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A New Equilibrium Optimization Method for Hybrid Uncertain Decision-Making System

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Abstract

This study deals with the optimization methodology in hybrid uncertain decision-making systems and develops a new equilibrium two-stage programming, in which both possibility and probability distributions are used to characterize uncertain parameters. The decision process is divided into two stages. The first-stage decision should be taken before knowing the realizations of uncertain parameters, while the second-stage decision must be taken after knowing the outcome of embedded objective uncertainty. On the basis of the proposed dynamic decision scheme, the second-stage problem is built via credibilistic optimization methods, and the objectives in the first-stage problem are constructed via stochastic optimization methods. For single objective equilibrium two-stage programming problem and bi-objective equilibrium two-stage programming problem, we define their wait-and-see solution, here-and-now solution and expected value solution, respectively. Two important indexes, the expected value of perfect random information (EVPRI) and the value of equilibrium recourse solution (VERS), are introduced, and their relations are illustrated via numerical examples.

Keywords: equilibrium two-stage optimization, wait-and-see solution, equilibrium recourse solution, expected value solution

1 Introduction

In literature, the recourse problem of stochastic programming has been studied extensively (see[1, 6, 29]), and has been applied to many real world decision problems, especially decision problems involving risk [5]. For a unit commitment problem with uncertain wind power output, Wang et al. [31] presented model includes both the two-stage stochastic program and the chance-constrained stochastic program features. Li and Chen [8] found risk-averse optimal ordering decisions in supply contracts problem with random demand by two-stage bi-objective stochastic optimization method. In order to provide an effective response and use resources efficiently, Moreno et al. [22] modelled uncertainty regarding demand, incoming supply, and availability of routes via a finite set of scenarios and presented an effective two-stage stochastic model to optimize location, transportation, and fleet sizing decisions. To design cost-optimal distributed energy systems under uncertainty, Mavromatidis et al. [21] presented a two-stage stochastic programming using multiple criteria. In a supply chain context, considering demand uncertainties with regard to product specifications and volumes and integrating the selection of new product designs and processing technologies, Stefansdottir and Grunow [25] presented a two-stage stochastic mixed integer linear programming model. The first-stage model selected the processing technologies, and the second-stage model took the detailed product designs and the production volumes as recourse actions. To maximize the profit under rent warehouse incentives decreasing over time and price-sensitive demands, Lin and Wang [10] proposed a two-stage stochastic optimization model, which firstly decided the optimal warehouse location for multiple markets and determining warehouse configuration design against stochastic demands, then designed an appropriate inventory policy with owned and rented warehouses for deteriorating items. For water resources management problem under uncertainty, an extended two-stage stochastic programming with fuzzy variables was developed in [24].

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The recourse problem of fuzzy programming was studied first by Liu [11]. He built a two-stage fuzzy expected value model. Liu and Zhu [20] presented a class of two-stage fuzzy minimum risk model for a location and allocation problem based on credibility theory. In 2009, Liu and Tian [19] presented a class of two-stage fuzzy programming with minimum-risk criteria in the sense of Value-at-Risk. These models take credibility theory as the theoretical foundation, the uncertain parameters in the model are fuzzy variables characterized by possibility distribution functions. Liu and Bai [15] studied two-stage fuzzy minimum risk problem and two-stage fuzzy value-at-risk problem, established the interconnections between their optimal objective values, and discussed the relations between their optimal solutions. As a consequence, it would be possible to solve one two-stage optimization problem indirectly by solving its counterpart.

Now the recourse problem of fuzzy programming has also been applied to many optimization problems. To reduce risks in location decisions from imprecise information, Wang et al. [33] proposed a two-stage fuzzy zero-one integer programming model with value-at-risk for an uncertain facility location problem. Yuan [36] developed a two-stage fuzzy optimization method to solve the multi-product multi-period production planning problem with fuzzy market demands and inventory costs. Sun et al. [27, 28] assumed the uncertain material demand, the spot market material unit price and the spot market material supply quantity to be fuzzy variables with known possibility distributions, and presented two-stage fuzzy material procurement planning models with minimum-risk criteria. Considering the uncertain customer demands, transportation costs and resource capacities, Yang and Liu [34] developed a mean-risk two-stage fuzzy optimization method for supply chain network design problem. In the proposed model, the standard semivariance is suggested to gauge the risk resulted from fuzzy uncertainty. They defined the value of fuzzy solution in the sense of mean-risk as the difference between the here-and-now solution and the expected value solution to demonstrate the advantages of the proposed optimization method. Characterizing the future returns of risky security by possibility distributions, Chen and Wang [3] studied a two-period portfolio selection problem with transaction costs and formulated it as a two-stage fuzzy programming model. The analytical optimal solution of the two-stage model is obtained.

Randomness and fuzziness coexist in many real-life fields. For instance, when measuring the depth of a lake, Negoita and Ralescu [23] assumed that the location was random and the measurement results were fuzzy. The demands of customers were assumed to have simultaneous randomness and fuzziness [38]. The uncertain travel times considered in [35] included both randomness and fuzziness simultaneously. The return rates of assets were characterized by joint normal distribution, and the parameters were fuzzy [30]. The uncertain demands for each commodity were also assumed to follow normal distribution with fuzzy mathematical expectation and covariance matrix in [37]. Chen et al. [2] used fuzzy random variables to characterize the lifetimes of components in some standby redundancy systems.

Motivated by two-stage stochastic programming and fuzzy programming with recourse problems, based on equilibrium chance theory [17, 18], two classes of fuzzy random optimization problem called two-stage fuzzy random programming or fuzzy random programming with recourse were first presented in [13, 14], and a class of random fuzzy programming with recourse was presented in [16]. These two-stage hybrid uncertain optimization methods had been applied to model some realistic optimization problems. For example, Wang and Watada [32] presented a recourse-based fuzzy random facility location model with fixed capacity. Zhai et al. [37] developed a two-stage uncapacitated hub location problem with recourse, in which uncertain parameters were characterized by both probability and possibility distributions. When demands were the only uncertain parameters, the proposed two-stage uncapacitated hub location model was equivalent to a static optimization problem subject to equilibrium constraint. Li et al. [9] modelled the uncertain demand by both probability distribution and possibility distribution, and developed a two-stage expected value optimization model for a supply contract problem. In the first decision-making stage, the distributor signed an options-futures contract with the supplier to determine the futures and options ordering quantities. After knowing the realization of uncertain demand, the distributor took the signed options as the recourse decision in the second stage.

In the fuzzy random programming with recourse and random fuzzy programming with recourse from literature, some uncertain parameters are characterized by known possibility and probability distributions. The first-stage decisions must be taken before the outcome of hybrid uncertain parameters are revealed and thus must be based on the knowledge of the distribution of the parameters only. After outcomes of all random fuzzy parameters or fuzzy random parameters have been observed, the second-stage decisions as some recourse (or corrective) actions may be taken. If there is an opportunity to adjust the decision, it is reasonable that the decision maker always hopes to make the recourse decision as soon as possible to reduce the loss. In some cases, the recourse decision must be made when partial information is unknown. The later the problem is solved, the higher the cost will be. To the best of our knowledge, the existing optimization methods can not solve this dynamic optimization problem. The aim of this paper is to develop a new two-stage equilibrium optimization method to deal with this class of dynamic optimization problem.

The main contributions of this study can be summarized as follows. First, integrating two different optimization methods into an optimization framework, this paper develops a new equilibrium optimization method with recourse. The first-stage decision should be taken before knowing the realizations of uncertain parameters. The second-stage decision must be taken after knowing the outcome of objective uncertainty embedded in uncertain parameters while before knowing the outcome of subjective uncertainty. On the basis of this dynamic decision scheme, the objectives and constraints in the first-stage are constructed via stochastic optimization methods, and the objective and constraints in the second-stage are built via credibilistic optimization methods. Second, we define three solution concepts, the wait-and-see solution, here-and-now solution and expected value solution, for single objective equilibrium two-stage programming problem and bi-objective equilibrium two-stage programming problem, respectively. Two indexes, the expected value of perfect random information (EVPRI) and the value of equilibrium recourse solution (VERS), are also introduced. Finally, we illustrate the relations among the wait-and-see solution, here-and-now solution and expected value solution via numerical examples.

The material is arranged into five sections. First, in Section 2, we give an equilibrium single objective programming model with recourse, and define its wait-and-see solution, here-and-now solution and expected value solution. The intent of Section 3 is to present an equilibrium bi-objective programming model with recourse, and define its three basic solution concepts. Section 4 discusses the method to solve the equilibrium two-stage programming problems. Numerical examples are covered in Section 5 to illustrate the three solutions' relations.

2 An Equilibrium Single Objective Model

2.1 Model Formulation

We consider the following problem:

$$\left.\begin{array}{ll} \min & c^T\boldsymbol{x} + q(\omega,\gamma)^T\boldsymbol{y} \\ \text{s.t.} & A\boldsymbol{x} = b \\ & T(\omega,\gamma)\boldsymbol{x} + W(\omega,\gamma)\boldsymbol{y} \geq h(\omega,\gamma) \\ & \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0, \end{array}\right\}$$

where some components of $q(\omega,\gamma)$, $h(\omega,\gamma)$, $T(\omega,\gamma)$ and $W(\omega,\gamma)$ are fuzzy random variables defined on a probability space $(\Omega,\, \Sigma,\, \Pr)$. We assume that the decision scheme is the following

$$\boxed{\begin{array}{c} \text{decision on } \boldsymbol{x} \text{ in fuzzy random environment} \\ \Downarrow \\ \text{observation of random event } \omega \\ \Downarrow \\ \text{decision on } \boldsymbol{y} \text{ in fuzzy environment.} \end{array}}$$

According to this scheme, we present a new equilibrium optimization problem with recourse, in which there are two optimization problems to be solved. Assuming $\boldsymbol{x}$ and ω to be fixed, we formulate the second-stage problem (or the recourse problem) as the following credibilistic programming model

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\omega,\gamma)^T\boldsymbol{y}] \\ \text{s.t.} & \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y} \geq h(\omega,\gamma) - T(\omega,\gamma)\boldsymbol{x}\} \geq \alpha \\ & \boldsymbol{y} \geq 0. \end{array}\right\} \tag{1}$$

Suppose that the decision vector $\boldsymbol{x}$ has to satisfy the following deterministic constraints:

$$A\boldsymbol{x} = b, \quad \boldsymbol{x} \geq 0.$$

We introduce additional constraint K on $\boldsymbol{x}$, which facilitates the discussion on the solution of problem (1). Let K be the set of all those $\boldsymbol{x}$ vectors for which problem (1) is feasible for almost every possibly realized random event ω. If we define the second-stage value function as

$$\begin{aligned}&Q(\boldsymbol{x},\xi(\omega,\gamma))\\ &=\begin{cases}\min\{E_\gamma[q(\omega,\gamma)^T\boldsymbol{y}] \mid \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y}\ge h(\omega,\gamma)-T(\omega,\gamma)\boldsymbol{x}\}\ge\alpha,\boldsymbol{y}\ge 0\}, & \text{if there is a feasible solution } \boldsymbol{y}\\ +\infty, \text{ if there is no a feasible solution } \boldsymbol{y},\end{cases}\end{aligned}$$

then K can be expressed as

$$K=\left\{\boldsymbol{x} \mid \boldsymbol{x}\in\Re^{n_1}, \Pr\{\omega \mid Q(\boldsymbol{x},\xi(\omega,\gamma))<\infty\}=1\right\},$$

where ξ is the fuzzy random vector obtained by piecing together the fuzzy random components of the second-stage problem data $q(\omega,\gamma)$, $h(\omega,\gamma)$, $W(\omega,\gamma)$ and $T(\omega,\gamma)$.

The constraint $\boldsymbol{x}\in K$ is called *induced constraint.* For convenience, in the rest of the paper, we will denote the second-stage value function $Q(\boldsymbol{x},\xi(\omega))$ by $Q(\boldsymbol{x},\omega)$.

The first-stage problem is formulated as follows

$$\left.\begin{array}{ll}\sup\limits_{\boldsymbol{x}} & c^T\boldsymbol{x}+\mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]\\ \text{s.t.} & A\boldsymbol{x}=b\\ & \boldsymbol{x}\ge 0\\ & \boldsymbol{x}\in K,\end{array}\right\} \tag{2}$$

where the expected second-stage value function $\mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]$ is called the *recourse function*, E_ω is the expected value operator with respect to random vector ω.

Combining problems (1) and (2), an equilibrium optimization with recourse problem can be built as

$$\left.\begin{array}{ll}\min\limits_{\boldsymbol{x}} & c^T\boldsymbol{x}+\mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]\\ \text{s.t.} & A\boldsymbol{x}=b\\ & \boldsymbol{x}\ge 0\\ & \boldsymbol{x}\in K,\end{array}\right\} \tag{3}$$

where

$$\left.\begin{array}{rl}Q(\boldsymbol{x},\omega)=\min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\omega,\gamma)^T\boldsymbol{y}]\\ \text{s.t.} & \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y}\ge h(\omega,\gamma)-T(\omega,\gamma)\boldsymbol{x}\}\ge\alpha\\ & \boldsymbol{y}\ge 0.\end{array}\right\} \tag{4}$$

The problem (3)–(4) is equivalent to the following programming problem:

$$\left.\begin{array}{ll}\min\limits_{\boldsymbol{x}} & c^T\boldsymbol{x}+\mathrm{E}_\omega\left[\min\limits_{\boldsymbol{y}} E_\gamma[q(\omega,\gamma)^T\boldsymbol{y}]\right]\\ \text{s.t.} & A\boldsymbol{x}=b\\ & \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y}\ge h(\omega,\gamma)-T(\omega,\gamma)\boldsymbol{x}\}\ge\alpha\\ & \boldsymbol{x}\ge 0,\boldsymbol{y}\ge 0.\end{array}\right\}$$

As the above discussion, a distinction is made between the first stage and the second stage. The first-stage decision is represented by an $n_1\times 1$ vector $\boldsymbol{x}$, while the second-stage decision is represented as an $n_2\times 1$ vector $\boldsymbol{y}$. In the second stage, a number of random events $\omega\in\Omega$ may realize, the second-stage problem data q, h, W and T are fuzzy matrices. Then the second-stage decision $\boldsymbol{y}$ must be taken in fuzzy environment.

Example 1. *Assume that the equilibrium two-stage programming problem is*

$$\left.\begin{array}{ll}\min\limits_{\boldsymbol{x}} & 2x_1+3x_2+\mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]\\ s.t. & x_1+x_2=1\\ & \boldsymbol{x}\ge 0\\ & \boldsymbol{x}\in K,\end{array}\right\} \tag{5}$$

where

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega)=\min\limits_{\boldsymbol{y}} & 2y_1+y_2 \\ s.t. & \mathrm{Cr}\{y_1\geq q(\omega,\gamma)-x_1-x_2\}\geq 0.8 \\ & y_1+y_2\geq 1-x_1 \\ & y_1,y_2\geq 0. \end{array}\right\} \tag{6}$$

In model (5)-(6), the fuzzy random parameter is defined as

$$q(\omega,\gamma)=\left\{\begin{array}{l} X_1 \textit{ with probability } 0.25 \\ X_2 \textit{ with probability } 0.75 \end{array}\right.$$

where

$$X_1\sim\left(\begin{array}{cccc} 3 & 4 & 5 & 6 \\ \frac{2}{5} & 1 & \frac{4}{5} & \frac{1}{5} \end{array}\right), \qquad X_2\sim\left(\begin{array}{ccc} 0 & 1 & 2 \\ \frac{1}{3} & 1 & \frac{2}{3} \end{array}\right).$$

Find its solution.

Note that the credibility level $0.8>0.5$*, then we have*

$$\mathrm{Cr}\{y_1\geq X_1-x_1-x_2\}\geq 0.8 \Longleftrightarrow y_1\geq 4$$

for given $q(\omega,\gamma)=X_1$ *and feasible* $\boldsymbol{x}$*,*

$$\mathrm{Cr}\{y_1\geq X_2-x_1-x_2\}\geq 0.8 \Longleftrightarrow y_1\geq 1$$

for given $q(\omega,\gamma)=X_2$ *and feasible* $\boldsymbol{x}$*.*

We have $Q(\boldsymbol{x},\omega)=8$ *with probability* 0.25*, and* $Q(\boldsymbol{x},\omega)=2$ *with probability* 0.75*. Hence model (5)-(6) is rewritten as*

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & 2x_1+3x_2+\frac{7}{2} \\ s.t. & x_1+x_2=1 \\ & \boldsymbol{x}\geq 0. \end{array}\right\}$$

The optimal solution is $x_1=1, x_2=0$*. The optimal value is* $11/2$*.*

2.2 The Value of Information for the Equilibrium Single Objective Model

Wait-and-See Solution

In the fuzzy random programming, the uncertainty is described by fuzzy random vector ξ. For each realization $\xi(\hat{\omega})$ of fuzzy random vector ξ, an optimization problem associated with this particular realized value $\xi(\hat{\omega})$ is defined as follows

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & z(x,\hat{\omega})=c^T\boldsymbol{x}+\min\limits_{\boldsymbol{y}} E_\gamma[q(\hat{\omega},\gamma)^T\boldsymbol{y}] \\ \text{s.t.} & A\boldsymbol{x}=b \\ & \mathrm{Cr}\{W(\hat{\omega},\gamma)\boldsymbol{y}\geq h(\hat{\omega},\gamma)-T(\hat{\omega},\gamma)\boldsymbol{x}\}\geq\alpha \\ & \boldsymbol{x}\geq 0,\boldsymbol{y}\geq 0. \end{array}\right\} \tag{7}$$

We assume that for all the realized values $\xi(\hat{\omega})$ of fuzzy random vector ξ, there is at least one x such that $z(x,\hat{\omega})<+\infty$. This assumption ensures the existence of optimal solution to problem (7) under $\hat{\omega}$. Let $\bar{x}(\hat{\omega})$ denote the corresponding optimal solution. We are interested in finding out all solutions $\bar{x}(\omega)$ of problem (7) and the associated optimal objective values $z(\bar{x}(\omega),\omega)$. We assume that these decisions $\bar{x}(\omega)$ and their objective values $z(\bar{x}(\omega),\omega)$ can be found. Therefore, in this assumption, we can calculate the expected value of the optimal value. We call

$$\mathrm{WS}=\mathrm{E}_\omega[\min_{\boldsymbol{x}} z(x,\omega)]=\mathrm{E}_\omega[z(\bar{x}(\omega),\omega)].$$

as the wait-and-see solution since decision x is made after the realized value of ω is known. WS is also called as the distribution solution since the distributions of both ω and $z(\bar{x}(\omega),\omega)$ with respect to ω can be found.

Here-and-Now Solution

Let

$$\mathrm{RP} = \min_{\boldsymbol{x}} \mathrm{E}_\omega[z(x,\omega)].$$

Since the first-stage decision $\boldsymbol{x}$ is made before the realized values of ω are known, RP is called as the here-and-now solution, also the equilibrium recourse solution, corresponding to the equilibrium recourse problem formulated as (3)-(4).

Expected Value Solution

Replacing the random parameter ω with its expected value $\bar{\omega}$, the corresponding problem is called the expected value problem or mean value problem. Let

$$\mathrm{EV} = \min_{\boldsymbol{x}} z(x,\bar{\omega}). \tag{8}$$

Let us denote by $\bar{x}(\bar{\omega})$ an optimal solution to model (8). Using $\bar{x}(\bar{\omega})$, the expected result is defined as

$$\mathrm{EEV} = \mathrm{E}_\omega[z(\bar{x}(\bar{\omega}),\omega)].$$

The quantity EEV, called the expected value solution, measures how $\bar{x}(\bar{\omega})$ performs in the sense of mean.

Generally, the values of WS, RP, and EEV are different, and their relations are expressed in the following proposition.

Proposition 1. *For any fuzzy random programming problem (3)–(4), we have* $\mathrm{WS} \le \mathrm{RP} \le \mathrm{EEV}$.

Proof. For any realization $\hat{\omega}$ of ω, one has

$$\min_{\boldsymbol{x}} z(x,\hat{\omega}) \le z(x,\hat{\omega}), \forall x.$$

Taking the expected value in ω on both sides of the above inequality, the following result

$$\mathrm{E}_\omega[\min_{\boldsymbol{x}} z(x,\omega)] \le \mathrm{E}_\omega[z(x,\omega)], \forall x$$

imply that

$$\mathrm{E}_\omega[\min_{\boldsymbol{x}} z(x,\omega)] \le \min_{\boldsymbol{x}} \mathrm{E}_\omega[z(x,\omega)].$$

As a consequence, $\mathrm{WS} \le \mathrm{RP}$.

Since

$$\min_{\boldsymbol{x}} \mathrm{E}_\omega[z(x,\omega)] \le \mathrm{E}_\omega[z(x,\omega)]$$

for any x, this inequality holds for $\bar{x}(\bar{\omega})$. Then

$$\min_{\boldsymbol{x}} \mathrm{E}_\omega[z(x,\omega)] \le \mathrm{E}_\omega[z(\bar{x}(\bar{\omega}),\omega)],$$

i.e. $\mathrm{RP} \le \mathrm{EEV}$.

The proof is complete.

Proposition 2. *If ω is discrete, and takes values $\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_N$, then*

$$\min_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i) \le \mathrm{WS} \le \max_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i).$$

Proof. According to the definition of the wait-and-see solution,

$$\mathrm{WS} = \mathrm{E}_\omega[\min_{\boldsymbol{x}} z(x,\omega)].$$

For any realization $\hat{\omega}$ of ω, one has

$$\min_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i) \le z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i) \le \max_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i).$$

Therefore,

$$\min_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i) \leq \sum_{i=1}^{N} z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i) \leq \max_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i),$$

i.e.,

$$\min_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i) \leq WS \leq \max_i z(\bar{x}(\hat{\omega}_i),\hat{\omega}_i).$$

The proof is complete.

Definition 1. *The expected value of perfect random information (*EVPRI*) is defined as the difference between* RP*and* WS*, i.e.,* EVPRI = RP − WS*.*

The EVPRI measures the maximum amount that a decision maker would be ready to pay in return for complete random information about the future.

Definition 2. *The value of the equilibrium recourse solution (*VERS*) is defined as the difference between* EEV*and* RP*, i.e.,* VERS = EEV − RP*.*

By Proposition 1, it is easy to know that EVPRI ≥ 0 and VERS ≥ 0.

3 An Equilibrium Bi-objective Model

3.1 Equilibrium Optimization Model

In many dynamic decision problems, a decision maker may want to optimize multiple objectives. For example, in portfolio selection problems, an investor want to maximize the return and minimize the risk. According to the decision scheme in above section, we can formulate an equilibrium bi-objective programming model with recourse. In the following, an equilibrium bi-objective programming model (9)-(10) is given as an example.

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & c^T\boldsymbol{x} + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)] \\ \min\limits_{\boldsymbol{x}} & d^T\boldsymbol{x} + \rho_\omega[Q(\boldsymbol{x},\omega)] \\ \text{s.t.} & A\boldsymbol{x} = b \\ & \boldsymbol{x} \geq 0 \\ & \boldsymbol{x} \in K, \end{array}\right\} \tag{9}$$

where $Q(\boldsymbol{x},\omega)$ denote the second-stage value function, that is to say,

$$\left.\begin{array}{rll} Q(\boldsymbol{x},\omega) = & \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\omega,\gamma)^T\boldsymbol{y}] \\ & \text{s.t.} & \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y} \geq h(\omega,\gamma) - T(\omega,\gamma)\boldsymbol{x}\} \geq \alpha \\ & & \boldsymbol{y} \geq 0. \end{array}\right\} \tag{10}$$

The problem (9)–(10) is equivalent to the following programming problem:

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & c^T\boldsymbol{x} + \mathrm{E}_\omega\left[\min\limits_{\boldsymbol{y}} E_\gamma[q(\omega,\gamma)^T\boldsymbol{y}]\right] \\ \min\limits_{\boldsymbol{x}} & d^T\boldsymbol{x} + \rho_\omega\left[\min\limits_{\boldsymbol{y}} E_\gamma[q(\omega,\gamma)^T\boldsymbol{y}]\right] \\ \text{s.t.} & A\boldsymbol{x} = b \\ & \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y} \geq h(\omega,\gamma) - T(\omega,\gamma)\boldsymbol{x}\} \geq \alpha \\ & \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0. \end{array}\right\}$$

In model (9)-(10), the constraint $\boldsymbol{x} \in K$ is the induced constraint, $\boldsymbol{x}$ is called the equilibrium recourse solution, $\boldsymbol{y}$ is called the recourse decision, E_ω is the expected value operator with respect to random vector ω, and ρ_ω is some measurement index, such as variance operator, with respect to random vector ω.

3.2 The Value of Information for Equilibrium Bi-objective Model

We denote

$$z_1(x,\omega) = c^T\boldsymbol{x} + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)],$$

$$z_2(x,\omega) = d^T\boldsymbol{x} + \rho_\omega[Q(\boldsymbol{x},\omega)],$$

and

$$X = \{\boldsymbol{x}|A\boldsymbol{x} = b, \boldsymbol{x} \geq 0, \boldsymbol{x} \in K\}.$$

For vector function $z = (z_1; z_2)$, we denote $x^1 \leq x^2$ if and only if $z(x^1)$ is less than $z(x^2)$, i.e., $\forall i = 1,2$, $z_i(x^1) \leq z_i(x^2)$, and $\exists j = 1$ or 2, $z_j(x^1) < z_j(x^2)$. We denote $x^1 < x^2$ if $\forall i = 1,2$, $z_i(x^1) < z_i(x^2)$,

Definition 3. *A solution $x^* \in X$ is said to be Pareto optimal in X if and only if there exist no $x \in X$ such that $x \leq x^*$.*

Definition 4. *For a given bi-objective programming with feasible region X, its Pareto optimal set P^* is defined as $P^* = \{x \in X | \nexists x^{'} \in X, x^{'} \leq x\}$. The mapping in the objective space of all solutions in the Pareto optimal set is called the Pareto front, denoted as PF^*.*

We cannot find the ideal optimal solution to problem (9)-(10), and usually obtain its Pareto optimal solutions [4]. In this case, we can employ multi-objective optimization methods such as weighting method, constraint method and goal programming [26]. In this section, we adopt the weighting method to turn problem (9)-(10) into single objective model:

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \lambda(c^T\boldsymbol{x} + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]) + (1-\lambda)(d^T\boldsymbol{x} + \rho_\omega[Q(\boldsymbol{x},\omega)]) \\ \text{s.t.} & \boldsymbol{x} \in X, \end{array}\right\} \tag{11}$$

where λ is weight coefficient, and $Q(\boldsymbol{x},\omega)$ denote the second-stage value function, that is to say,

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\omega,\gamma)^T\boldsymbol{y}] \\ \text{s.t.} & \mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y} \geq h(\omega,\gamma) - T(\omega,\gamma)\boldsymbol{x}\} \geq \alpha \\ & \boldsymbol{y} \geq 0. \end{array}\right\} \tag{12}$$

According to the results in Section 2, we can find the wait-and-see solution, here-and-now solution and expected value solution of model (11)-(12).

Definition 5. *The wait-and-see solution, here-and-now solution and expected value solution of model (11)-(12), denoted as WS_λ, RP_λ and EEV_λ, are called as the Pareto wait-and-see solution, Pareto here-and-now solution and Pareto expected value solution under weighted coefficient λ of model (9)–(10), respectively.*

Similarly, EVPRI and VERS for model (11)-(12) are called as EVPRI and VERS under weighted coefficient λ for model (9)–(10), respectively. This two indexes can measure the expected value of perfect random information and the value of equilibrium recourse solution.

4 Solving Method

One key of solving problem (3)–(4) is to solve the second-stage programming problem (4). This section will deal with this issue in two cases.

4.1 Translating into Equivalent Deterministic Programming

For any realization ω, we assume that $W(\omega,\gamma)$, $T(\omega,\gamma)$ and $q(\omega,\gamma)$ in model (4) are $1 \times n_2$, $1 \times n_1$ and $n_2 \times 1$ fuzzy vectors, respectively. The components of $W(\omega,\gamma)$, $T(\omega,\gamma)$ and $h(\omega,\gamma)$ are independent fuzzy variables. The components of $q(\omega,\gamma)$ are also independent fuzzy variables. Hence, the credibility constraint

$$\mathrm{Cr}\{W(\omega,\gamma)\boldsymbol{y} \geq h(\omega,\gamma) - T(\omega,\gamma)\boldsymbol{x}\} \geq \alpha$$

is equivalent to

$$\mathrm{Cr}\{\sum_{i=1}^{n_2} W_i(\omega,\gamma)y_i - h(\omega,\gamma) + \sum_{i=j}^{n_1} T(\omega,\gamma)x_i \geq 0\} \geq \alpha.$$

Since $\alpha > 0.5$, the credibility constraint is further equivalent to

$$\sum_{i=1}^{n_2}(W_i(\omega,\gamma))_{inf}(2-2\alpha)y_i + (-h(\omega,\gamma))_{inf}(2-2\alpha) + \sum_{j=1}^{n_1}(T_j(\omega,\gamma))_{inf}(2-2\alpha)x_j \geq 0,$$

where $(\cdot)_{inf}(\beta)$ denotes β pessimistic value of fuzzy variable.

Objective function

$$\mathrm{E}_\gamma[q(\omega,\gamma)^T\boldsymbol{y}]$$

is equivalent to

$$\sum_{k=1}^{n_2} \mathrm{E}_\gamma[q_k(\omega,\gamma)]y_k.$$

Therefore the second-stage programming problem (4) is translated into the following model

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & \sum_{k=1}^{n_2} \mathrm{E}_\gamma[q_k(\omega,\gamma)]y_k \\ \text{s.t.} & \sum_{i=1}^{n_2}(W_i(\omega,\gamma))_{inf}(2-2\alpha)y_i + (-h(\omega,\gamma))_{inf}(2-2\alpha) \\ & +\sum_{j=1}^{n_1}(T_j(\omega,\gamma))_{inf}(2-2\alpha)x_j \geq 0 \\ & \boldsymbol{y} \geq 0. \end{array}\right\} \tag{13}$$

It is a deterministic programming. So, problem (3)–(4) is translated into a two-stage stochastic programming problem (3)–(13) equivalently. The equivalent two-stage stochastic programming problem can be solved by some traditional solution methods, such as Bender's decomposition method.

If the random parameter ω is discrete and takes finite number of values, then the derived two-stage stochastic programming can usually be reduced equivalently to a deterministic programming model.

4.2 Hybrid Intelligent Algorithm

If the second-stage programming problem (4) cannot be translated into a deterministic programming, then approximation techniques, credibility simulation and expected value simulation will be used to evaluate the optimal value $Q(\boldsymbol{x},\omega)$ of the second-stage problem. We adapt the SAA method to evaluate the recourse function $\mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]$.

More precisely, let M be the number of the sample size, and $\hat{\omega}_1,\hat{\omega}_2,\ldots,\hat{\omega}_M$ the independent identically distributed sample of M realizations of random parameter ω. As a result, we can turn the original recourse function $\mathrm{E}_\omega[Q(\boldsymbol{x},\omega)]$ into SAA recourse function

$$\frac{1}{M}\sum_{i=1}^{M} Q(\boldsymbol{x},\hat{\omega}_i).$$

So, we can turn the original two-stage programming problem (3)–(4) into its associated SAA model

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & c^T\boldsymbol{x} + \frac{1}{M}\sum_{i=1}^{M} Q(\boldsymbol{x},\hat{\omega}_i) \\ \text{s.t.} & A\boldsymbol{x} = b \\ & \boldsymbol{x} \geq 0 \\ & \boldsymbol{x} \in K, \end{array}\right\} \tag{14}$$

where

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\hat{\omega}_i) = \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\hat{\omega}_i,\gamma)^T\boldsymbol{y}] \\ \text{s.t.} & \mathrm{Cr}\{W(\hat{\omega}_i,\gamma)\boldsymbol{y} \geq h(\hat{\omega}_i,\gamma) - T(\hat{\omega}_i,\gamma)\boldsymbol{x}\} \geq \alpha \\ & \boldsymbol{y} \geq 0. \end{array}\right\} \tag{15}$$

For simplicity, we denote fuzzy event $\{W(\hat{\omega}_i,\gamma)\boldsymbol{y} \geq h(\hat{\omega}_i,\gamma) - T(\hat{\omega}_i,\gamma)\boldsymbol{x}\}$ as $\{g(\xi(\hat{\omega}_i)) \geq 0\}$, and denote $q(\hat{\omega}_i,\gamma)^T\boldsymbol{y}$ as $f(\xi(\hat{\omega}_i))$. For any random realization $\hat{\omega}_i$ and $\boldsymbol{x}$, we approximate the continuous fuzzy vector

$\xi(\hat{\omega}_i)$ by a sequence $\{\zeta_n(\hat{\omega}_i)\}$ of discrete fuzzy vectors [12]. Let the discretization $\zeta_n(\hat{\omega}_i)$ takes on J values $\hat{\zeta}_n^j(\hat{\omega}_i)$ with possibility ν_j, $j = 1, 2, \ldots, J$. The approximating programming problem of model (15) is as follows

$$\left.\begin{array}{rl} Q(\boldsymbol{x}, \hat{\omega}_i) = \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[f(\zeta_n(\hat{\omega}_i))] \\ \text{s.t.} & \mathrm{Cr}\{g(\zeta_n(\hat{\omega}_i)) \geq 0\} \geq \alpha \\ & \boldsymbol{y} \geq 0. \end{array}\right\} \tag{16}$$

In order to check the credibilistic constraint in model (16), it is required to compute the credibility

$$\mathrm{Cr}\{g(\zeta_n(\hat{\omega}_i)) \geq 0\}.$$

Then

$$L = \frac{1}{2}\left(1 + \max\{\nu_j | g(\hat{\zeta}_n^j) \geq 0\} - \max\{\nu_j | g(\hat{\zeta}_n^j) < 0\}\right) = \mathrm{Cr}\{g(\zeta_n(\hat{\omega}_i)) \geq 0\} \tag{17}$$

is the estimation value of

$$\mathrm{Cr}\{W(\hat{\omega}_i, \gamma)\boldsymbol{y} \geq h(\hat{\omega}_i, \gamma) - T(\hat{\omega}_i, \gamma)\boldsymbol{x}\}$$

for any decision $\boldsymbol{y} \geq 0$. It is easy to know that $\boldsymbol{y} \geq 0$ is feasible if $L \geq \alpha$. Otherwise, $\boldsymbol{y}$ is unfeasible.

For any feasible decision $\boldsymbol{y}$, we estimate its objective value to obtain $Q(\boldsymbol{x}, \hat{\omega}_i)$. Write $f_j = f(\hat{\zeta}_n^j(\hat{\omega}_i))$ for $j = 1, 2, \ldots, J$. Rearrange the subscript j of ν_j and f_j such that $f_1 \leq f_2 \leq \cdots \leq f_J$. For $j = 1, 2, \ldots, J$, calculate the weights

$$p_j = \frac{1}{2}\left(\max_{j \leq i \leq J} \nu_i - \max_{j+1 \leq i \leq J+1} \nu_i\right) + \frac{1}{2}\left(\max_{1 \leq i \leq j} \nu_i - \max_{0 \leq i \leq j-1} \nu_i\right), \ j = 1, 2, \ldots, J$$

with $\nu_0 = \nu_{J+1} = 0$. After that, the expected value $\mathrm{E}_\gamma[f(\zeta_n(\hat{\omega}_i))]$ is calculated by the formula

$$P_n = \sum_{j=1}^{J} p_j f_j. \tag{18}$$

Consequently, one can estimate the expected value $\mathrm{E}_\gamma[f(\xi(\hat{\omega}_i))]$ by formula (18) provided n is sufficiently large.

Based on the above discussion, the solution process of the equilibrium two-stage programming can be summarized as Algorithm 1. About Particle Swarm Optimization, the interested reader can refer to [7].

Algorithm 1:	Hybrid Particle Swarm Optimization Algorithm for equilibrium two-stage programming
Step 1	Set the initial parameters.
Step 2	Randomly generate the initial feasible *Pop* particles (solutions).
Step 3	Randomly generate independent identically distributed sample $\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_M$ of M realizations of random parameter ω. The SAA model (14)-(15) is obtained.
Step 4	For any selected feasible solution $\boldsymbol{x}$, go to Step 5.
Step 5	For any realization $\hat{\omega}_i$, generate the discretization $\zeta_n(\hat{\omega}_i)$ which takes on J values $\hat{\zeta}_n^j(\hat{\omega}_i)$ with possibility ν_j, $j = 1, 2, \ldots, J$.
Step 6	Solve the approximating two-stage programming (16) based on (17) and (18).
Step 7	Repeat Step 5 to Step 6 for M cycles and obtain $Q(\boldsymbol{x}, \hat{\omega}_i)$, $i = 1, 2, \ldots, M$.
Step 8	Repeat Step 4 to Step 7 for *Pop* cycles.
Step 9	For each particle, compare the current objective value with that of its pbest. If the current objective value is smaller than that of pbest, then renew pbest and its objective value with the current position and objective value.
Step 10	Find the best particle of the current swarm with the smallest objective value. If the objective value is smaller than that of gbest, then renew gbest and its objective value with the position and objective value of the current best particle.
Step 11	Repeat Step 4 to Step 10 for a given number of cycles.
Step 12	Return the gbest and its objective value as the optimal solution and the optimal value.

5 Numerical Example

This section will give Example 2, in which EVPRI $\neq 0$ and VERS $= 0$, Example 3, in which EVPRI $= 0$ and VERS $\neq 0$, and Example 4, in which there are two objective functions.

Example 2. *Assume that the equilibrium two-stage programming problem is*

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & 2x_1 + 9x_2 + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)] \\ s.t. & x_1 + 3x_2 = 7 \\ & \boldsymbol{x} \geq 0 \\ & \boldsymbol{x} \in K, \end{array}\right\} \tag{19}$$

where

$$\left.\begin{array}{ll} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\omega,\gamma)y] \\ s.t. & \mathrm{Cr}\{W(\omega,\gamma)y \geq x_1\} \geq \alpha \\ & y \geq 0. \end{array}\right\} \tag{20}$$

In model (19)-(20), the credibility level $\alpha = 0.8$, the uncertain parameters $q(\omega,\gamma) = (1,2,3)+\omega$ and $W(\omega,\gamma) = \omega(2,3,4)$ are triangular fuzzy random variables, where

$$\omega \sim \begin{pmatrix} 1 & 2 \\ 0.3 & 0.7 \end{pmatrix}.$$

Find the wait-and-see solution, here-and-now solution and expected value solution.

Note that the credibility level $\alpha = 0.8 > 0.5$ and $W(\omega,\gamma) = \omega(2,3,4)$, we have

$$\mathrm{Cr}\{W(\omega,\gamma)y \geq x_1\} \geq \alpha \Longleftrightarrow x_1 \leq (4-2\alpha)\omega y$$

for any given $\omega > 0$ and $y > 0$. Also, we have $\mathrm{E}_\gamma[q(\omega,\gamma)y] = (2+\omega)y$ since $q(\omega,\gamma) = (1,2,3)+\omega$.

If $\omega = 1$, then model (19)-(20) can be rewritten as

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x},y} & 2x_1 + 9x_2 + 3y \\ s.t. & x_1 + 3x_2 = 7 \\ & y \geq \frac{x_1}{4-2\alpha} \\ & \boldsymbol{x} \geq 0, y \geq 0 \\ & \boldsymbol{x} \in K. \end{array}\right\}$$

It is obviously that

$$Q(\boldsymbol{x},1) = \frac{3x_1}{4-2\alpha}.$$

Therefore, under scenario $\omega = 1$, the optimal value of equilibrium two-stage model is

$$z\left(\left(0,\frac{7}{3}\right)^T,1\right) = 21.$$

If $\omega = 2$, model (19)-(20) is rewritten as

$$\left.\begin{array}{ll} \min\limits_{,y} & 2x_1 + 9x_2 + 4y \\ s.t. & x_1 + 3x_2 = 7 \\ & y \geq \frac{x_1}{8-4\alpha} \\ & \boldsymbol{x} \geq 0, y \geq 0 \\ & \boldsymbol{x} \in K. \end{array}\right\}$$

It is easy to know that

$$Q(\boldsymbol{x},2) = \frac{x_1}{2-\alpha}.$$

Consequently, corresponding to scenario $\omega = 2$, the optimal value of equilibrium two-stage model is

$$z\left((7,0)^T,2\right) = \frac{119}{6}.$$

In summary, according to the definition, the wait-and-see solution is

$$\mathrm{WS} = \mathrm{E}_\omega[\min_{\boldsymbol{x}} z(x,\omega)] = \mathrm{E}_\omega[z(\bar{x}(\omega),\omega)] = \frac{1211}{60}.$$

Based on the definition

$$\mathrm{RP} = \min_{\boldsymbol{x}} \mathrm{E}_\omega[z(x,\omega)],$$

we have

$$\left.\begin{array}{rl} \mathrm{RP} = \min\limits_{\boldsymbol{x}} & 2x_1 + 9x_2 + 0.3Q(\boldsymbol{x},1) + 0.7Q(\boldsymbol{x},2) \\ s.t. & x_1 + 3x_2 = 7 \\ & \boldsymbol{x} \geq 0 \end{array}\right\}$$

with the optimal solution $x_1 = 7,\ x_2 = 0$. Accordingly the here-and-now solution is

$$\mathrm{RP} = \frac{497}{24}.$$

Substituting $\bar{\omega} = 1.7$ into model (19)-(20), then we have the expected value problem

$$\left.\begin{array}{rl} \min\limits_{\boldsymbol{x}} & 2x_1 + 9x_2 + 3.7y \\ s.t. & x_1 + 3x_2 = 7 \\ & y \geq \frac{x_1}{2.4*1.7} \\ & \boldsymbol{x} \geq 0. \end{array}\right\} \tag{21}$$

The optimal solution of model (21) is $\bar{x}(\bar{\omega}) = (7,0)^T$.

For $\bar{x}(\bar{\omega}) = (7,0)^T$, model (19)-(20) can be rewritten as

$$\min \quad 14 + \mathrm{E}_\omega[Q(\bar{x}(\bar{\omega}),\omega)] \tag{22}$$

where

$$\left.\begin{array}{rl} Q(\bar{x}(\bar{\omega}),\omega) = \min\limits_{y} & (2+\omega)y \\ s.t. & y \geq \frac{7}{2.4} \\ & y \geq 0. \end{array}\right\} \tag{23}$$

It is obviously that the optimal values of (22)-(23) under different scenarios are

$$z(\bar{x}(\bar{\omega}),1) = 14 + \frac{21}{2.4},\ z(\bar{x}(\bar{\omega}),2) = 14 + \frac{28}{4.8}.$$

According to the definition $\mathrm{EEV} = \mathrm{E}_\omega[z(\bar{x}(\bar{\omega}),\omega)]$, *the expected result is*

$$\mathrm{EEV} = 0.3z(\bar{x}(\bar{\omega}),1) + 0.7z(\bar{x}(\bar{\omega}),2) = \frac{497}{24}.$$

Example 3. *Assume that the equilibrium two-stage programming problem is*

$$\left.\begin{array}{rl} \min\limits_{\boldsymbol{x}} & x_1 + 4x_2 + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)] \\ s.t. & x_1 + x_2 = 1 \\ & \boldsymbol{x} \geq 0 \\ & \boldsymbol{x} \in K, \end{array}\right\} \tag{24}$$

where

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[\frac{10}{4} + 11y_1 + 20y_2 - 10x_1 + 20x_2 - 10q(\omega,\gamma)] \\ s.t. & \mathrm{Cr}\{-x_1 + 2x_2 + y_1 + y_2 + \frac{6}{5} \geq q(\omega,\gamma)\} \geq \alpha \\ & 0 \leq y_1 \leq 2 \\ & y_2 \geq 0. \end{array}\right\} \tag{25}$$

In model (24)-(25), the credibility level $\alpha = 0.8$, the uncertain parameters $q(\omega,\gamma) = (\omega-1, \omega, \omega+2)$ is triangular fuzzy random variables, where $\omega \sim U(1,3)$ Find the here-and-now solution, wait-and-see solution and expected value solution.

Note that the credibility level $\alpha = 0.8 > 0.5$, then we have

$$\mathrm{Cr}\{-x_1 + 2x_2 + y_1 + y_2 + \frac{6}{5} \geq q(\omega,\gamma)\} \geq 0.8 \Longleftrightarrow y_1 + y_2 \geq \omega + x_1 - 2x_2$$

for any given $\omega \in [1,3]$ and $\boldsymbol{x} \geq 0$. Introducing surplus variable $y_3 \geq 0$, this credibilistic constraint is equivalently rewritten as

$$y_1 + y_2 - y_3 = \omega + x_1 - 2x_2.$$

We also have

$$\mathrm{E}_\gamma[\tfrac{10}{4} + 11y_1 + 20y_2 - 10x_1 + 20x_2 - 10q(\omega,\gamma)] = y_1 + 10y_2 + 10y_3.$$

Then the equilibrium two-stage programming problem (24)-(25) is translated into the following two-stage stochastic programming problem

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & x_1 + 4x_2 + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)] \\ s.t. & x_1 + x_2 = 1 \\ & \boldsymbol{x} \geq 0 \\ & \boldsymbol{x} \in K, \end{array}\right\} \tag{26}$$

where

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & y_1 + 10y_2 + 10y_3 \\ s.t. & y_1 + y_2 - y_3 = \omega + x_1 - 2x_2 \\ & 0 \leq y_1 \leq 2 \\ & y_2, y_3 \geq 0. \end{array}\right\} \tag{27}$$

Substituting $\bar\omega = 2$ into model (26)-(27), we have $\bar x(\bar\omega) = (0,1)^T$ and $EV = 4$.

Using $\bar x(\bar\omega) = (0,1)^T$ in model (26)-(27), one has

$$\min_{\boldsymbol{x}} \quad 4 + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)] \tag{28}$$

where

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & y_1 + 10y_2 + 10y_3 \\ s.t. & y_1 + y_2 - y_3 = \omega - 2 \\ & 0 \leq y_1 \leq 2 \\ & y_2, y_3 \geq 0. \end{array}\right\} \tag{29}$$

The optimal value of model (28)-(29) is $z(\bar x(\bar\omega),\omega) = 24 - 10\omega$ when $\omega \in [1,2]$, and $z(\bar x(\bar\omega),\omega) = \omega + 2$ when $\omega \in [2,3]$. Hence the expected value solution of model (24)-(25) is

$$\mathrm{EEV} = \frac{27}{4}.$$

Model (26)-(27) is simplified as

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & 4 - 3x_1 + \mathrm{E}_\omega[Q(\boldsymbol{x},\omega)] \\ s.t. & 0 \leq x_1 \leq 1 \end{array}\right\} \tag{30}$$

where

$$\left.\begin{array}{rl} Q(\boldsymbol{x},\omega) = \min\limits_{\boldsymbol{y}} & y_1 + 10y_2 + 10y_3 \\ s.t. & y_1 + y_2 - y_3 = \omega - 2 + 3x_1 \\ & 0 \leq y_1 \leq 2 \\ & y_2, y_3 \geq 0. \end{array}\right\} \tag{31}$$

In the first case of $x_1 \in [0,1/3]$, the optimal value of (31) is $10(-\omega + 2 - 3x_1)$ for $\omega \in [1, 2-3x_1]$, and the optimal value of (31) is $\omega - 2 + 3x_1$ for $\omega \in [2-3x_1, 3]$.

In the second case of $x_1 \in [1/3,1]$, the optimal value of (31) is $\omega - 2 + 3x_1$ for $\omega \in [1, 4-3x_1]$, and the optimal value of (31) is $10\omega - 38 + 30x_1$ for $\omega \in [4-3x_1, 3]$.

Now model (30)-(31) can be decomposed into two subproblems, in which only use $0 \le x_1 \le 1/3$ *and* $1/3 \le x_1 \le 1$ *to replace* $0 \le x_1 \le 1$*, respectively. The subproblems are equivalent to*

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \frac{99x_1^2-66x_1+27}{4} \\ s.t. & 0 \le x_1 \le \frac{1}{3}, \end{array}\right\}$$

and

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \frac{81x_1^2-54x_1+25}{4} \\ s.t. & \frac{1}{3} \le x_1 \le 1. \end{array}\right\}$$

Therefore, we have the optimal solution $x_1^* = 1/3$*, and the here-and-now solution*

$$\mathrm{RP} = 4.$$

For any realization of ω*, model (30)-(31) is rewritten as*

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{y}} & -3x_1 + y_1 + 10y_2 + 10y_3 + 4 \\ s.t. & -3x_1 + y_1 + y_2 - y_3 = \omega - 2 \\ & 0 \le x_1 \le 1 \\ & 0 \le y_1 \le 2 \\ & y_2, y_3 \ge 0. \end{array}\right\} \tag{32}$$

Solving model (32) for any realization of ω*, we have the optimal value* $\omega + 2$*. Hence, the wait-and-see solution of model (24)-(25) is*

$$\mathrm{WS} = 4.$$

Example 4. *Assume that the two-stage problem is defined as*

$$\left.\begin{array}{ll} \min\limits_{x} & 5x + \mathrm{E}_\omega[Q(x,\omega)] \\ \min\limits_{x} & \mathrm{V}_\omega[Q(x,\omega)] \\ s.t. & x \ge 0 \\ & x \in K, \end{array}\right\} \tag{33}$$

where $Q(\boldsymbol{x}, \omega)$ *denotes the second-stage value function, that is to say,*

$$\left.\begin{array}{rl} Q(x,\omega) = \min\limits_{\boldsymbol{y}} & \mathrm{E}_\gamma[q(\omega,\gamma)^T y] \\ s.t. & \mathrm{Cr}\{y \le T(\omega,\gamma)\} \ge 0.8 \\ & \mathrm{Cr}\{y \ge h(\omega,\gamma) + x\} \ge 0.9 \\ & y \ge 0. \end{array}\right\} \tag{34}$$

In model (33)-(34), $\mathrm{Pr}\{\omega = \omega_1 = 1\} = 0.4$*,* $\mathrm{Pr}\{\omega = \omega_2 = 2\} = 0.6$*,* $q(\omega,\gamma) = (0,1,2)\omega$*,* $T(\omega,\gamma) = \omega + (5,6,7)$ *and* $h(\omega,\gamma) = (2,3.5,4,5) - \omega$ *are trapezoidal fuzzy random variables.*

The objective function in the second-stage problem is rewritten as $\mathrm{E}_\gamma[q(\omega,\gamma)^T \boldsymbol{y}] = \omega y$*. The constraint* $\mathrm{Cr}\{\boldsymbol{y} \le T(\omega,\gamma)\} \ge 0.8$ *is equivalent to* $y \le 5.4 + \omega$*, constraint* $\mathrm{Cr}\{\boldsymbol{y} \ge h(\omega,\gamma) + \boldsymbol{x}\} \ge 0.9$ *is equivalent to* $y \ge 4.8 - \omega + x$*.*

When $\omega_1 = 1$*, we have* $y \le 6.4$ *and* $y \ge 3.8 + x$*. When* $\omega_2 = 2$*, we have* $y \le 7.4$ *and* $y \ge 2.8 + x$*. As a consequence,*

$$K = \{x | x \le 2.6\}.$$

For any given x *and* $\omega_1 = 1$*, the optimal value in the second-stage is*

$$Q(x, \omega_1) = 3.8 + x.$$

Similarly, for any given x *and* $\omega_2 = 2$*, the optimal value in the second-stage problem is*

$$Q(x, \omega_2) = 5.6 + 2x.$$

Consequently,

$$\mathrm{E}_\omega[Q(x,\omega)] = 0.4Q(x,\omega_1) + 0.6Q(x,\omega_2) = 4.88 + 1.6x.$$

$$\mathrm{V}_{\omega}[Q(x,\omega)] = 0.4(1.08+0.6x)^2 + 0.6(0.72+0.4x)^2.$$

The equivalent static programming problem is

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & 4.88+6.6x \\ \min\limits_{\boldsymbol{x}} & 0.4(1.08+0.6x)^2+0.6(0.72+0.4x)^2 \\ s.t. & 0 \le x \le 2.6. \end{array}\right\}$$

Under different values of parameter λ, we solve the following programming problem

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \lambda(4.88+6.6x)+(1-\lambda)(0.4(1.08+0.6x)^2+0.6(0.72+0.4x)^2) \\ s.t. & 0 \le x \le 2.6. \end{array}\right\}$$

We get the Pareto here-and-now solutions RP_λ *under weighted coefficient λ, some of them are shown in Table 1.*

Table 1: The solution results under different weighted coefficient λ

λ	WS_λ	RP_λ	EEV_λ
0.0	0	0.7776	0.7776
0.1	0.488	1.18784	1.18784
0.2	0.976	1.59808	1.59808
0.3	1.464	2.00832	2.00832
0.4	1.952	2.41856	2.41856
0.5	2.44	2.8288	2.8288
0.6	2.928	3.23904	3.23904
0.7	3.416	3.64928	3.64928
0.8	3.904	4.05952	4.05952
0.9	4.392	4.46976	4.46976
1.0	4.88	4.88	4.88

Since the expected value of random parameter ω is $\bar{\omega} = 1.6$, the expected value problem is

$$\left.\begin{array}{ll} \min\limits_{\boldsymbol{x}} & 5x+1.6y \\ s.t. & 0 \le x \le 2.6 \\ & y \ge 3.2+x \\ & 0 \le y \le 7. \end{array}\right\}$$

Submitting its optimal solution $\bar{x}(\bar{\omega}) = 0$ into model (33)-(34), we have the following problem

$$\left.\begin{array}{ll} \min\limits_{x} & \mathrm{E}_{\omega}[Q(0,\omega)] \\ \min\limits_{x} & \mathrm{V}_{\omega}[Q(0,\omega)] \end{array}\right\}$$

where

$$Q(0,\omega) = \left.\begin{array}{ll} \min\limits_{\boldsymbol{y}} & \omega y \\ s.t. & y \le 5.4+\omega \\ & y \ge 4.8-\omega \\ & y \ge 0. \end{array}\right\}$$

Since $Q(0,\omega_1) = 3.8$ and $Q(0,\omega_2) = 5.6$, the Pareto expected value solution under weighted coefficient λ is $\mathrm{EEV}_\lambda = 0.7776 + 4.1024\lambda$*, some of them are shown in Table 1.*

Finally, since

$$z(x,\omega_1) = \left.\begin{array}{ll} \min\limits_{\boldsymbol{y}} & 5x+y \\ s.t. & 3.8+x \le y \le 6.4 \\ & 0 \le x \le 2.6, \end{array}\right\}$$

and

$$\left.\begin{array}{rll} z(x,\omega_2) = \min\limits_{\boldsymbol{y}} & 5x+2y \\ s.t. & 2.8+x \le y \le 7.4 \\ & 0 \le x \le 2.6, \end{array}\right\}$$

then $z(x,\omega_1) = 3.8$ *and* $z(x,\omega_2) = 5.6$*. As a consequence, the Pareto wait-and-see solution under weighted coefficient* λ *is* $\mathrm{WS}_\lambda = 4.88\lambda$*, some of them are shown in Table 1.*

6 Conclusions

The major new results in this study can be summarized as follows:

Based on credibility theory and probability theory, a new single objective equilibrium programming model, called equilibrium two-stage programming, was proposed. The wait-and-see solution, here-and-now solution and expected value solution were defined, respectively. A new bi-objective equilibrium programming model was also developed. The model's Pareto wait-and-see solution, Pareto here-and-now solution and Pareto expected value solution under weighted coefficient were defined, respectively. The proposed optimization methodology can address twofold uncertainty arisen in hybrid uncertain decision-making systems.

The relations among the wait-and-see solution, here-and-now solution and expected value solution were discussed (Proposition 1). The wait-and-see solution is the smallest, and the expected value solution is the largest. These results illustrate the expected value of perfect random information and the value of equilibrium recourse solution. Some numerical examples were provided to illustrate the relations.

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