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Safe Tractable Approximations of Ambiguous Expected Model with Chance Constraints and Applications in Transportation Problem

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Abstract

This paper studies an ambiguous expected model with chance constraints, which assumes limited information about the distributions of uncertainties. The uncertainties are characterized by a given family of distributions. This setting leads to ambiguous expected objective function and chance constraints are typically computationally intractable. We propose a new safe tractable approximations of chance constraints, and employ refinement robust counterpart under the intersection of three perturbation sets (i.e., Box-Ball-Budget and Box-Ellipsoid-Budget). Such refinement can better reflect the randomness of the data than traditional intersection of two perturbations. Finally, we apply the proposed method in a real case of transportation problem to illustrate the effectiveness of our model.

Keywords: robust optimization, chance constraint, safe approximation, transportation problem

1 Introduction

The real-life optimal problems in the physical, engineering or economic systems always contain uncertain data. The reasons for data uncertainty include many aspects: imprecise data caused by the physical factors, or environmental conditions; experiment errors come from the physical impossibility to exactly implement a computed solution; measurement errors that come from the lack of knowledge of the parameters; system errors come from different environmental conditions. Some uncertain data's distribution rules can be relatively easily summarized from history data, but some are impossible to identify. How to deal with these uncertainties is one of the most important issues for managers and designers. There are two major optimization approaches to deal with the data uncertainties: stochastic optimization and robust optimization.

Stochastic optimization requires the exact probability distributions of the model uncertainties. In this way, stochastic optimization method can solve the uncertainty of the data and practical problems in life. The class of stochastic optimization problems with chance constraints can date back to Charns and Cooper [11]. Birge and Louveaux [10] introduced the methods to obtain analytic solutions to this class of problems. In this respect, many scholars have studied different aspects. Shapiro and Kleywegt [21] pointed out the stochastic programs by mini-max analysis, we get that the solution is conservation for the traditional. Next, Shaoiro and Nemirovski [22] employed the complexity of stochastic. When people use stochastic optimization to solve problem, the probability distribution is partially known. That is we don't know the exact distribution of perturbation vector and it can not guarantee chance contrains are computationally tractable. Dyer and Stougie [15] considered the computational complexity of stochastic programming problems. Ma and Liu [20] considered a stochastic chance constrained closed-loop supply chain network design model which objective is VaR, they made the stochastic parameters with known joint distributions.

In robust optimization, the uncertainties of the data can be reflected. The key of robust optimization is to establish the corresponding robust counterpart model to obtain the optimal solution, which can preserve computational tractability. In 1973, Soyster [23] aimed at original liner programming problem, and had a new idea that the resulting solution would be feasible for all possible realization of the uncertain model.

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Ben-Tal and Nemirovski [5, 6] studied the robust optimization within deterministic uncertainty sets. In 2004, Ben-Tal et al. [4] introduced adjustable robust counterpart with an linear problem and applied it to a multi-stage inventory management problem. Subsequently, Chen and Zhang [13] employed the extended affinely adjustable robust counterpart by an affine reparameterization means. Bertsimas and Goyal [8] proposed adjustable robust versions of convex optimization problems with uncertain constrains, and use it to revenue management, semi-definite optimization. Yang and Yang [26] proposed the robust optimization for the single allocation p-hub median problem under discount factor uncertainty, for the parameters, they employed robust counterpart to approach the constraint. For a detailed overview of the RO framework, the readers can refer to the literature [2, 3, 18].

Compared with the traditional stochastic optimization and robust optimization, distributionally robust optimization is more suitable to deal with the problem with partially characterized uncertainties. Delage and Ye [14] described uncertainty in both the distribution form (discrete, Gaussian, exponential, etc.) and moments (mean and covariance matrix). Goh and Sim [17] presented a framework for the distributionally robust optimization of linear programs under uncertainty, by using linear based decision rules to model the recourse variables. Wiesemann and Kuhn [24] discussed the computationally tractability of distributionally robust optimization problems, and presented some tractable conservative approximations for the problems. Popecu and Bertsimas [9] derived tractable reformulations of robust individual chance constraints, this way can deal with ambiguity sets which contain all distributions with a known mean value and covariance matrix. In addition, Arkadi [1] considered some computationally tractable approximations of chance constrained convex programs, Li and Floudas [19] studied the solution quality of robust optimization problems and used "Interval-Ellipsoidal" uncertainty set induced robust counterpart model as safe approximations.

In this paper, we present an ambiguous expected model with chance constraints based on distributionally robust optimization. Unlike the above literature, facing the computationally intractable ambiguous chance constraints, we consider new robust counterpart with refined uncertainty sets intersected from three perturbation sets("Box-Ball-Budget" and "Box-Ellipsoid-Budget"). This refinement setting can better reflect the randomness of data and do not rely on the distribution of information. Finally, we applied our proposed model and method in transportation problem [7, 25] to illustrate the effectiveness of our method.

The rest of this paper is organized as follows. In Section 2, an ambiguous expected model with chance constraints is built. Section 3 presents safe tractable approximations under Ball-Box-Budget. In Section 4, safe tractable approximations under Box-Ellipsoid-Budget. Section 5 applies our proposed model into robust transportation problem. The conclusions and future research are presented in the final section.

2 Ambiguous Expected Model with Chance Constraints

The classic expected model with chance constraints under stochastic environment are widely studied [11, 16, 12]. Moreover, such models require the exact information about the uncertain variables' distributions in advance. However, in many real cases, we make decisions under high uncertainty with only limited distribution information. In order to deal with these cases, we extend the classic expected model into an ambiguous expected model with chance constraints as follows,

$$\begin{array}{lll} \min & \mathrm{E}_{(c,d)\sim p}[c^T x + d] & p \in \mathcal{P} \\ \text{s.t.} & \mathrm{Pr}_{(a_i,b_i)\sim P_i}\{a_i^T x \le b_i\} \ge 1-\varepsilon_i, \quad i = 1\dots M, & P_i \in \mathcal{P}, \end{array} \tag{1}$$

where $\mathrm{E}_{(c,d)\sim p}$ and $\mathrm{Pr}_{(a_i,b_i)\sim P_i}$ are the expected value and probability associated with the distributions of random perturbations, respectively. Next we will simplify the expected objective and chance constraints.

For the chance constraint, we first focus on a single uncertainty-affected linear inequality–a family

$$\{a_i^T \le b_i\}_{[a_i:b_i]\in\mathcal{U}}, \tag{2}$$

of linear inequalities with the date varying in the uncertainty set

$$\mathcal{U} = \Big\{[a_i; b_i] = [a_i^0; b_i^0] + \sum_{l=1}^{L} \zeta_l [a_i^l; b_i^l] : \forall \zeta_l \in Z\Big\}. \tag{3}$$

Among them, ζ_l is random variables, and the vectors a_i and b_i have the following forms,

$$a_i^T x \leq b_i, \quad a_i = a_i^0 + \sum_{l=1}^{L} \zeta_l a_i^l, \quad b_i = b_i^0 + \sum_{l=1}^{L} \zeta_l b_i^l, \tag{4}$$

it is easily seen that a given x satisfies (4) if and only if x is robust feasible and the closed convex hull of the support is P_i. To satisfy the constraint with probability at least $1 - \varepsilon_i$, where $\varepsilon_i \in (0, 1)$. This approach associates with the randomly perturbed constraint (4), and the chance constraint becomes:

$$\Pr_{(\zeta)\sim P_i}\Big\{\zeta : [a_i^0]^T x + \sum_{l=1}^{L} \zeta_l [a_i^l]^T x > b_i^0 + \sum_{l=1}^{L} \zeta_l [b_i^l]\Big\} \leq \varepsilon_i, \tag{5}$$

it is reasonable to use this constraint when we do know this distribution. But sometimes, we only know the partial information on P_i, in other words, we know only that P belongs a given family $\mathcal{P}$ of distributions. Thus we make it sense to the ambiguous chance constraint

$$\forall(P_i \in \mathcal{P}) : \Pr_{\zeta\sim P_i}\Big\{\zeta : [a_i^0]^T x + \sum_{l=1}^{L} \zeta_l [a_i^l]^T x > b_i^0 + \sum_{l=1}^{L} \zeta_l [b_i^l]\Big\} \leq \varepsilon_i. \tag{6}$$

In model (1), assuming the expectations of c and d are $c^{'} \in [c^-, c^+]$, and $d^{'} \in [d^-, d^+]$ respectively, we can simplify the expected objective as follows,

$$\min \mathrm{E}_{(c,d)\sim p}[c^T x + d], \qquad p \in \mathcal{P},$$

it has the following equivalent form:

$$\begin{aligned} &\min \ t \\ &\text{s.t.} \ \max_{c',d'}\{c^{'} x + d^{'}\} \leq t. \end{aligned}$$

Finally, we reformulate the general model (1) as the following form

$$\begin{aligned} &\min \ t \\ &\text{s. t.} \ \max_{c',d'}\{c^{'} x + d^{'}\} \leq t, \qquad c^{'} \in [c^-, c^+], d^{'} \in [d^-, d^+] \\ &\qquad \Pr_{(a_i,b_i)\sim P_i}\Big\{[a_i^0]^T x + \sum_{l=1}^{L} \zeta_i^h [a_i^l]^T x > b_i^0 + \sum_{l=1}^{L} \zeta_i^l b_i^l\Big\} \leq \varepsilon_i, \qquad i = 1 \ldots M, \qquad P_i \in \mathcal{P}. \end{aligned} \tag{7}$$

Model (1) is a typical robust chance constrained problem, which is severely computationally intractable. In the next sections, we will replace the chance constraints with their safe convex approximations. In the next sections, we present the robust counterparts under two intersections of three perturbation sets (such as Ball-Box-Budget and Box-Ball-Budget), and propose the safe approximations under the corresponding perturbation sets.

3 Safe Tractable Approximations under Ball-Box-Budget

3.1 Robust Counterpart of Uncertain Set Ball-Box-Budget

In this section, we build safe approximations of a randomly perturbed linear constraint. Now consider a rather general case when the perturbation set $\mathcal{Z}$ is given by a conic representation. We can define the $\mathcal{Z}$ is Box-Ball-Budget

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^K : P\varsigma + Qu + p \in K\}, \tag{8}$$

where K is closed convex pointed cone in $\mathbb{R}^N$ with a nonempty interior, P and Q are the given matrices and p is a given vector. When K is not a polyhedral cone, assume that this representation is strictly feasible

$$\exists(\zeta^-, u^-) : P\varsigma^- + Qu^- + p \in intK. \tag{9}$$

Theorem 1. *Let the perturbation set $\mathcal{Z}$ be given by (2), and in the case of non-polyhedral K, let also (3) take place. Then the semi-infinite constraint can be represented by the following system of conic inequalities in variables $x \in \mathbb{R}^n,\ y \in \mathbb{R}$,*

$$\begin{aligned} & p^T y + [a^0]^T x \leq b^0, \\ & Q^T y = 0, \\ & (P^T y)_l + [a^l]^T x = b^l, l = 1,\ \ldots,\ L, \\ & y \in K_*. \end{aligned} \tag{10}$$

Now, we consider the random variables ζ_l of chance constraint, assume that random variables ζ_l satisfy the following conditions

$$E\{\zeta_l\} = 0\ \&\ |\zeta_l| \leq 1,\ l = 1, \ldots, L, \&\{\zeta_l\}_l^L = 1 \textit{ are independent}. \tag{11}$$

Consider a rather general case when the perturbation set $\mathcal{Z}$ is given as the intersection of three uncertain sets (Ball, Box and Budget),

$$\begin{aligned} & \textit{Box} \quad \{\mathcal{Z} = \zeta \in R^l : \|\zeta\|_\infty \leq 1\}, \\ & \textit{Ball} \quad \{\mathcal{Z} = \zeta \in R^l : \|\zeta\|_2 \leq \Omega\}, \\ & \textit{Budget} \quad \{\mathcal{Z} = \zeta \in R^l : \|\zeta\|_\infty \leq 1 \quad \|\zeta\|_1 \leq \gamma\}, \\ & \mathcal{Z}_1 = \{\zeta \in R^l : \|\zeta\|_\infty \leq 1 \quad \|\zeta\|_2 \leq \Omega \quad \|\zeta\|_1 \leq \gamma\}, \\ & \mathcal{P}_1\zeta = [\zeta : 0] \qquad p_1 = [0:1] \qquad \mathcal{K}_*^1 = \{(z:t) \in R^l \times R \quad t \geq \|z\|_1\}, \\ & \mathcal{P}_2\zeta = [\zeta : 0] \qquad p_1 = [0:1] \qquad \mathcal{K}_*^2 = \{(z:t) \in R^l \times R \quad t \geq \|z\|_2\}, \\ & \mathcal{P}_3\zeta = [\zeta : 0] \qquad p_1 = [0:1] \qquad \mathcal{K}_*^3 = \{(z:t) \in R^l \times R \quad t \geq \|z\|\infty\}. \end{aligned}$$

Setting $y^1 = [z : \tau_1],\ y^1 = [w : \tau_1],\ y^1 = [v : \tau_1]$, based on Theorem 1, the system of conic constraints in variables x, $y^1, \ldots, y^s$ are as follows

$$\begin{aligned} & \sum_{s=1}^{S} \mathcal{P}_s^T y^s + [a^0]^T x \leq b^0, \quad Q_s^T y^s = 0, \quad s = 1, \ldots, S. \\ & \sum_{s=1}^{S} (\mathcal{P}_s^T y^s)_l + [a^l]^T x = b^l, \quad l = 1, \ldots, L. y^s \in k_*^s\ ,\ s = 1, \ldots, S. \end{aligned} \tag{12}$$

Setting y^1, y^2, y^3 with one-dimensional τ^1, τ^2, τ^3 and L-dimensional z, w, v, x,

$$\begin{aligned} & \tau^1 + \tau^2 + \tau^3 \leq b^0 - [a]^0 x, \\ & (z + w + v)_l = b^l - [a^l]^T x \quad l = 1, \ldots, L, \\ & \|z\|_1 \leq \tau_1, \\ & \|w\|_\Omega \leq \tau_2, \\ & \|v\|_\infty \leq \tau_3. \end{aligned} \tag{13}$$

We eliminate form this system the variables τ_1, τ_2, τ_3 for every feasible solution to the system, and get

$$\begin{aligned} & \sum_{l=1}^{L} |z_l| + \Omega \sqrt{\sum_{l=1}^{L} w_l^2} + \gamma \max_l |v_l| \leq b_i^0 - [a_i^0]^T x, \\ & z_l + w_l + v_l = b_i^l - [a_i^l]^T x \qquad l = 1, \ldots, L \qquad i = 1, \ldots, M. \end{aligned} \tag{14}$$

Finally, we obtain the robust counterpart of the perturbation set Box-Ball-Budget.

3.2 Safe Approximations of Chance Constraint under Box-Ball-Budget

The robust counterpart of the uncertain linear constraint $a^T x \leq b$, $[a; b] = [a^0; b^0] + \sum_{l=1}^{L} [a^l; b^l]$ with the uncertainty set $\mathcal{Z} = \{\zeta \in R^l : \|\zeta\|_\infty \leq 1 \quad \|\zeta\|_2 \leq \Omega \quad \|\zeta\|_1 \leq \gamma\}$ is equivalent to the system of conic quadratic constraints

$$\begin{aligned} &(a) \quad \sum_{l=1}^{L} |z_l| + \Omega \sqrt{\sum_{l=1}^{L} w_l^2} + \gamma \max_l |v_l| \leq b_i^0 - [a_i^0]^T x, \\ &(b) \quad z_l + w_l + v_l = b_i^l - [a_i^l]^T x \qquad l = 1, \ldots, L \qquad i = 1, \ldots, M. \end{aligned} \tag{15}$$

In the case of (11), x is component of every feasible solution to this system satisfying the randomly perturbed inequality with probability at least $1 - \exp\{-\Omega^2/2\}$. In particular, with $\Omega \geq \sqrt{\ln(1/\varepsilon)}$, the robust counterpart is a tractable safe approximation of the chance constraint.

Theorem 2. *Let z_l, $l = 1, \ldots, L$, be deterministic coefficients and $\zeta_l, l = 1, \ldots, l$, be independent random variables with zero mean taking values in $[-1,\ 1]$. Then for every $\Omega \geq 0$ it holds that*

$$\Pr\Big\{ -\sum_{l=1}^{L} z_l \zeta_l \geq \Omega \sqrt{\sum_{l=1}^{L} (z_l^2)^2} \Big\} \leq \exp\{-\Omega^2/2\}. \tag{16}$$

Proof. Expanding the original form of chance constraint $a^T x \leq b$, we have

$$\sum_{i=1}^{L} [[a^l]^T x - b_l] \zeta_l > b^0 - [a^0]^T x.$$

Based on (15.b), we get

$$-\sum_{l=1}^{L} z_l \zeta_l - \sum_{l=1}^{L} w_l \zeta_l - \sum_{l=1}^{L} v_l \zeta_l > b^0 - [a^0]^T x.$$

By (15.a), we obtain

$$-\sum_{i=1}^{L} w_l \zeta_l - \sum_{i=1}^{L} v_l \zeta_l > \Omega \sqrt{\sum_{i=1}^{L} w_l^2} + \gamma \max |v_l|.$$

Assume that $\phi = \min(\Omega, \gamma/\sqrt{L})$, $\|\zeta\|_2 \leq \sqrt{L} \|\zeta\|_\infty$, and we have

$$-\sum_{l=1}^{L} w_l \zeta_l - \sum_{l=1}^{L} v_l \zeta_l > \phi \sqrt{\sum_{l=1}^{L} w_l^2} + \phi \sqrt{\sum_{l=1}^{L} v_l^2}.$$

Finally, we get the final form

$$-\sum_{l=1}^{L} w_l \zeta_l - \sum_{l=1}^{L} v_l \zeta_l > \phi \sqrt{\sum_{l=1}^{L} w_l^2 + \sum_{l=1}^{L} v_l^2}.$$

Based on Theorem 2, we have

$$\Pr\Big\{ -\sum_{l=1}^{L} \zeta_l (w_l + v_l) \geq \phi \sqrt{\sum_{i=1}^{L} (w_l + v_l)^2} \Big\} \leq \exp\{\phi^2/2\}. \tag{17}$$

□

We can obtain the safe approximations of model (7)

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \max\{c^{'}x + d^{'}\} \le t, \qquad c^{'} \in [c^-, c^+], d^{'} \in [d^-, d^+] \\ & \sum_{l=1}^{L} |z_l| + \Omega \sqrt{\sum_{l=1}^{L} w_l^2} + \gamma \max_l |v_l| \le b_i^0 - [a_i^0]^T x, \\ & z_l + w_l + v_l = b_i^l - [a_i^l]^T x, \qquad l = 1, \dots, L \qquad i = 1, \dots, M. \end{array} \tag{18}$$

Note that the model (18) is a second order cone programming, which can be solved by traditional methods.

4 Safe Tractable Approximations under Box-Ellipsoid-Budget

4.1 Robust Counterpart of Uncertain Set Box-Ellipsoid-Budget

The distributions P_l of the components ζ_l are such that:

$$P_1 : \zeta_l, L = 1, \dots, L, \text{are independent random variables.}$$

$$P_2 : \int \exp\{ts\} dP_l(s) \le \exp\big\{ \max[u_l^+, u_t^-] + \frac{1}{2}\sigma_l^2 t^2 \big\} \quad \forall t \in R,$$

with known $u_l^- \le u_l^+$ and $\sigma_l \ge 0$. In the solving process, u_l^+, u_l^- is 0. Since

$$Ellipsoid \quad \{\mathcal{Z} = \zeta \in R^l : \|\frac{\zeta}{\sigma}\|_2 \le \Omega\},$$

we denote the uncertain set Box-Ellipsoid-Budget as

$$\mathcal{Z}_2 = \{\zeta \in R^l : \|\zeta\|_\infty \le 1 \quad \|\frac{\zeta}{\sigma}\|_2 \le \Omega \quad \|\frac{\zeta}{\sigma}\|_1 \le \gamma\}. \tag{19}$$

By the Theorem 1, referring to the sovling process in Section 3.1, we get the following form,

$$\begin{array}{ll} (a) & \sum_{l=1}^{L} |z_l| + \Omega \sqrt{\sum_{l=1}^{L} w_l^2 \sigma_1^2} + \gamma \max_l |v_l \sigma_1| \le b_i^0 - [a_i^0]^T x \\ (b) & z_l + w_l + v_l = b_i^l - [a_i^l]^T x \qquad l = 1, \dots, L \qquad i = 1, \dots, M. \end{array} \tag{20}$$

This is the robust counterpart under uncertain set Box-Ellipsoid-Budget.

4.2 Safe Approximations of Chance Constraint under Box-Ellipsoid-Budget

The robust counterpart of the uncertain linear constraint $a^T x \le b$, $[a; b] = [a^0; b^0] + \sum_{l=1}^{L} [a^l; b^l]$ with the uncertainty set $\mathcal{Z}_2$ is equivalent to the system of conic quadratic constraints. According to Theorem 2, we can sort it out

$$\mathcal{Z}_2 = \Big\{\zeta \in R^l : -1 \le \zeta_l \le 1, l \le L, \sqrt{\sum_{l=1}^{l} \zeta_l^2 / \sigma_l^2} \le \Omega \quad \sum_{l=1}^{L} |\frac{\zeta_l}{\sigma_l}| \le \gamma\Big\}.$$

Theorem 3. *Let ζ_l, $l = 1, \dots, L$, be independent random variables with distributions satisfying P_2, than, for every deterministic vector $\zeta_l, l = 1, \dots, l$ and constant $\Omega \ge 0$ one has*

$$\Pr\Big\{ -\sum_{l=1}^{L} z_l \zeta_l \ge \sum_{l=1}^{L} \max[u_l^- z_l, u_l^+ z_l] + \Omega \sqrt{\sum_{l=1}^{L} (z_l^2)^2} \Big\} \le \exp\{-\Omega^2/2\}. \tag{21}$$

Proof. The corresponding proof is similar to the one of Theorem 2. □

Based on the Theorem 3, we obtain the approximation model as follows,

$$\begin{aligned} &\min\ t \\ &\text{s.t. } \max\{c^{'}x+d^{'}\}\le t, \qquad c^{'}\in[c^-,c^+], d^{'}\in[d^-,d^+], \\ &\quad \sum_{l=1}^{L}|z_l|+\Omega\sqrt{\sum_{l=1}^{L}w_l^2\sigma_l^2}+\gamma\max_l|v_l\sigma_l|\le b_i^0-[a_i^0]^Tx, \\ &\quad z_l+w_l+v_l=b_i^l-[a_i^l]^Tx, \qquad l=1,\dots,L \qquad i=1,\dots,M. \end{aligned} \tag{22}$$

Model (22) is a tractable second order cone programming.

5 Robust Transportation Problem

In this section, we employ the proposed model into a real transportation problem. Transportation problem is a traditional optimization problem solving business problem in the physical distribution of products. Basically, the purpose is to minimize the freight of shipping goods from one location to another so that the needs of each arrival area are met and every shipping location operates within its capacity. We consider the transportation problem about a new kind of products, and the manager cannot forecast the demand exactly. Thus we assume that the demand only has limited information and is characterized by a given family of distributions. At last, we formulate the robust transportation model as follows,

$$\begin{aligned} &\min\ \mathrm{E}_{(c_{ij}\sim p)}\Big[\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\Big] \\ &\text{s.t. } \sum_{j=1}^{n}x_{ij}=a_i, \qquad i=1,\dots,m \\ &\quad \Pr{}_{(a_i,b_i)\sim P_i}\Big\{\sum_{i=1}^{m}x_{ij}\ge b_j\Big\}\ge 1-\varepsilon, \qquad j=1,\dots,n \\ &\quad x_{ij}\ge 0, \end{aligned} \tag{23}$$

where x_{ij} is the transportation quantity between locations; c_{ij} is the unit transportation freight; a_i is the supply quantity of factory; and b_j is the demand.

We analyse the transportation problem of a company whose business mainly in the north of China. It has five factories ($m = 5$) located in BeiJing, TianJing, BaoDing, ShiJiaZhuang and ChengDe for processing goods, and seven sales points ($n = 7$) located in TaiYuan, QinHuangDao, JiNan, YinChuan, XiAn, HuHeHaoTe and ZhengZhou. The locations of factories and sales points are shown in Figure 1.

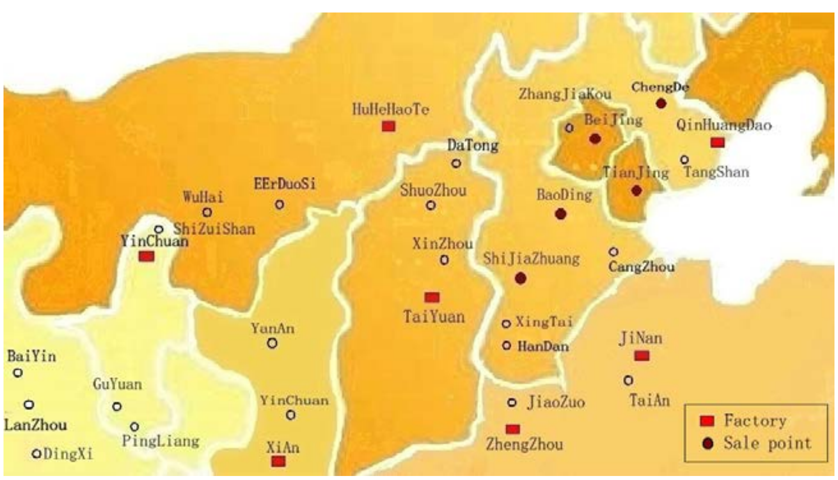


Figure 1: The locations of factories and sales points

The unit freight for the products from factories to the sale points c_{ij} is derived from distance of factories and sale points, and changes between $[\tilde{c}_{ij}(1-5\%), \tilde{c}_{ij}(1+5\%)]$, where its reference value $\tilde{c}_{ij}$ is shown in the Table 1. Table 2 presents the supply quantity of city a_i. In our case, the demand of the sale points is uncertain. Based on the population of seven cities, The uncertain demand b_j are described in Table 3.

Table 1: The reference value of unit freight ($\tilde{c}_{ij}$)

	TaiYuan	QinHuangDao	JiNan	YinChuan	XiAn	HuHeHaoTe	ZhengZhou
BeiJing	1474.57	1138.93	1197.23	1842.66	1566.49	1334.12	1165.03
TianJing	2231.73	398.89	1666.57	2399.15	1927.00	2399.15	1684.40
BaoDing	510.62	651.62	2100.00	1548.00	2144.00	1950.00	916.57
ShiJiaZhuang	2000.00	898.84	1925.14	2399.99	1915.73	1871.89	1930.17
ChengDe	1074.98	332.09	961.89	2006.28	1907.31	2200.00	1800.00

Table 2: The supply quantity of city (a_i)

	BeiJing	TianJing	BaoDing	ShiJiaZhuang	ChengDe
Supply Quantity	110	97	114	109	125

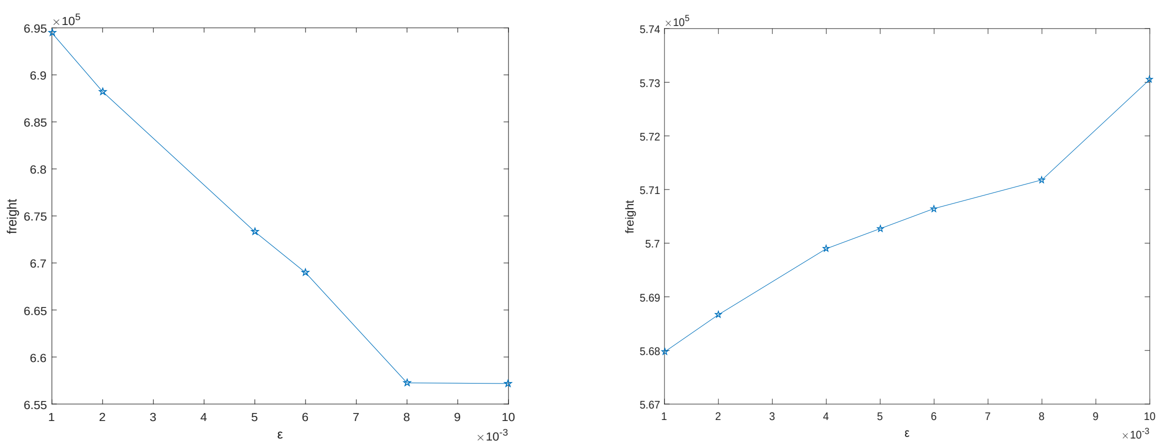
Table 3: The demand of city (b_j)

	TaiYuan	QinHuangDao	JiNan	YinChuan	XiAn	HuHeHaoTe	ZhengZhou
b_j^0	75	59	89	45	80	75	60
b_j^l	10	12	5	10	5	8	9
Demand	[65,85]	[47,71]	[84,94]	[35,55]	[75,85]	[63,83]	[51,69]

We take robust counterpart as a safe approximation of chance constraints under perturbation set Box-Ball-Budget. Based on the case of the model (18), we assume that when ε is known, $\Omega = \sqrt{2\ln(1/\varepsilon)}$, $\gamma = \sqrt{L}\Omega$ and $\sigma = 2$. Table 4 presents the total freight in this case with different ε.

Table 4: The total freight under uncertain set Box-Ball-Budget

ε	0.01	0.02	0.04	0.05	0.06	0.08	0.1
Objective Value	694463.9	688214.8	743511.5	673321.6	668969.8	657250.3	657179.1



(a) The fright of robust optimization

(b) The fright of stochastic optimization

Figure 2: The total freights of robust optimization and stochastic optimization

We compare the results of our proposed robust optimization with the stochastic optimization. Assume that $\zeta_1, \ldots, \zeta_L$ are independent Gaussian random variables with partially known expectations μ, variances σ^2 and $\varepsilon \sim \mathcal{N}(\mu, \sigma^2)$. The stochastic optimization results are shown in the right part of Figure 2 (part (a)). When the variable is Gaussian perturbation, we can get the exact probability. Meanwhile, the left part of Figure 2 (part (b)) is the robust results, the results solved by the two methods have completely opposite trend with ε. The total freight decreases with respect to ε in robust method, while increases with ε in stochastic method.

Table 5: The total freight under stochastic optimization

ε	0.01	0.02	0.04	0.05	0.06	0.08	0.1
Objective Value	567971.6	568666.1	569896.5	570270.9	570645.4	571180.3	573052.7

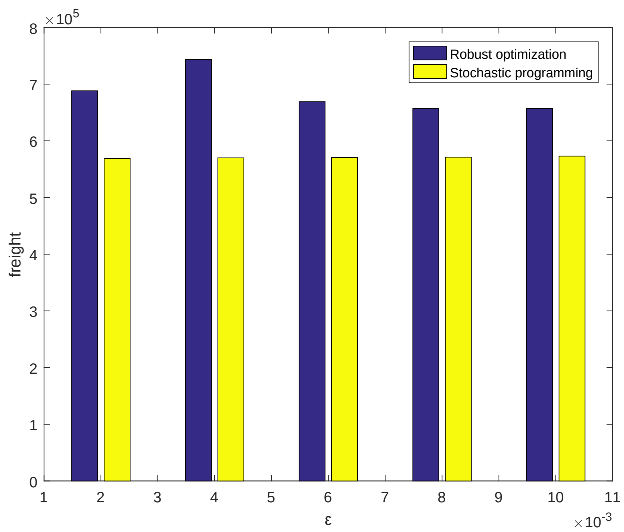


Figure 3: The freights under robust and stochastic models

Finally, in order to observe and contrast, we plot a bar chart about the solutions of robust and stochastic optimizations in Figure 3. We find that the freights of robust optimization is higher than the stochastic optimization. It is because the robust result contains "worst-case-oriented" criterion. In addition, stochastic optimization must rely on the distribution, this illustrates the stochastic optimization has limitation.

6 Conclusions

In this paper, we focus on the uncertainties with partially distribution information, and extend traditional expected model with chance constraints into an ambiguous model. The uncertainties are described by a given family of distributions. This setting gives rise to ambiguous expected objective function and ambiguous chance constraints, which are typically computationally intractable. We analyse refined robust counterpart under the intersection of three perturbation sets (i.e., Box-Ball-Budget and Box-Ellipsoid-Budget). Moreover, we propose the safe tractable approximations of ambiguous chance constraints under the corresponding uncertain perturbation set. Such refinement can better reflect the randomness of the data than traditional intersection of two perturbations. Consequently, we apply our proposed model and method into a case of transportation problem with five factories and seven sale points. We use our methods and theories to deal with the uncertainty of demand, and obtain the quantity of goods delivered to the sale points.

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