf{U} = \left(u_{ijk}\right)_{N\times Q\times K} = 10 \cdot \Theta\left(N, Q, K\right) + 5 \tag{19}$$

by a function $\Theta\left(N, Q, K\right)$ returning a pseudorandom $N \times Q \times K$ matrix drawn from the standard uniform distribution on the open interval $\left(0;1\right)$. Stochastic matrix $\mathbf{P} = \left(p_{ijk}\right)_{N\times Q\times K}$ and probabilities $\mathbf{W} = \left(w_k\right)_{1\times K}$ of metastates are formed similarly. Figure 5 shows a sample of montaged histograms of single-criterion decisions (in light color, for the five criteria used in the $4\times 3\times 3$ example above) and positions of the hybridized-criterion decision (in black color). It is well seen that the case when the best decision (by the hybridized criterion) does not coincide with any of the standalone criteria is rare. Nonetheless, the case when the best decision does not coincide with the highest bar occurs much more frequently. In particular, Figure 6 shows that the number of coincidences with the most frequent standalone-criterion decision for $N\times 10\times 10$ matrices decreases as the size increases. The number of coincidences with any of the standalone criteria decreases also, although a little bit slower.

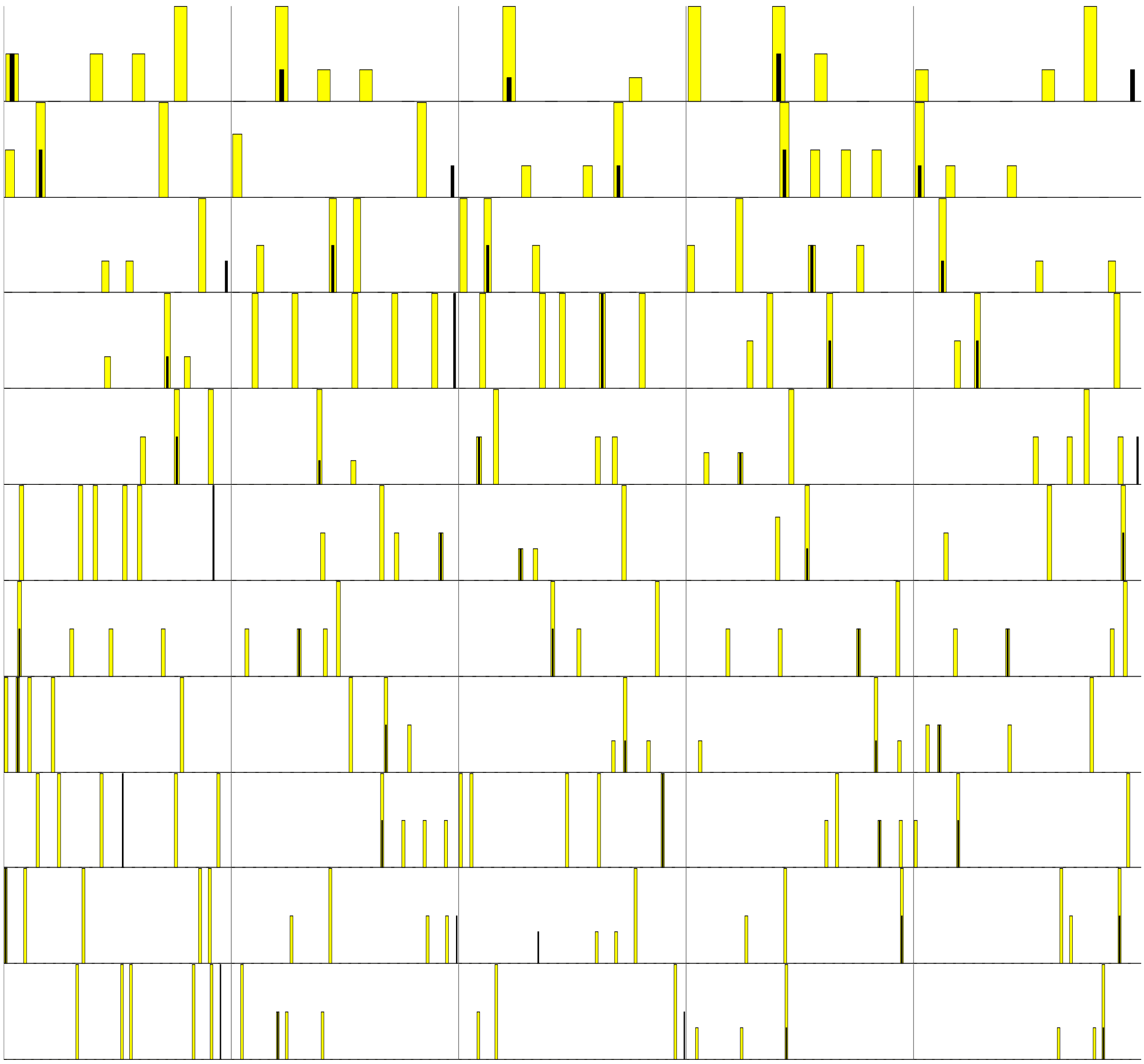

Figure 5: A sample of montaged histograms for $N\times 10\times 10$ matrices by $N \in \left\{6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26\right\}$ (results for these eleven sizes are given row-wise, five versions for every size, descending as the size increases)

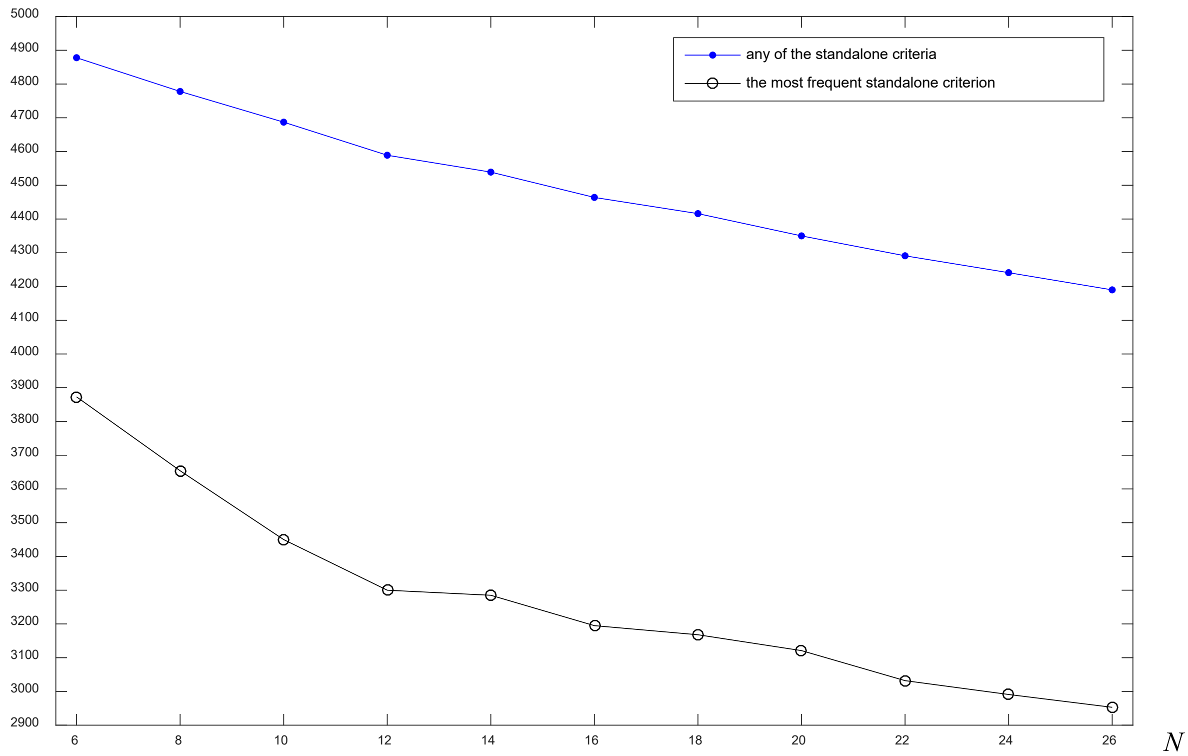

Figure 6: The number of coincidences with the most frequent standalone-criterion decision and the number of coincidences with any of the standalone criteria for 5000 $N \times 10 \times 10$ matrices (at each point)

Apparently, the said coincidence numbers depend on the number of states and metastates also. The decreasing dependence is similar to that in Figure 6. This is revealed in Figures 7 and 8 whose meshes are plotted for 50000 $N \times Q \times Q$ matrices (having the same numbers of states and metastates, for simplification) at each node.

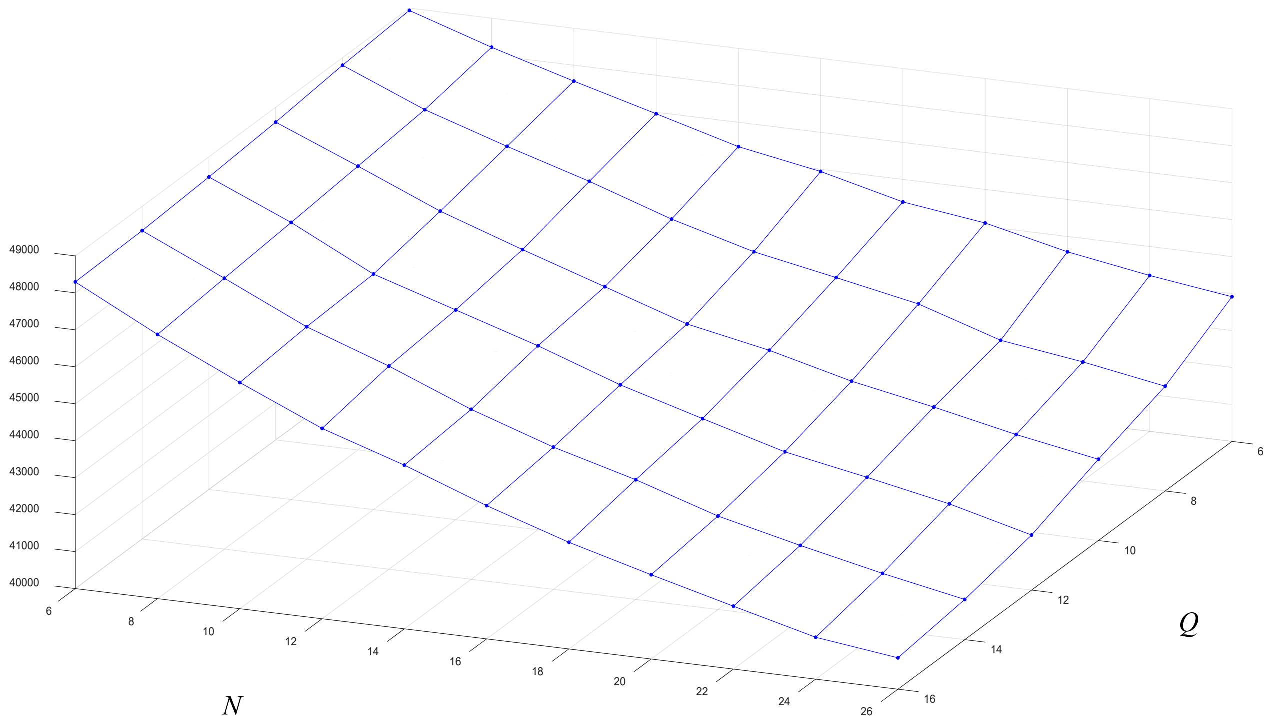

Figure 7: The decreasing number of coincidences with any of the standalone criteria

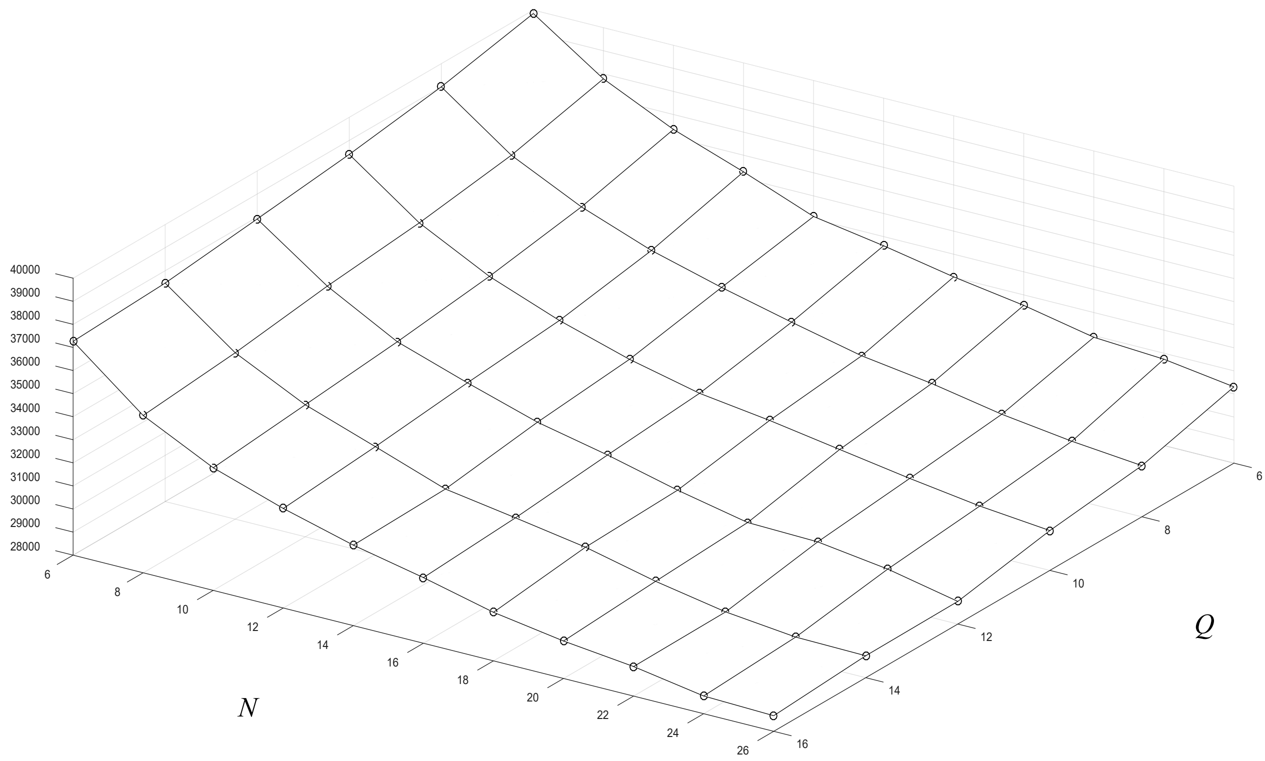

Figure 8: The faster decreasing number of coincidences with the most frequent standalone-criterion decision

The shown results of numerical experiments validate the substantiated routine for straightforwardly calculating the normalized expected utility (Figure 3) and criteria combination formulae (13)–(17). The main aspect of the validation is that standalone-criteria decisions are badly scattered (they do not have a "direction"), while the hybridized-criterion decision is almost always single. Standalone-criteria and hybridized-criterion decisions coincide less for bigger sizes of decision matrices. Besides, non-coincidence with the most frequent decision decreases faster.

8 Conclusion

Decision making criteria hybridization by (15) or (17) is intended for combining criteria of various natures, and so straightforward calculation of normalized expected utility is preferable. Variety of their natures may be also expressed with that how severely the ordinary decision function changes through metastates, that is differently perceived by diverse criteria (for instance, by those who operate with a probabilistic measure over metastates and by those who do not). Standardization (12) does make its sense only when weights (16) determining importance of criteria are to be deduced. The deduction is accomplished with comparing magnitude and smoothness of the expected utility curves.

Before hybridizing, normalization (13) is accomplished by condition (14). Criteria violating that condition are excluded and not taken into hybridization. Even if some methods of eliminating states and metastates are controversial, or not all changes of the decision function are regarded appropriately, hybridization of a great number of criteria must expectedly disregard such controversies. This reminds an effect of the law of large numbers.

Acknowledgments

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