Journal of Uncertain Systems
Vol.12, No.3, pp.208-214, 2018
Online at: www.jus.org.uk

In Fuzzy Decision Making, General Fuzzy Sets can be Replaced by Fuzzy Numbers

Christian Servin[1], Olga Kosheleva[2], Vladik Kreinovich[3,*]

[1] *Computer Science and Information Technology Systems Department, El Paso Community College*
919 Hunter, El Paso, TX 79915, USA

[2] *Department of Teacher Education, University of Texas at El Paso*
500 W. University, El Paso, Texas 79968, USA

[3] *Department of Computer Science, University of Texas at El Paso*
500 W. University, El Paso, Texas 79968, USA

Received 1 March 2017; Revised 21 March 2017

Abstract

In many real decision situations, for each of the alternatives, we only have fuzzy information about the consequences of each action. This fuzzy information can be described by a fuzzy number, i.e., by a membership function with a single local maximum, or it can be described by a more complex fuzzy set, with several local maxima. We show that, from the viewpoint of decision making, it is sufficient to consider only fuzzy numbers. To be more precise, the decisions will be the same if we replace each original fuzzy set with the smallest fuzzy number of all fuzzy numbers of which the original fuzzy set is a subset.

Keywords: decision making, fuzzy uncertainty, fuzzy numbers

1 Formulation of the Problem

Decision making: deterministic case. How do we make decisions? Let us start with the simplest case, when the outcome is the amount of money. A typical such situation is an auction:

- we have an item that we want to sell, and
- we have several possible buyers who propose different prices.

In this case, it is easy to decide which buyer to select: the one who proposes the largest amount of money.

In particular, for two buyers, it is easy to decide which of them provides a better alternative:

- if the amount of money a corresponding to the alternative of selecting the first buyer is larger than the amount of money b corresponding to the alternative of selecting the second buyer ($a \geq b$), then the first alternative is better;
- on the other hand, if the amount of money b corresponding to the alternative of selecting the second buyer is larger than the amount of money a corresponding to the alternative of selecting the first buyer ($b \geq a$), then the second alternative is better.

In the non-financial situations, we can use a similar comparison of two numbers, because it is known (see, e.g., [1, 3, 4, 6]) that decisions of a rational person can be described as maximizing a certain quantity called *utility*.

In practice, we have uncertainty. In most real-life situations, we do not know the exact consequences of each action. For each action a, we can, in principle, get different income values (or, more generally, different values of utility).

*Corresponding author.
Emails: cservin@gmail.com (C. Servin), olgak@utep.edu (O. Kosheleva), vladik@utep.edu (V. Kreinovich).

In many cases, the only information that we have about possible outcomes come from the experts, and the experts often formulate their knowledge by using imprecise ("fuzzy") words from natural language such as "small", "approximately equal to", etc. To formalize this knowledge, it is reasonable to use techniques that were specifically designed for describing this knowledge: namely, fuzzy techniques; see, e.g., [2, 5, 7].

In this description, the consequences of a possible action a are characterized by a *membership function* μ_a that assigns, to each possible outcome x, the degree $\mu_a(x)$ to which this value is possible. A membership function is also known as a *fuzzy set*.

Usually, small changes in the quantity x lead only to small changes in the expert's degree of confidence, so the membership functions are usually continuous.

How do we make decisions under such fuzzy uncertainty? A natural question is: if we have two such fuzzy alternatives, how do we make a decision?

In the next section, we propose a natural formula for this decision, and in Section 3, we show that under this natural formula, there is no need to consider general fuzzy sets, it is sufficient to consider *fuzzy numbers*, i.e., membership functions that first increase to 1 and then decrease back to 0.

2 Analysis of the Problem

Let us start with reminders. To come up with a natural formula, let us start with a few reminders.

Fuzzy logic: a brief reminder. First, let us recall what is fuzzy logic. In the standard true-false logic, if for each of the two statements A and B, we know whether each of these statements is true or false, then we can determine whether the composite statements $A\,\&\,B$ and $A \vee B$ are true or false.

In the fuzzy case, when we have general degrees of certainty, it is not sufficient to know:

- the degree of certainty $a = d(A)$ in a statement A and
- the degree of certainty $b = d(B)$ in a statement B

to determine the degree of certainty of $A\,\&\,B$ or $A \vee B$.

For example, let S_1 be "coin falls heads" and S_2 be "coin falls tail". Then, it is reasonable to take $s_1 = d(S_1) = 0.5$ and $s_2 = d(S_2) = 0.5$. Let us now consider two pairs (A, B):

- If we take $A = S_1$ and $B = S_2$, then $A\,\&\,B$ is impossible, so our degree of belief in $A\,\&\,B$ is
$$d(A\,\&\,B) = 0.$$
- On the other hand, if we take $A = B = S_1$, we also have
$$d(A) = d(B) = 0.5,$$
but, since in this case $A\,\&\,B$ is simply equivalent to A, we have
$$d(A\,\&\,B) = d(A) = 0.5.$$

In both cases, we have the same values of $d(A) = 0.5$ and $d(B) = 0.5$, but we have different values of $d(A\,\&\,B)$:

- $d(A\,\&\,B) = 0$ in the first case, and
- $d(A\,\&\,B) = 0.5 > 0$ in the second case.

So, ideally, if we want to know the expert's degree of certainty in different propositional combinations of the original statements, we have to ask the expert about these combinations one by one.

The problem is that the number of such propositional combinations grows exponentially with the number of statements in the knowledge base. As a result, even for a reasonable size knowledge base, with a few hundred statement, the number of possible combinations becomes astronomical – and it is not realistic to ask billions of question to the expert.

Since we cannot elicit the expert's degree of certainty in each composite statement, we need to be able to estimate the degree of confidence of a composite statement based on the expert's degrees of confidence in each individual statement. In other words, we need an algorithm $f_\&(a,b)$ that, given the expert's degrees of confidence a and b in individual statements A and B, provides an estimate $f_\&(a,b)$ of the expert's degree of confidence in the composite statement $A \,\&\, B$.

Since the composite statement $A \,\&\, B$ implies both A and B, our degree of confidence in $A \,\&\, B$ cannot exceed the degrees of confidence a and b in statements A and B. Thus, we must have $f_\&(a,b) \le a$ and $f_\&(a,b) \le b$.

What is the simplest operation with this property? In the computer, hardware supported operations are, in increasing order of complexity:

- computing min and max – which do not require any arithmetic operations at all,
- addition and subtraction,
- multiplication – which, as when we do it by hand, is implemented by performing several additions, and
- division, which, similar to the way we do it by hand, is implemented by performing several multiplications.

The simplest possible operations are min and max. Out of these two operations, only $f_\&(a,b) = \min(a,b)$ satisfied the inequalities $f_\&(a,b) \le a$ and $f_\&(a,b) \le b$. Thus, the simplest possible "and"-operation is

$$f_\&(a,b) = \min(a,b).$$

Similarly, we need an algorithm $f_\vee(a,b)$ that, given the expert's degrees of confidence a and b in individual statements A and B, provides an estimate $f_\vee(a,b)$ of the expert's degree of confidence in the composite statement $A \vee B$.

Since each of the statements A and B implies the composite statement $A \vee B$, our degree of confidence in $A \vee B$ cannot be smaller than the degrees of confidence a and b in statements A and B. Thus, we must have $f_\vee(a,b) \ge a$ and $f_\vee(a,b) \ge b$.

What is the simplest operation with this property? As we have mentioned earlier, the simplest possible operations are min and max. Out of these two operations, only $f_\vee(a,b) = \max(a,b)$ satisfied the inequalities $f_\vee(a,b) \ge a$ and $f_\vee(a,b) \le b$. Thus, the simplest possible "or"-operation is

$$f_\vee(a,b) = \max(a,b).$$

Propagating fuzzy uncertainty through algorithms: Zadeh's extension principle.

- Suppose that we know the relation $y = f(x_1, \cdots, x_n)$ between a quantity y and quantities $x_1, \ldots, x_n$, and
- suppose that for each i from 1 to n, we know the corresponding membership function $\mu_i(x_i)$ that describes the expert's information about x_i.

What can we then say about y?

Of course, it is, in principle, possible that we do not know anything about one of the inputs. In this case, all values of x_i are equally possible, so we have $\mu_i(x_i) = 1$ for all possible values x_i. In such situations, we cannot conclude anything about y. The above problem makes sense only if we know approximate value of each input, i.e., if, as each x_i increases or decreases, the corresponding degree of possibility drops to 0:

$$\lim_{x_i \to +\infty} \mu(x_i) = 0 \text{ and } \lim_{x_i \to -\infty} \mu_i(x_i) = 0.$$

In the following text, we will only consider such membership functions.

For each possible value of y, we want to find the degree $\mu(y)$ to which this value is possible. For the same y, we have several tuples $(x_1, \cdots, x_n)$ for which $y = f(x_1, \cdots, x_n)$. Thus, the given number is a possible value of the quantity y if:

- either for one of these tuples $(x_1, \cdots, x_n)$,
  - x_1 is a possible value of the first input,

  - x_2 is a possible value of the second input,
  - $\ldots$, and
  - x_n is a possible value of the n-th input,

- or for the second of these tuples $(x_1, \cdots, x_n)$,
  - x_1 is a possible value of the first input,
  - x_2 is a possible value of the second input,
  - $\ldots$, and
  - x_n is a possible value of the n-th input,
- etc.

For each i and for each x_i, we know the degree $\mu_i(x_i)$ to which this value x_i is a possible value of the i-th input. So, if we use the above-described simplest "and"- and "or"-operations, then:

- for each tuple $(x_1, \cdots, x_n)$, the degree to which
  - x_1 is a possible value of the first input,
  - x_2 is a possible value of the second input,
  - $\ldots$, and
  - x_n is a possible value of the n-th input,

  is equal to
  $$\min(\mu_1(x_1), \cdots, \mu_n(x_n)),$$
  and
- the degree to which y is possible is equal to the maximum of such degrees over all such tuples:
$$\mu(y) = \max(\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : f(x_1, \ldots, x_n) = y).$$

This formula was originally proposed by Zadeh; it is known as *Zadeh's extension principle*; see, e.g., [2].

Zadeh's extension principle in terms of α-cuts: a brief reminder. It is known (see, e.g., [2, 5]) that from the computational viewpoint, Zadeh's extension principle becomes much easier if we describe it in terms of the corresponding *alpha-cuts*, i.e., sets

$$\mathbf{x_i}(\alpha) \stackrel{\text{def}}{=} \{x_i : \mu_i(x_i) \geq \alpha\}$$

and
$$\mathbf{y}(\alpha) \stackrel{\text{def}}{=} \{y : \mu(y) \geq \alpha\}.$$

Indeed, according to the above formula, $\mu(y)$ is greater than of equal to α if and only if there exists a tuple $(x_1, \cdots, x_n)$ for which $y = f(x_1, \cdots, x_n)$ and

$$\min(\mu_1(x_1), \cdots, \mu(x_n)) \geq \alpha.$$

This inequality is, in turn, equivalent to having $\mu_i(x_i) \geq \alpha$ for all i, i.e., to $x_i \in \mathbf{x}_i(\alpha)$ for all i. Thus,

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \cdots, \mathbf{x}_n(\alpha)),$$

where for every n sets $X_1, \ldots, X_n$, the set $f(X_1, \cdots, X_n)$ denotes the range

$$f(X_1, \cdots, X_n) \stackrel{\text{def}}{=} \{f(x_1, \cdots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

Comment. In the following text, it will be important to know that for continuous membership functions, α-cuts are closed sets, i.e., sets that contain all their limit points.

Let us apply Zadeh's extension principle to decision making. In our case, we have two membership functions $\mu_a(x)$ and $\mu_b(x)$, and we are interested in describing to what extend $a \ge b$. Here, we have two inputs $n = 2$, and the corresponding algorithm $f(x_1, x_2)$ simply returns "true" ($= 1$) or "false" ($= 0$) depending on whether $x_1 \ge x_2$ or not.

Thus, the degree $d(a \ge b)$ to which $a \ge b$ can be computed as follows:

$$d(a \ge b) = \max(\min(\mu_a(x_1), \mu_b(x_2)) : x_1 \ge x_2).$$

This is is the main formula that we will use to describe fuzzy decision making.

Comment. In the precise case, if we have $a \ge b$, then the only possibility to have $b \ge a$ is when $a = b$. In contrast, in the fuzzy case, we can have both $a \ge b$ and $b \ge a$ to some degree without having $a = b$.

Thus, in the fuzzy case, to make a decision:

- it is not sufficient to know the degree $d(a \ge b)$ to which a is greater than or equal to b,
- we also need to know the degree $d(b \ge a)$ to which b is greater than or equal to a.

3 Main Result: In Fuzzy Decision Making, General Fuzzy Sets Can be Replaced by Fuzzy Numbers

Let us reduce the problem to α-cuts. According to the above formula, $d(a \ge b) \ge \alpha$ if and only if there exists a pair (x_1, x_2) for which $x_1 \ge x_2$ and $\min(\mu_a(x_1), \mu_b(x_2)) \ge \alpha$. This inequality, in its turn, means that $\mu_a(x_1) \ge \alpha$ and $\mu_b(x_2) \ge \alpha$, i.e., that $x_1 \in \mathbf{a}(\alpha)$ and $x_2 \in \mathbf{b}(\alpha)$, where

$$\mathbf{a}(\alpha) \stackrel{\text{def}}{=} \{x : \mu_a(x) \ge \alpha\}$$

and

$$\mathbf{b}(\alpha) \stackrel{\text{def}}{=} \{x : \mu_b(x) \ge \alpha\}.$$

In other words:

$$d(a \ge b) \ge \alpha \Leftrightarrow \exists x_1 \in \mathbf{a}(\alpha)\, \exists x_2 \in \mathbf{b}(\alpha)\, (x_1 \ge x_2).$$

Let us analyze the result. Since the membership functions $\mu_a(x)$ and $\mu_b(x)$ both tend to 0 as x tends to plus or minus infinity, for all $\alpha > 0$, the α-cuts are bounded sets.

For the first fussy set $\mu_a(x)$:

- let $\underline{a}(\alpha)$ denote the greatest lower bound (infimum) of the alpha-cut $\mathbf{a}(\alpha)$, and
- let $\overline{a}(\alpha)$ denote the least upper bound (supremum) of this alpha-cut.

Similarly, for the second fuzzy set $\mu_b(x)$:

- let $\underline{b}(\alpha)$ denote the greatest lower bound (infimum) of the alpha-cut $\mathbf{b}(\alpha)$, and
- let $\overline{b}(\alpha)$ denote the least upper bound (supremum) of this alpha-cut.

Since the membership functions are continuous, the alpha-cuts are closed and thus, contain the corresponding bounds:

$$\underline{a}(\alpha) \in \mathbf{a}(\alpha), \quad \overline{a}(\alpha) \in \mathbf{a}(\alpha), \quad \underline{b}(\alpha) \in \mathbf{b}(\alpha), \quad \overline{b}(\alpha) \in \mathbf{b}(\alpha).$$

Let us now show that

$$d(a \ge b) \ge \alpha \Leftrightarrow \overline{a}(\alpha) \ge \underline{b}(\alpha).$$

Indeed, if $\overline{a}(\alpha) \ge \underline{b}(\alpha)$, then we have $x_1 \in \mathbf{a}(\alpha)$ and $x_2 \in \mathbf{b}(\alpha)$ for which $x_1 \ge x_2$: namely, we have $x_1 = \overline{a}(\alpha)$ and $x_2 = \underline{b}(\alpha)$. Thus, by the last formula of the previous subsection, we have $d(a \ge b) \ge \alpha$.

Vice versa, suppose that $d(a \geq b) \geq \alpha$. This means that there exist $x_1 \in \mathbf{a}(\alpha)$ and $x_2 \in \mathbf{b}(\alpha)$ for which $x_1 \geq x_2$. Since $\overline{a}(\alpha)$ is the least upper bound (supremum) of the set $\mathbf{a}(\alpha)$, we have $\overline{a}(\alpha) \geq x_1$. Similarly, since $\underline{b}(\alpha)$ is the greatest lower bound (infimum) of the set $\mathbf{b}(\alpha)$, we have $x_2 \geq \underline{b}(\alpha)$. From

$$\overline{a}(\alpha) \geq x_1 \geq x_2 \geq \underline{b}(\alpha),$$

we can now conclude that

$$\overline{a}(\alpha) \geq \underline{b}(\alpha).$$

The equivalence has been proven.

This proves our main result. What we have proved, in effect, is as follows:

- We have started with the general membership functions $\mu_a(x)$ and $\mu_b(x)$ (which are not necessarily fuzzy sets).
- We have shown that for these two general membership functions, the degree $d(a \geq b)$ would remain the same if:
  - instead of the original membership functions,
  - we consider the corresponding fuzzy numbers,

  namely, the fuzzy numbers A and B for which,
  - for all α,
  - the corresponding α-cut is the interval $[\underline{a}(\alpha), \overline{a}(\alpha)]$ or, correspondingly, $[\underline{b}(\alpha), \overline{b}(\alpha)]$.

Conclusion. Thus, indeed, *in fuzzy decision making, general fuzzy sets can be replaced by fuzzy numbers.*

Comment. One can easily show that for each membership function $\mu_a(x)$, the corresponding fuzzy number $\mu_A(x)$ can be obtained in one of the two ways:

- we can describe $\mu_A(x)$ as the smallest fuzzy number of which the original fuzzy set is a subset, i.e., for which

  $$\mu_a(x) \leq \mu_A(x) \text{ for all } x,$$

  or
- we can describe $\mu_A(x)$ explicitly, as

  $$\mu_A(x) = \min\left(\max_{y \leq x} \mu_a(y), \max_{y \geq x} \mu_a(y)\right).$$

For example, the first equivalence comes from the fact that for two fuzzy sets A and B,

- A is a subset of B if and only
- each alpha-cut of A is a subset of the corresponding alpha-cut of B.

A fuzzy set if a fuzzy number if and only if its alpha-cuts are intervals. Thus:

- the smallest fuzzy number of which the original fuzzy set a is a subset means that
- for each α, we have the smallest of all intervals that contain the alpha-cut $\mathbf{a}(\alpha)$.

Of course, each such interval should contain the infimum and the supremum points and thus, contain the whole interval $[\underline{a}(\alpha), \overline{a}(\alpha)]$.

Clearly, this interval itself is the smallest of all such intervals – which proves the equivalence.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

References

[1] Fishburn, P.C., *Utility Theory for Decision Making*, John Wiley & Sons Inc., New York, 1969.

[2] Klir, G., and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[3] Luce, R.D., and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.

[4] Nguyen, H.T., Kosheleva, O., and V. Kreinovich, Decision making beyond Arrow's 'impossibility theorem', with the analysis of effects of collusion and mutual attraction, *International Journal of Intelligent Systems*, vol.24, no.1, pp.27–47, 2009.

[5] Nguyen, H.T., and E.A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.

[6] Raiffa, H., *Decision Analysis*, Addison-Wesley, Reading, Massachusetts, 1970.

[7] Zadeh, L.A., Fuzzy sets, *Information and Control*, vol.8, pp.338–353, 1965.