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Uncertain Network Interdiction Problem

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Abstract

Indeterminacy is an intrinsic characteristic of a decision making process. Sometimes there are no samples, and historical data are not enough for estimating an appropriate probability distribution for an indeterminate variable. In these situations, an option could be referring to an expert on the subject, and uncertainty theory is a potentially powerful framework to manage this sort of indeterminacy. In this theory, an undetermined input parameter is referred to as an uncertain variable and its distribution is constructed based on the opinion of an expert. This is the case in many practical problems such as in the smuggling networks and drug-trafficking networks. This paper considers the network interdiction problem that aims to minimize the maximum flow through a capacitated network from a source to a sink where the arc capacities are uncertain variables. It is proved that there exists an equivalent deterministic model to the uncertain network interdiction problem. The proposed method is applied on a test problem, and the results are compared with the associated deterministic counterpart.

Keywords: network Interdiction, uncertainty theory, bi-level programming, mixed integer programming

1 Introduction

Network Interdiction Problem (NIP) is classically defined in the interdictor-evader framework as a bi-level mixed integer programming, which is one of the hardest problems in optimization. The primary purpose of an NIP is to minimize the evader's maximum flow through a capacitated network by an interdictor using a set of limited resources. This goal could be replaced by "limiting the evader's feasible actions set" or by "increasing the cost of associated activities".

Historically, the initial deterministic NIP was formulated to investigate in a military context by Wollmer[17] with the goal of removing n arcs from a network to minimize the maximum flow between the source and the sink nodes. An extension of this problem, by adding a budget constraint, was considered to prevent enemy's arsenal transmission [13]. Wood solved this problem using a mathematical method by devising an integer programming formulation of the problem, and provided a proof of NP-completeness [18]. The author also mentioned that the model could be generalized to other possibilities such as multicommodity networks, undirected networks, networks with multiple sources and multiple sinks.

Let us briefly review the deterministic NIP. For $G = (N, A)$ as a directed graph, where N and A are the sets of nodes and arcs respectively, two specific nodes are considered as the source and the sink nodes. We denote them by s and d, correspondingly. To each arc $(i, j) \in A$ the capacity u_{ij} is associated, and $x_{ij} \geq 0$ is the possible amount of flow on this arc that could not exceed the arc's capacity. Let R denote the available resource units of the interdictor that can be expended to potentially interdict some arcs, and r_{ij} units of resource is required for interdicting of the arc (i, j). Corresponding to the nodes s and d, an artificial arc (d, s) is defined with $u_{ds} = \infty$ and $r_{ds} = \infty$. This assumption is valid since this artificial arc should not be interdicted. Let y_{ij} be a binary decision variable of the interdictor that is 1 when the arc (i, j) is interdicted and 0 otherwise. Moreover, the set of feasible decisions of the interdictor is denoted by

$$Y = \big\{y_{ij} \mid \sum_{(i,j)\in A} y_{ij} r_{ij} \leq R,\ y_{ij} \in \{0, 1\},\ \forall (i, j) \in A\big\}.$$

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Respecting this notation, the deterministic NIP is in the following form [18]:

$$\begin{array}{lll} \min\limits_{y \in Y} \max\limits_{x} & x_{ds} & \\ \text{subject to} & \sum\limits_{j:(i,j)\in A'} x_{ij} - \sum\limits_{j:(j,i)\in A'} x_{ji} = 0 & \forall i \in N \\ & 0 \le x_{ij} \le u_{ij}(1 - y_{ij}) & \forall (i,j) \in A, \end{array}$$

where $A' = A \cup \{(d,s)\}$. Here, the first restriction is the mass balance constraint at each node, and the second one expresses the arc capacity constraint.

Cormican et al. extended the deterministic NIP to the stochastic case [2]. In their model, capacity of arcs and success in their interdiction are uncertain when the interdiction successes are binary random variables. In their work, the authors developed sequences of upper and lower bounds to the value function achieved by the interdictor. These values are improved during the solution procedure and used to construct a sequence of solutions approaching to the optimal attack performance value. Later on, Woodruff [19] devised a similar model, whence the interdictor must put detecting sensors for narcotic drugs without any information on the exact location of the source and the sink nodes. Pan [15], in his dissertation, has developed a class of stochastic NIP models. In these models, the interdictor installs detectors on arcs subject to a budget constraint. Then, the smuggler's random origin-destination pair is revealed and the smuggler selects a maximum reliable path in the residual network. The interdictor's goal is to minimize the reliability of the smuggler's maximum reliable path. Morton et al. [14] studied stochastic network interdiction models for nuclear smuggling interdiction, where the evader's origin-destination pair is random, and the interdictor and the attacker have asymmetric information on network parameters.

Zheng et al. [20] studied a stochastic dynamic NIP where the interdictor's actions achieve indeterminate effects. They developed a new solution algorithm based on a modified branch and bound framework, which combines the idea of using sequentially improving lower bounds with the branch-and-bound search. The resulting algorithm was faster than the algorithms proposed in [2]. The authors further considered a dynamic two stage version of the stochastic NIP problem where the attack on the network occurs in two stages. The outcome of the interdiction on the first stage attack could be observed by the attacker and is used to adapt the second stage attack.

Ramirez and Rocco [16] proposed a stochastic network interdiction problem (SNIP) with capacitated network reliability, and introduced an evolutionary optimization approach to solve SNIPs. They assumed that the nominal capacity of each link as well as the associated cost of interdiction can change from link to link and that such interdiction has a probability of being successful. This version of the SNIP was modeled as a capacitated network reliability problem for the first time by these authors. Later on, Carrigy et al. [1] considered the multi-state SNIP for the problem of maximizing the reliability associated with an interdiction strategy. They presented an effective approach for solving SNIPs with the inclusion of multi-state stochastic behavior of link flow in a two-terminal network.

All of these works have considered the probability theory on their modeling scheme. It is important to restate that probability theory is only applicable for modeling frequencies based on historical data and existence of enough trusted observations. However, such data rarely exists in practice for NIP instances, such as the arc capacities in narcotic trafficking networks, and as a consequence, utilizing the probability theory in these situations may lead to incorrect decision with irreversible consequences. The uncertainty theory initiated by Liu [8] could be a well-founded powerful axiomatic counterpart in such circumstances. Liu clarified that it is inappropriate to model belief degrees by probability theory, since it may lead to counterintuitive results [11] (See also Page 6 in [12]).

Uncertain network problem was first proposed by Liu in order to model the project scheduling problem with uncertain duration times [10]. Here, capacity of arcs were represented by uncertain variables and the 99-method was considered for solving the maximum flow problem with uncertain capacities on arcs. This method has been used for the first time by Liu [10] in calculating the uncertainty distribution of monotone function of uncertain variables. Han et al. [4] investigated the maximum flow problem in an uncertain environment. They also used the 99-method for producing the uncertainty distributions as well as the expected value of the maximum flow in an uncertain network.

In this paper, we consider the NIP with uncertain capacity on some arcs, and reformulate the problem as an uncertain network problem based on the uncertainty theory. Since our approach is independent of the type of uncertain variable, it is assumed that the capacity of arcs are only independent linear or zigzag uncertain variables in the sense of the uncertainty theory (See Appendix A). The proposed uncertain problem

Table 1: The common notation

u_{ij}	The capacity of arc (i,j).
s, d	The source node and the sink node, respectively.
(d,s)	An artificial arc with $u_{ds}=\infty$ and $r_{ds}=\infty$.
x_{ij}	The amount of flow on the arc (i,j).
x_{ds}	The maximum flow on network from s to d.
R	The total resource units for the interdictor.
y_{ij}	Binary interdicting decision variable.
r_{ij}	Units of resource is required to interdict the arc (i,j).
A'	The set of arcs where $A'=A\cup\{(d,s)\}$.
B	A Borel set.
α_{ij}	The confidence level within $(0,1)$.
ξ_{ij}	Uncertain variable associated to the capacity of arc(i,j).
Φ	The (linear or zigzag) uncertainty distribution.
$\mathcal{M}$	An uncertain measure on $\mathcal{L}$.
$\mathcal{L}$	σ-Algebra over Γ where Γ be a nonempty set.
$(\Gamma,\mathcal{L},\mathcal{M})$	An uncertainty space.

can be transformed into a corresponding deterministic one. The produced deterministic model is a bi-level mixed integer program which can be converted to a single-level one. This problem is identical with the original model in optimality. A simple test problem is considered and the results are compared with the corresponding deterministic version.

The remainder of paper is organized as follows. Section 2 is devoted to the formulation of NIP with uncertain arc capacities. The section continues with presenting an equivalent deterministic problem, and finally a single-level mixed integer linear problem. The equivalence of these problems are proved in this section. Section 4 is devoted to the existing challenges in applying the probability theory in the absence of enough trusted samples. In Section 3, we present a numerical example to illustrate the differences of the results in uncertain environment with the deterministic situation. The paper terminates with concluding remarks and possible future work directions. The appendix contains some necessary preliminary concepts and results from the uncertainty theory.

2 Problem Formulation

For easily access, the notation used in this paper are summarized in Table 1. In the graph $G(N,A)$, let ξ_{ij} denote an uncertain variable corresponding to u_{ij}, the capacity of the arc (i,j). Considering a confidence level α_{ij} for the constraint associated to the uncertain capacity ξ_{ij}, the following optimization problem is defined for the uncertain NIP.

$$\min_{y\in Y}\max_{x} \quad x_{ds}$$

subject to

$$\textstyle\sum_{j:(i,j)\in A'} x_{ij} - \sum_{j:(j,i)\in A'} x_{ji} = 0 \quad \forall i\in N \tag{1}$$

$$\mathcal{M}\{x_{ij}\le \xi_{ij}(1-y_{ij})\}\ge \alpha_{ij} \quad \forall (i,j)\in A \tag{2}$$

$$x_{ij}\ge 0 \quad \forall (i,j)\in A. \tag{3}$$

Constraint (2) reads as the belief degree that "the interdictor's constraint holds" is not less than α_{ij}. In other words, the belief degree that "When there is no interdicting action on the arc (i,j), potential flow through this arc does not exceed the uncertain value ξ_{ij}." is greater than of equal to the pre-specified confidence level α_{ij}. Further, the belief degree that "When there is an interdicting action on the arc (i,j), the flow through this arc vanishes." is again at least the confidence level α_{ij}.

Let f denote the value of objective function. From this formulation, the value of the objective function is $f(\xi)$. Observe that as a function of ξ, f is also an uncertain variable.

The following theorem presents a substitute to constraint (2) when the uncertain variable ξ_{ij} has linear uncertainty distribution.

Theorem 2.1. *Let ξ_{ij} be a linear uncertain variable $\mathcal{L}(a_{ij}, b_{ij})$ of the capacity on the arc (i, j). For a given confidence level $\alpha_{ij} \in (0, 1)$, constraint (2) is identical to*

$$x_{ij} \leq \big(\alpha_{ij} a_{ij} + (1 - \alpha_{ij}) b_{ij}\big)(1 - y_{ij}). \tag{4}$$

Proof. Observe that for $y_{ij} = 1$, constraint (2) reads as

$$\mathcal{M}\left\{x_{ij} \leq 0\right\} \geq \alpha_{ij}. \tag{5}$$

Regarding (3), it holds $\mathcal{M}\left\{x_{ij} \leq 0\right\} = 1$. In this case, (4) reduces to $x \leq 0$, accompanying with (3) completes the proof.

Thus, without loss of generality, constraint (2) can be rewritten as

$$\mathcal{M}\left\{(x_{ij})/(1 - y_{ij}) \leq \xi_{ij}\right\} \geq \alpha_{ij}. \tag{6}$$

From the linearity of ξ_{ij}, the uncertainty distribution Φ_{ij} is

$$\Phi(\frac{x_{ij}}{1-y_{ij}}) = \begin{cases} 0, & \text{if } \ \dfrac{x_{ij}}{1-y_{ij}} \leq a_{ij} \\ \dfrac{\frac{x_{ij}}{1-y_{ij}} - a_{ij}}{b_{ij} - a_{ij}}, & \text{if } \ a \leq \dfrac{x_{ij}}{1-y_{ij}} \leq b_{ij} \\ 1, & \text{if } \ \dfrac{x_{ij}}{1-y_{ij}} \geq b_{ij}, \end{cases} \tag{7}$$

and considering (16) (see Appendix A), constraint (6) converts to the following deterministic form.

$$\frac{x_{ij}}{1-y_{ij}} \leq \Phi_{ij}^{-1}(1 - \alpha_{ij}) = \alpha_{ij} a_{ij} + (1 - \alpha_{ij}) b_{ij}. \tag{8}$$

The proof is complete. □

Observe that constraint (4) forces the flow on the arc (i, j) to be zero when an interdicting action is carried out on (i, j), and leave the flow to be at most $(\alpha_{ij} a_{ij} + (1 - \alpha_{ij}) b_{ij})$ when there is no interdiction on the arc. Further, the upper bound for the possible flow depends both on the uncertainty interval and confidence level.

The following theorem presents a substitute to constraint (2) when the uncertain variable ξ_{ij} has zigzag uncertainty distribution.

Theorem 2.2. *Let ξ_{ij} be a zigzag uncertain variable $\mathcal{Z}(a_{ij}, b_{ij}, c_{ij})$ of the capacity on the arc (i, j). For a given confidence level $\alpha_{ij} \in (0, 1)$, constraint (2) is identical to*

$$x_{ij} \leq \big(2\alpha_{ij} b_{ij} + (1 - 2\alpha_{ij}) c_{ij}\big)(1 - y_{ij}), \qquad \text{if } \alpha_{ij} \geq 0.5, \tag{9}$$

$$x_{ij} \leq \big((2\alpha_{ij} - 1) a_{ij} + (2 - 2\alpha_{ij}) b_{ij}\big)(1 - y_{ij}), \qquad \text{if } \alpha_{ij} < 0.5. \tag{10}$$

Proof. Let $y_{ij} \neq 1$ (the case of $y_{ij} = 1$ can be proved as in Theorem 2.1). Observe that the uncertain distribution Φ_{ij} of the zigzag uncertain variable ξ_{ij} is

$$\Phi(\frac{x_{ij}}{1-y_{ij}}) = \begin{cases} 0, & \text{if } \ x \leq a \\ \dfrac{\frac{x_{ij}}{1-y_{ij}} - a}{2(b-a)}, & \text{if } \ a \leq x \leq b \\ \dfrac{\frac{x_{ij}}{1-y_{ij}} + c - 2b}{2(c-b)}, & \text{if } \ b \leq x \leq c \\ 1, & \text{if } \ x \geq c, \end{cases} \tag{11}$$

and considering (17) (see Appendix A), constraint (6) converts to the following deterministic form.

$$\frac{x_{ij}}{1-y_{ij}} \leq \Phi_{ij}^{-1}(1-\alpha_{ij}) = \begin{cases} 2\alpha_{ij}b_{ij} + (1-2\alpha_{ij})c_{ij}, & \text{if } \alpha_{ij} \geq 0.5 \\ (2\alpha_{ij}-1)a_{ij} + (2-2\alpha_{ij})b_{ij}, & \text{if } \alpha_{ij} < 0.5. \end{cases} \tag{12}$$

The proof is complete. □

Let A_1 be the set of arcs that the capacity on these arcs are independent linear uncertain variables, and A_2 denote the set of arcs, where the uncertain variables on these arcs are independent zigzag. Theorems 2.1 and 2.2 enable us to have the following deterministic model instead of the uncertain NIP form.

$$\text{(MOD1)} \quad \left\{ \min_{y \in Y} \max_{x} x_{ds} \middle| \text{ constraints (1), (4), (9), (10), and (3)} \right\}.$$

Note that $x = 0$ is a feasible solution of the inner maximization problem of (MOD1). In addition, the feasible region of this problem is bounded. Therefore, by the strong duality theorem, there is no duality gap and the primal and dual objective values are identical in optimality. Therefore, the inner maximization problem can be replaced with its dual with no effect on the optimal solution. Let π_i $(i \in N)$, β_{ij}, θ_{ij}, and ω_{ij} be dual variable vectors corresponding to (1), (4), (9), and (10), respectively. Thus, this substitution reduces (MOD1) to the following single-level minimization problem.

$$\begin{aligned} \text{(MOD2)} \min \quad & \sum_{(i,j)\in A} \Big[\beta_{ij}\big(\alpha_{ij}a_{ij} + (1-\alpha_{ij})b_{ij}\big) + \theta_{ij}\big(2\alpha_{ij}b_{ij} + (1-2\alpha_{ij})c_{ij}\big) \\ & +\omega_{ij}\big((2\alpha_{ij}-1)a_{ij} + (2-2\alpha_{ij})b_{ij}\big)\Big](1-y_{ij}) \\ \text{subject to} \quad & \\ & \pi_i - \pi_j + \beta_{ij} + \theta_{ij} + \omega_{ij} \geq 0 \qquad \forall (i,j) \in A \\ & \pi_d - \pi_s \geq 1 \\ & \beta_{ij} \geq 0 \qquad \forall (i,j) \in A_1 \\ & \theta_{ij}, \omega_{ij} \geq 0 \qquad \forall (i,j) \in A_2 \\ & y \in Y. \end{aligned}$$

The following theorem summarizes this argument asserting that the bi-level linear mixed integer program (MOD1) is identical with the single-level mixed integer nonlinear problem (MOD2).

Theorem 2.3. *Two optimization problems (MOD1) and (MOD2) are equivalent in optimality.*

Observe that problem (MOD2) has a subtle nonlinearity in the objective function. Recall that β_{ij} is the shadow price corresponding to the constraint (4). Increasing the value of $\alpha_{ij}a_{ij} + (1-\alpha_{ij})b_{ij}$ by one unit permits at most one more unit of flow through the network. According to the strong duality in optimality, the increase in the objective function of problem (MOD2) will be at most one, too. Thus $\beta_{ij} \leq 1$ for all $(i,j) \in A$. Similar conclusion is also valid for θ_{ij} and ω_{ij}.

In the sequel, we only describe the standard method for linearization of $\beta_{ij}y_{ij}$. Analogous results can be unambiguously derived for $\theta_{ij}y_{ij}$ and $\omega_{ij}y_{ij}$, too. Observe that $\beta_{ij}y_{ij}$, when $0 \leq \beta_{ij} \leq 1$ and $y_{ij} \in \{0,1\}$, is either equal to zero (if $y_{ij} = 0$) or to β_{ij} (if $y_{ij} = 1$). By introducing the auxiliary variable $\gamma_{ij} \geq 0$ and adding the constraints

$$\gamma_{ij} \leq y_{ij}, \tag{13}$$

$$\gamma_{ij} \leq \beta_{ij} - y_{ij} + 1, \tag{14}$$

$$\gamma_{ij} \geq \beta_{ij} + y_{ij} - 1 \tag{15}$$

to the problem, one can replace the nonlinear term $\beta_{ij}y_{ij}$ by γ_{ij}.

Validity of this substitution is justified in the following theorem.

Theorem 2.4. *Constraints (13)-(15) justify the substitution $\gamma_{ij} = \beta_{ij}y_{ij}$.*

Proof. Let us complete the proof in two cases.
Case 1: When $y_{ij} = 0$ (no interdicting action happens), the nonnegativity of γ_{ij} and (13), implies $\gamma_{ij} = 0$.
Case 2: When $y_{ij} = 0$, constraints (14) and (15), leads to $\gamma_{ij} = \beta_{ij}$. □

Considering Theorem 2.4, by replacing of $\gamma_{ij} = \beta_{ij} y_{ij}$, $\lambda_{ij} = \theta_{ij} y_{ij}$, and $\mu_{ij} = \omega_{ij} y_{ij}$, the following mixed integer linear optimization problem is devised which is equivalent to (MOD2) in optimality. This problem can be solved by standard solvers.

$$
\begin{aligned}
\text{(MOD)} \quad \min \quad & \sum_{(i,j)\in A} \big(\alpha_{ij} a_{ij} + (1-\alpha_{ij}) b_{ij}\big)(\beta_{ij} - \gamma_{ij}) \\
& + \big(2\alpha_{ij} b_{ij} + (1-2\alpha_{ij}) c_{ij}\big)(\theta_{ij} - \lambda_{ij}) \\
& + \big((2\alpha_{ij} - 1) a_{ij} + (2 - 2\alpha_{ij}) b_{ij}\big)(\omega_{ij} - \mu_{ij}) \\
\text{subject to} \quad & \\
& \pi_i - \pi_j + \beta_{ij} + \theta_{ij} + \omega_{ij} \geq 0 \qquad \forall (i,j) \in A \\
& \pi_d - \pi_s \geq 1 \\
& \gamma_{ij} \leq \beta_{ij} - y_{ij} + 1 \qquad \forall (i,j) \in A_1 \\
& \gamma_{ij} \geq \beta_{ij} + y_{ij} - 1 \qquad \forall (i,j) \in A_1 \\
& \lambda_{ij} \leq \theta_{ij} - y_{ij} + 1 \qquad \forall (i,j) \in A_2 \\
& \lambda_{ij} \geq \theta_{ij} + y_{ij} - 1 \qquad \forall (i,j) \in A_2 \\
& \mu_{ij} \leq \omega_{ij} - y_{ij} + 1 \qquad \forall (i,j) \in A_2 \\
& \mu_{ij} \geq \omega_{ij} + y_{ij} - 1 \qquad \forall (i,j) \in A_2 \\
& \gamma_{ij},\, \lambda_{ij},\, \mu_{ij} \leq y_{ij} \qquad \forall (i,j) \in A \\
& \beta_{ij},\, \theta_{ij},\, \omega_{ij} \geq 0 \qquad \forall (i,j) \in A \\
& \gamma_{ij},\, \lambda_{ij},\, \mu_{ij} \geq 0 \qquad \forall (i,j) \in A \\
& y \in Y.
\end{aligned}
$$

3 An Illustrative Example

The proposed model is implemented on the problem introduced in [3] which consists 20 nodes and 30 arcs. The authors considered this problem as an NIP with deterministic arcs' capacity, while we assumed that some of them are uncertain linear variables $\mathcal{L}(a_{ij}, b_{ij})$ with $a_{ij} = u_{ij} - 2$ and $b_{ij} = u_{ij} + 2$, and the others are zigzag variables $\mathcal{Z}(a_{ij}, b_{ij}, c_{ij})$ with $a_{ij} = u_{ij} - 2$ and $b_{ij} = u_{ij} + 1$ and $c_{ij} = u_{ij} + 2$. The information of uncertain variables and necessary resource to interdict each arc are given in Table 2. A total of 9 units of resource is available for interdicting all the arcs (i. e., $R = 9$).

The belief degree α_{ij} is specified by experienced persons of the field which could be distinct for different arcs. Here, they are set identical as $\alpha_{ij} = \alpha = 0.9$ for all $(i, j) \in A$.

Recall that for a given value of the confidence level, only one of the constraints (9) and (10) exists in the problem. Therefore, the final mixed integer linear problem has 192 constraints and 170 variables and the $s-d$ maximum flow without interdicting action is 36 in an uncertain flow network. This amount equals to 44 units where there is no uncertainty assumption on the network [3].

Optimal solution of problem has been calculated by the branch-and-bound algorithm using GAMS 23.5.2 with the solver CPLEX 12.5. In both uncertain and deterministic environment, interdicted arcs are $(8, 16)$ and $(18, d)$, while the optimal objective value reduces to 23.6 units in uncertain environment, and to 29 units where there is no imposed uncertainty assumption on the problem [3].

The uncertainty distribution of Objective Value (OV) of (MOD) is listed in Table 3 and plotted in Fig.3 for different selections of α. It is clear from Table 3 that the best solution of (MOD) is obtained when $\alpha = 0.9$.

Table 2: Network data

Arc	Capacity	Resource	Arc	Capacity	Resource
(1,6)	$\mathcal{L}(6,10)$	4	(9,17)	$\mathcal{Z}(8,11,12)$	3
(1,7)	$\mathcal{L}(2,6)$	4	(10,17)	$\mathcal{L}(9,13)$	3
(1,8)	$\mathcal{L}(7,11)$	3	(11,17)	$\mathcal{Z}(11,14,15)$	2
(2,8)	$\mathcal{Z}(11,14,15)$	5	(12,d)	$\mathcal{L}(11,15)$	6
(2,9)	$\mathcal{Z}(5,8,9)$	4	(13,18)	$\mathcal{L}(11,15)$	3
(2,10)	$\mathcal{L}(4,8)$	5	(14,18)	$\mathcal{Z}(2,5,6)$	4
(3,10)	$\mathcal{L}(3,7)$	5	(15,18)	$\mathcal{L}(7,11)$	4
(3,11)	$\mathcal{Z}(10,13,14)$	5	(16,d)	$\mathcal{Z}(13,16,17)$	8
(3,12)	$\mathcal{Z}(5,8,9)$	5	(17,d)	$\mathcal{L}(9,13)$	6
(4,13)	$\mathcal{Z}(6,9,10)$	3	(18,d)	$\mathcal{Z}(12,15,16)$	7
(4,14)	$\mathcal{L}(13,17)$	2	(s,1)	$\mathcal{L}(9,13)$	8
(5,15)	$\mathcal{L}(6,10)$	6	(s,2)	$\mathcal{Z}(6,9,10)$	9
(6,16)	$\mathcal{L}(8,12)$	3	(s,3)	$\mathcal{L}(9,13)$	7
(7,16)	$\mathcal{Z}(2,5,6)$	4	(s,4)	$\mathcal{L}(11,15)$	10
(8,16)	$\mathcal{L}(5,9)$	2	(s,5)	$\mathcal{Z}(4,7,8)$	12

Table 3: The value of (MOD) corresponding to the different choosing of α

α	interdicted arcs	OV
0.9	$(8,16)$, $(18,d)$	23.6
0.8	$(8,16)$, $(18,d)$	24.8
0.7	$(8,16)$, $(18,d)$	26.2
0.6	$(8,16)$, $(18,d)$	27.4
0.5	$(8,16)$, $(18,d)$	28.5
0.4	$(18,d)$	30.6
0.3	$(18,d)$	31.7
0.2	$(18,d)$	32.8
0.1	$(18,d)$	33.9

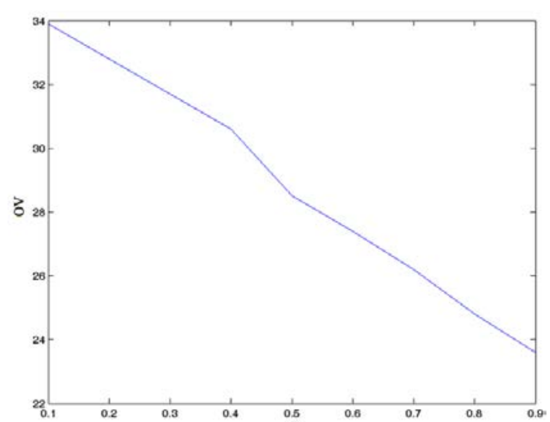

Figure 1: Uncertainty distribution of (MOD)

4 Challenges in Stochastic Case

Liu declared that uncertainty theory is only legitimate approach when only belief degrees are available [11].

In the sequel, we first review the model in stochastic case and mention some existing chalenges. Then we show that using probability theory to model the belief degree in our problem could lead to incorrect conclusion.

Let us consider the general probabilistic constraint as

$$G(x) = \mathrm{Pr}_\xi(t^T x \geq h(\xi)),$$

where t is a deterministic vector, x is a decision vector, $h(\xi)$ is the right-hand-side depending to the random variable ξ that has the probability distribution function F_ξ. Here, x is a decision vector independent of the random variable xi. As a result, the associate chance constraint would be $G(x) \geq \alpha$ where $\alpha \in [0,1]$.

Let $B(\alpha) = \{x|G(x) \geq \alpha\}$. It was proved that this set is closed [6]. Moreover, it is convex for $\alpha = 0$ and $\alpha = 1$ [5]. In the case of $\alpha = 0$ the probability constraint is clearly redundant. If $\alpha = 1$, then the solution of the associated optimization problem can be interpreted as a "fat solution", in a probabilistic sense [7]. The problem with this class of constraints can lead, under appropriate assumptions, to convex optimization problems. Obviously we have $G(x) = F_\xi(t^T x)$, and this is equivalent to a linear constraint

$$\mathrm{Pr}(t^T x \geq \xi) \geq \alpha \Leftrightarrow F(t^T x) \geq \alpha \Leftrightarrow t^T x \geq Q^- \xi(\alpha),$$

where $Q^-\xi(\alpha)$ denotes the left end-point of the closed interval of αquantiles of F_ξ. some care is needed if ξ does not have a continuous distribution. For instance, a finite discrete distribution forξ , the theoretically correct reformulation may consist of the strict inequality, which from the modeling point of view this is usually not a real problem. Moreover, finding exact representation of $Q^-\xi(\alpha)$ is impossible in some cases (e.g. in Normal distribution), while it has a clear representation in uniform distribution. Observe that this is not the case in uncertainty theory when ξ is treated as an uncertain variable.

As mentioned beforehand, using probability theory in the case when only expert opinion is the base of decision making process will lead to misleading results. Forexample, let the continuous uniform distributions $\mathcal{U}(a_{ij}, b_{ij})$, and constraint (2) is replaced by

$$\mathrm{Pr}(x_{ij} \leq \xi_{ij}(1 - y_{ij})) \geq p_{ij}, \quad \forall (i,j) \in A,$$

where p_{ij} is the minimum prespecified probability level that the constraint $\mathrm{Pr}(x_{ij} \leq \xi_{ij}(1 - y_{ij}))$ holds. Consequently by

$$F^{-1}_{\xi_{ij}}(p_{ij}) = a_{ij}(1 - p_{ij}) + b_{ij}p_{ij} \quad \text{for } 0 < p_{ij} < 1,$$

where $F_{\xi_{ij}}$ is its probability distribution function. Though the final form of the problem is similar to problem (MOD), the interpretations are completely different. Independency in probability theory implies a wrong conclusion as stated in [12]. For example, let the uncertain network G have m arcs, and for k of them, let $u_{ij} = b_{ij}$ and $x_{ij} = (a_{ij} + b_{ij})/2$. In the absence of interdicting, from the uncertainty theory

$\mathcal{M}${"The amount of passing flow from these k arcs is not greater than the uncertain predefined value"}=1,

whereas, in probability theory, for enough large k, it holds

Pr{"The amount of passing flow from these k arcs is not greater than the uncertain predefined value"}

=$(0.5)^k \simeq 0$.

Observe that this does not make sense in most practical situations.

5 Conclusion

This paper proposed a model of the network interdiction problem with uncertain arc capacity. It is proved that the model can be transformed into the corresponding deterministic bi-level mixed integer optimization problem when the uncertain variable is linear or zigzag. The solution approach consisted of transforming the bi-level problem to a single-level mixed-integer problem in the first stage and then applying a standard linearization method to produce a mixed integer linear optimization problem. The model is implemented on an example to illustrate the theoretical considerations.

There are other kinds of uncertain variables introduced in [12] such as empirical variables. Moreover, we assumed that all of the uncertain variables are independent and identically distributed. What would be the result if these are not the cases? Respecting these concerns would be of special interest and the possible direction of further study.

Appendix A. Preliminaries

In this part, some necessary concepts of uncertainty theory are briefly reviewed. For more details, we refer to [12].

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. A set function $\mathcal{M} : \mathcal{L} \longrightarrow [0,1]$ is called an *uncertain measure* if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, it holds

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertainty space*. Furthermore, Liu defined a product uncertain measure as the fourth axiom as follows.
Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is the satisfying

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\},$$

where Λ_k is an arbitrarily chosen event from $\mathcal{L}_k$, for $k = 1, 2, \ldots$.

It is of value to remind that the probability measure satisfies the first three but not the last axiom [12]. An *uncertain variable* is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\gamma | \xi(\gamma) \in B\}$ is an event for any Borel set B. The *uncertainty distribution* Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x. $\Phi(x)$ denotes the belief degree that the uncertain variable ξ is at least x.

An uncertain variable ξ is called *linear* and denoted by $\mathcal{L}(a, b)$ if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ \dfrac{x-a}{b-a}, & \text{if} \quad a \leq x \leq b \\ 1, & \text{if} \quad x \geq b, \end{cases}$$

where a and b are real numbers with $a < b$. An uncertain variable ξ is called *zigzag* if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ \dfrac{x-a}{2(b-a)}, & \text{if} \quad a \leq x \leq b \\ \dfrac{x+c-2b}{2(c-b)}, & \text{if} \quad b \leq x \leq c \\ 1, & \text{if} \quad x \geq c, \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$ where a, b, and c are real numbers with $a < b < c$.

An uncertainty distribution $\Phi(x)$ is said to be *regular* if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \qquad \lim_{x \to +\infty} \Phi(x) = 1.$$

For an uncertain variable ξ with regular uncertainty distribution $\Phi(x)$, the inverse function $\Phi^{-1}(\alpha)$ is well-defined and called the inverse uncertainty distribution of ξ. The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is [12]

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b, \tag{16}$$

and the inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is [12]

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \geq 0.5. \end{cases} \tag{17}$$

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\Big\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\Big\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

References

[1] Carrigy, A., Ramirez-Marquez, J.E., and C.M. Rocco, Multi-state stochastic network interdiction via reliability modeling & evolutionary optimization, *Journal of Risk and Reliability*, vol.224, no.1, pp.27–42, 2009.

[2] Cormican, K.J., Morton, D.P., and R.K. Wood, Stochastic network interdiction, *Operations Research*, vol.46, pp.184–197, 1998.

[3] Dai, Y., and K. Poh, Solving the network interdiction problem with genetic algorithms, *Proceedings of the Fourth Asia-Pacific Conference on Industrial Engineering and Management System*, 2002.

[4] Han, S.W., Peng, Z.X., and S.Q. Wang, The maximum flow problem of uncertain network, *Information Sciences*, vol.265, pp.167–175, 2014.

[5] Kall, P., *Stochastic Linear Programming*, Springer, 1976.

[6] Kall, P., and S.W. Wallace, *Stochastic Programming*, John Wiley & Sons, Chichester, 1994.

[7] Kall, p., and J. Mayer, *Stochastic Linear Programming: Models, Theory, and Computation*, Springer US, 2011.

[8] Liu, B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[9] Liu, B., Some research problems in uncertainty theory, *Journal of Uncertain Systems*, vol.3, no.1, pp.3–10, 2009.

[10] Liu, B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertain*, Springer-Verlag, Berlin, 2010.

[11] Liu, B., Why is there a need for uncertainty theory? *Journal of Uncertain Systems*, vol.6, no.1, pp.3–10, 2012.

[12] Liu, B., *Uncertainty Theory*, 5nd Edition, Uncertainty Theory Laboratory, 2015.

[13] McMasters, A.W., and T.M. Mustin, Optimal interdiction of a supply network, *Naval Research Logistics Quarterly*, vol.17, pp.261–268, 1970.

[14] Morton, D., Pan, F., and K. Saeger, Models for nuclear smuggling interdiction, *IIE Transactions*, vol.39, pp.3–14, 2007.

[15] Pan, F., *Stochastic Network Interdiction: Models and Methods*, Ph.D. Thesis, The University of Texas, Austin, 2005.

[16] Ramirez-Marquez, J.E., and S.C.M. Rocco, Stochastic network interdiction optimization via capacitated network reliability modeling and probabilistic solution discovery, *Reliability Engineering and System Safety*, vol.94, no.5, pp.913–921, 2008.

[17] Wollmer, R.D., Removing arcs from a network, *Operations Research*, vol.12, pp.934–940, 1964.

[18] Wood, R.K., Deteministic network interdiction, *Mathematical and Computer Modeling*, vol.17, pp.1–18, 1993.

[19] Woodruff, D.L., *Network Interdiction and Stochastic Integer Programming*, Kluwer Academic Publishers, 2003.

[20] Zheng, J., and D.A. Castanon, Stochastic dynamic network interdiction games, *American Control Conference*, 2012.