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Chance Order of Two Uncertain Random Variables

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Abstract

Comparing and ordering of uncertain random variables give a guideline to make decisions in uncertain random environments, so we study the comparison of two uncertain random variables. For this purpose, we define a new order which we call the chance order and study some of its basic properties. We then apply the results to order the lifetime of the k-out-of-n systems when the lifetimes of some components of the system are random variables while the others are uncertain variables. It is worth mentioning that when two uncertain random variables are both random variables or uncertain variables, the chance order becomes the stochastic order and the uncertain dominance, respectively. That is, the concept of chance order of uncertain random variables extends the concepts of the stochastic order of random variables and the uncertain dominance of uncertain variables.

Keywords: uncertain random variable, uncertain dominance, stochastic order, chance order, uncertain random system

1 Introduction

When there are enough samples to estimate the probability distributions, probability theory provides an effective mathematical tool to deal with such a condition. However, sometimes no samples are available. In this case, we have to invite some domain experts to give their belief degrees about the indeterminacy quantity. It has been shown that the belief degree function has a much larger variance than the real cumulative frequency. For instance, Liu [22] showed that human beings usually estimate a much wider range of values than the object actually takes. The conservative estimation of belief degrees by human beings deviates far from the frequency. Hence, the belief degree could not be modeled by the probability measure. Thus probability theory is no longer applicable. Some other convincing examples may be found in Liu [20]. In order to deal with belief degrees, uncertainty theory was proposed by Liu [16] and is becoming a branch of axiomatic mathematics for modeling human uncertainty. Nowadays, uncertainty theory is well developed in both theoretical and practical aspects, for further details, see [18, 19]. In real situations, some quantities may be modeled by random variables while some others by uncertain variables. Consequently, it is reasonable to assume that randomness and uncertainty co-exist in a complex system. In order to deal with this type of indeterminacy, Liu [24] proposed the concept of uncertain random variable that is a measurable function from a chance space to the set of real numbers. The concept of the chance measure, the expected value, and the variance of an uncertain random variable were also presented by Liu [24]. As an important contribution to the chance theory, Liu [23] put forward the operational law for uncertain random variables. In addition, Guo and Wang [11] proved a formula to calculate the variance of the uncertain random variable using the uncertainty distribution. Sheng and Yao [28] verified some formulas to calculate the variance of the uncertain random variable by using the inverse uncertainty distribution. Yao and Gao [30] proved the law of large numbers for independent uncertain random variables with a common chance distribution in the sense of distribution. Gao and Sheng [7] proved another law of large numbers for uncertain random variables with different chance distributions. And Gao and Ralescu [6] proved the convergence in distribution for a sequence of uncertain random variables without a common

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chance distribution. Also, two types of concepts of convergence in mean and convergence in distribution for the sequence of uncertain random variables were put forward by Gao and Ahmadzade [3]. The chance theory has been also successfully applied to another problems such as the uncertain random programming [14, 23], the uncertain random risk analysis [25], the uncertain random reliability analysis [8, 9, 29], the uncertain random graph and the uncertain random network [21], the uncertain random process [10], and the uncertain random logic [26].

Comparison of two variables is a very important topic in many fields for instance in the fields of reliability theory, finance, and survival studies. Comparison of two random variables has been widely studied and used. In this comparison, we sometimes face with the problem that there are just a few or even no samples so that we can not estimate their probability distributions via statistics. Therefore, comparison of two variables based on uncertainty theory has been proposed [5, 31]. The aim of this work is to deal with problems that have both kinds of variables (random variables and uncertain variables) together. Therefore, this paper proposes an order of comparing based on chance theory which is a generalization of both probability theory and uncertainty theory. For more clarification, this order will also be applied to compare the lifetime of the k-out-of-n system.

The rest of this paper is organized as follows: Some basic definitions and properties with respect to probability theory, uncertainty theory, and chance theory are reviewed in Section 2. Section 3 is devoted to the presentation of our order of uncertain random variables and the investigation of some basic properties of this order. The application of this order to k-out-of-n system is also studied in section 4. Finally, some remarks are made in Section 5.

2 Preliminaries

In this section, we review some concepts of probability theory, uncertainty theory and chance theory, including probability measure, random variable, probability distribution, uncertain measure, uncertain variable, uncertainty distribution, chance measure, uncertain random variable, chance distribution, operational law, expected value, variance.

2.1 Probability Theory

In this subsection, we provide some elementary definitions of probability theory that will be used in the next sections (For details, see [12]).

Let $\mathscr{F}$ be a σ-algebra over a nonempty set Ω. A set function $\Pr : \mathscr{F} \longrightarrow [0,1]$ is called a probability measure if it satisfies the following axioms:
(i) (Normality Axiom) $\Pr\{\Omega\} = 1$ for the universal set Ω.
(ii) (Nonnegativity Axiom) $\Pr\{A\} \geq 0$ for any event $A \in \mathscr{F}$.
(iii) (Additivity Axiom) For every countable sequence of mutually disjoint events $A_1, A_2, \ldots$, we have

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}.$$

The triple $(\Omega, \mathscr{F}, \Pr)$ is called a probability space. The set function Pr is called a probability measure if it satisfies the normality, nonnegativity, and additivity axioms. Besides, the product probability on the product σ-algebra $\mathscr{F}$ is defined as follows. Let $(\Omega_k, \mathscr{F}_k, \Pr_k)$ be probability spaces for $k = 1, 2, \ldots$. The product probability measure Pr is a probability measure satisfying

$$\Pr\{\prod_{k=1}^{\infty} A_k\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\}, \tag{2.1}$$

where A_k are arbitrary events chosen from $\mathscr{F}_k$ for $k = 1, 2, \ldots$, respectively. This conclusion is called product probability theorem. And such a product probability measure is denoted by

$$\Pr = \prod_{k=1}^{\infty} \Pr_k. \tag{2.2}$$

A random variable η is a measurable function from a probability space $(\Omega, \mathscr{F}, \Pr)$ to the set of real numbers such that $\{\eta \in B\}$ is an event for any Borel set B of real numbers. In order to describe a random variable in practice, a concept of probability distribution function is defined as

$$\Psi_\eta(y) = \Pr(\eta \le y) \tag{2.3}$$

for any real number y. If there is a function ψ satisfying

$$\Psi_\eta(y) = \int_{-\infty}^{y} \psi_\eta(t)\mathrm{d}t \tag{2.4}$$

for any real number y, then $\psi_\eta(\cdot)$ is called the probability density function of a continuous random variable η. The random variables $\eta_1, \eta_2, \ldots, \eta_m$ are said to be independent if

$$\Pr\left\{\bigcap_{i=1}^{m}\{\eta_i \in B_i\}\right\} = \prod_{k=1}^{m} \Pr\{\eta_i \in B_i\} \tag{2.5}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Theorem 2.1 ([8]) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distribution functions $\Psi_{\eta_1}(\cdot)$, $\Psi_{\eta_2}(\cdot)$, $\ldots$, $\Psi_{\eta_m}(\cdot)$, respectively, and $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ a measurable function. then the random variable*

$$\eta = f(\eta_1, \eta_2, \cdots, \eta_m) \tag{2.6}$$

has a probability distribution function

$$\Psi_\eta(y) = \int_{f(y_1,y_2,\cdots,y_m)\le y} \mathrm{d}\Psi_{\eta_1}(y_1)\mathrm{d}\Psi_{\eta_2}(y_2)\cdots\mathrm{d}\Psi_{\eta_m}(y_m). \tag{2.7}$$

If $\eta_1, \eta_2, \ldots, \eta_m$ have probability density functions $\psi_{\eta_1}(\cdot)$, $\psi_{\eta_2}(\cdot)$, $\ldots$, $\psi_{\eta_m}(\cdot)$, respectively, then $\eta = f(\eta_1, \eta_2, \cdots, \eta_m)$ has a probability distribution function

$$\Psi_\eta(y) = \int_{f(y_1,y_2,\cdots,y_m)\le y} \psi_{\eta_1}(y_1)\psi_{\eta_2}(y_2)\cdots\psi_{\eta_m}(y_m)\mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_m. \tag{2.8}$$

Definition 2.1 ([27]) *Let η_1 and η_2 be two random variables such that*

$$\Pr(\eta_1 > t) \le \Pr(\eta_2 > t) \ \textit{for any}\ t \in \mathbb{R}. \tag{2.9}$$

Then η_1 is said to be smaller than η_2 in stochastic order, denoted by $\eta_1 \preceq_{st} \eta_2$. Note that (2.9) *is equivalent to*

$$\Pr(\eta_1 \le t) \ge \Pr(\eta_2 \le t) \ \textit{for any}\ t \in \mathbb{R}. \tag{2.10}$$

Theorem 2.2 ([2]) *Let η_1 and η_2 be two random variables. Then $\eta_1 \preceq_{st} \eta_2$ if and only if $E[f(\eta_1)] \le E[f(\eta_2)]$ for all real valued increasing function f such that the expectations exist.*

Theorem 2.3 ([27]) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be a set of independent random variables and let $\eta_1^{'}, \eta_2^{'}, \ldots, \eta_m^{'}$ be another set of independent random variables. If $\eta_i \preceq_{st} \eta_i^{'}$ for $i = 1, 2, \ldots, m$, then for any increasing function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$, we have*

$$f(\eta_1, \eta_2, \cdots, \eta_m) \preceq_{st} f(\eta_1^{'}, \eta_2^{'}, \cdots, \eta_m^{'}). \tag{2.11}$$

In particular,

$$\sum_{i=1}^{m} \eta_i \preceq_{st} \sum_{i=1}^{m} \eta_i^{'}. \tag{2.12}$$

Definition 2.2 ([8]) *Let $\eta_1, \ldots, \eta_m$ be random variables, and let k be an index with $1 \leq k \leq m$. Then*

$$\eta^{(k)} = k - \min[\eta_1, \cdots, \eta_m]$$

is called the kth order statistic of $\eta_1, \ldots, \eta_m$, where k-min represents kth smallest value.

Theorem 2.4 ([8]) *Let $\eta_1, \eta_2 \ldots, \eta_m$ be independent random variables with probability distribution functions $\Psi_{\eta_1}(\cdot)$, $\Psi_{\eta_2}(\cdot)$, …, $\Psi_{\eta_m}(\cdot)$, respectively. Then the kth order statistic of $\eta_1, \ldots, \eta_m$ has a probability distribution function*

$$\Psi_{\eta^{(k)}}(y) = \int_{\mathbb{R}^m} k - \max[I(y_1 \leq y), \cdots, I(y_m \leq y)] \mathrm{d}\Psi_{\eta_1}(y_1) \cdots \mathrm{d}\Psi_{\eta_m}(y_m),$$

where k-max represents kth largest value and $I(\cdot)$ is an indictor function.

2.2 Uncertainty Theory

In this subsection, we recall some elementary definitions of uncertainty theory which are used in the next sections. (For more details, see [16, 17]).

Let $\mathcal{L}$ be a σ-algebra over a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:
(i) (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
(ii) (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
(iii) (Subadditivity Axiom) $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$.
(iv) (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots,$ the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$.

Definition 2.3 ([19]) *The events $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\},$$

such that Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \ldots, n$.

Definition 2.4 ([22]) *An uncertain variable τ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\tau \in B\}$ is an event for any Borel set B of real numbers.*

Definition 2.5 ([22]) *The uncertain variables $\tau_1, \tau_2, \ldots, \tau_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \{\tau_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 2.5 ([22]) *Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables, and f_1, f_2, …, f_n be measurable functions. Then $f_1(\tau_1)$, $f_2(\tau_2)$, …, $f_n(\tau_n)$ are independent uncertain variables.*

Definition 2.6 ([16]) *Let τ be an uncertain variable. Its uncertainty distribution function is*

$$\Upsilon_\tau(x) = \mathcal{M}(\tau \leq x)$$

for any real number x.

Theorem 2.6 ([19]) *Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with continuous uncertainty distribution functions $\Upsilon_{\tau_1}(\cdot), \Upsilon_{\tau_2}(\cdot), \ldots, \Upsilon_{\tau_n}(\cdot)$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to x_{m+1}, x_{m+2}, ..., x_n, then the uncertain variable*

$$\tau = f(\tau_1, \tau_2, \cdots, \tau_n)$$

has an uncertainty distribution function as

$$\Upsilon_\tau(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \left(\min_{1\le i\le m} \Upsilon_{\tau_i}(x_i) \wedge \min_{m+1\le i\le n} (1 - \Upsilon_{\tau_i}(x_i)) \right).$$

Definition 2.7 ([17]) *Let τ be an uncertain variable with regular uncertainty distribution function $\Upsilon_\tau(x)$. Then the inverse function $\Upsilon_\tau^{-1}(x)$ is called the inverse uncertainty distribution function of τ.*

Theorem 2.7 ([17]) *Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with regular uncertainty distribution functions $\Upsilon_{\tau_1}(\cdot), \Upsilon_{\tau_2}(\cdot), \ldots, \Upsilon_{\tau_n}(\cdot)$, respectively. If f is a component-wise strictly increasing function, then $\tau = f(\tau_1, \tau_2, \cdots, \tau_n)$ is an uncertain variable with inverse uncertainty distribution function*

$$\Upsilon_\tau^{-1}(\alpha) = f(\Upsilon_{\tau_1}^{-1}(\alpha), \Upsilon_{\tau_2}^{-1}(\alpha), \cdots, \Upsilon_{\tau_n}^{-1}(\alpha)).$$

Definition 2.8 ([15]) *Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set*

$$\{\tau_t \in B\} = \{\gamma \in \Gamma | \tau_t(\gamma) \in B\}$$

is an event.

Definition 2.9 ([5]) *Let $(\tau_1, \tau_2, \cdots, \tau_n)$ and $(\tau_1^{'}, \tau_2^{'}, \cdots, \tau_n^{'})$ be uncertain vectors. If*

$$\mathcal{M}(\tau_1 \le x_1, \cdots, \tau_n \le x_n) = \mathcal{M}(\tau_1^{'} \le x_1, \cdots, \tau_n^{'} \le x_n) \tag{2.13}$$

for any real numbers $x_1, x_2, \cdots, x_n \in \mathbb{R}$, then $(\tau_1, \tau_2, \cdots, \tau_n)$ is said to be identically distributed with $(\tau_1^{'}$, $\tau_2^{'}, \cdots, \tau_n^{'})$, denoted by $(\tau_1, \tau_2, \cdots, \tau_n) =_D (\tau_1^{'}, \tau_2^{'}, \cdots, \tau_n^{'})$.

Definition 2.10 ([5]) *Let τ_1 and τ_2 be two uncertain variables such that*

$$\mathcal{M}(\tau_1 > t) \le \mathcal{M}(\tau_2 > t) \ for\ any\ t \in \mathbb{R}, \tag{2.14}$$

then τ_1 is said to be smaller than τ_2 in uncertain dominance, denoted by $\tau_1 \preceq_{un} \tau_2$. Note that (2.14) is equivalent to

$$\mathcal{M}(\tau_1 \le t) \ge \mathcal{M}(\tau_2 \le t) \ for\ any\ t \in \mathbb{R}. \tag{2.15}$$

Theorem 2.8 ([5]) *Let $\tau_1, \tau_2, \ldots, \tau_n$ be a set of independent uncertain variables and let $\tau_1^{'}, \tau_2^{'}, \ldots, \tau_n^{'}$ be another set of independent uncertain variables. If $\tau_i \preceq_{un} \tau_i^{'}$, for $i = 1, 2, \ldots, n$, then, for any component-wise strictly increasing function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we have*

$$f(\tau_1, \tau_2, \cdots, \tau_n) \preceq_{un} f(\tau_1^{'}, \tau_2^{'}, \cdots, \tau_n^{'}). \tag{2.16}$$

In particular,

$$\sum_{i=1}^{n} \tau_i \preceq_{un} \sum_{i=1}^{n} \tau_i^{'}. \tag{2.17}$$

Similar to random variables, the values of uncertain variables also might be arranged in an ascending order for providing us with useful summary information. Consequently, Gao et al. [4] described the concept of order statistics for uncertain variables and presented the uncertainty distribution function of the kth order statistic.

Definition 2.11 ([4]) *Let $\tau_1, \ldots, \tau_n$ be uncertain variables, and let k be an index with $1 \le k \le n$. Then*

$$\tau^{(k)} = k - \min[\tau_1, \cdots, \tau_n]$$

is called the kth order statistic of $\tau_1, \ldots, \tau_n$.

Theorem 2.9 ([4]) *Let $\tau_1, \ldots, \tau_n$ be independent uncertain variables with uncertainty distribution functions $\Upsilon_{\tau_1}(\cdot), \ldots, \Upsilon_{\tau_n}(\cdot)$, respectively. Then the kth order statistic of $\tau_1, \ldots, \tau_n$ has an uncertainty distribution function*

$$\Upsilon_{\tau^{(k)}}(x) = k - \max[\Upsilon_{\tau_1}(x), \Upsilon_{\tau_2}(x), \cdots, \Upsilon_{\tau_n}(x)].$$

2.3 Chance Theory

The chance space is refer to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathscr{F}, \Pr)$, where $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathscr{F}, \Pr)$ is a probability space.

Definition 2.12 ([24]) *Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathscr{F}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathscr{F}$ be an uncertain random event. Then the chance measure of Θ is defined as*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \ge r\} \mathrm{d}r.$$

Note that the chance measure is in fact the expected value of the random variable $\mathcal{M}(\xi(\cdot) \in \Theta)$, i.e.,

$$\mathrm{Ch}(\xi \in \Theta) = E_{\Pr}(\mathcal{M}(\xi(\cdot) \in \Theta)), \tag{2.18}$$

where $E_{\Pr}(\cdot)$ denotes the expected value operator for random variable under probability measure $\Pr(\cdot)$.

Liu [24] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is
(i) $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$;
(ii) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ for any event Θ;
(iii) $\mathrm{Ch}\{\Theta_1\} \le \mathrm{Ch}\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$.

Besides, Hou [13] proved the subadditivity of chance measure, that is,

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \le \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\} \tag{2.19}$$

for a sequence of events $\{\Theta_n, n \ge 1\}$.

Definition 2.13 ([24]) *An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathscr{F}, \Pr)$ to the set of real numbers, i.e., $\{\xi \in B\}$ is an event for any Borel set B of real numbers.*

To calculate the chance measure, Liu [23] presented a definition of chance distribution function.

Definition 2.14 ([23]) *Let ξ be an uncertain random variable. Then its chance distribution function is defined by*

$$\Phi_\xi(x) = \mathrm{Ch}(\xi \le x)$$

for any $x \in \mathbb{R}$.

The chance distribution function of a random variable is simply its probability distribution function, and the chance distribution function of an uncertain variable is simply its uncertainty distribution function.

Theorem 2.10 ([23]) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distribution functions $\Psi_{\eta_1}(\cdot)$, $\Psi_{\eta_2}(\cdot), \ldots, \Psi_{\eta_m}(\cdot)$, respectively, and let $\tau_1, \tau_2, \ldots, \tau_n$ be uncertain variables. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ has a chance distribution function*

$$\Phi_\xi(x) = \int_{\mathbb{R}^m} F(x, y_1, \cdots, y_m) \mathrm{d}\Psi_{\eta_1}(y_1) \cdots \mathrm{d}\Psi_{\eta_m}(y_m),$$

where $F(x, y_1, y_2, \cdots, y_m)$ is the uncertainty distribution function of uncertain variable $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \ldots, y_m$.

Definition 2.15 ([10]) *Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathscr{F}, \Pr)$ be a chance space. An uncertain random process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathscr{F}, \Pr)$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set*

$$\{\xi_t \in B\} = \{(\omega, \gamma) \in \Omega \times \Gamma | \xi_t(\omega, \gamma) \in B\}$$

is an event.

Definition 2.16 ([8]) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables, $\tau_{m+1}, \tau_{m+2}, \ldots, \tau_n$ be uncertain variables, and $1 \le k \le n$. Then*

$$(\eta\tau)^{(k)} = k - \min[\eta_1, \eta_2, \cdots, \eta_m, \tau_{m+1}, \tau_{m+2}, \cdots, \tau_n]$$

is called the kth order statistic of $\eta_1, \eta_2, \ldots, \eta_m, \tau_{m+1}, \tau_{m+2}, \ldots, \tau_n$.

Theorem 2.11 ([8]) *Let $\eta_1, \ldots, \eta_m$ be independent random variables with probability distribution functions $\Psi_{\eta_1}(\cdot), \ldots, \Psi_{\eta_m}(\cdot)$, and $\tau_{m+1}, \ldots, \tau_n$ be uncertain variables with uncertainty distribution functions $\Upsilon_{\tau_{m+1}}(\cdot), \ldots, \Upsilon_{\tau_n}(\cdot)$, respectively. Then the kth order statistic of $\eta_1, \ldots, \eta_m, \tau_{m+1}, \ldots, \tau_n$ has the chance distribution function*

$$\Phi_{(\eta\tau)^{(k)}}(x) = \int_{\mathbb{R}^m} k - \max\left\{I(y_1 \le x), \cdots, I(y_m \le x), \Upsilon_{\tau_{m+1}(x)}, \cdots, \Upsilon_{\tau_n(x)}\right\} \mathrm{d}\Psi_{\eta_1}(y_1) \cdots \mathrm{d}\Psi_{\eta_m}(y_m).$$

3 Definition and Properties

In the real world, a complex system always includes not only random variables but also uncertain variables. To compare such systems, we need an order. Naturally, this order must be based on uncertain random variables, as an extension of orders of uncertain and random variables. In this section, we introduce an order, called the chance order, and study some of its basic properties.

Definition 3.1 *The uncertain random variable ξ_1 is said to be smaller than the uncertain random variable ξ_2 in chance ordering, denoted by $\xi_1 \preccurlyeq_{ch} \xi_2$, if*

$$\mathrm{Ch}(\xi_1 > t) \le \mathrm{Ch}(\xi_2 > t) \ \textit{for any} \ t \in \mathbb{R}.$$

Remark 3.1 *When ξ_1 and ξ_2 are both random variables or uncertain variables, chance order becomes the stochastic order (Definition 2.1) and the uncertain dominance (Definition 2.10), respectively. Therefore, the concept of the chance order of uncertain random variables extends the stochastic order of random variables and the uncertain dominance of uncertain variables.*

Proposition 3.1 *If $\xi_1 \preceq_{ch} \xi_2$, and $f(\cdot)$ is any increasing (decreasing) function, then*

$$f(\xi_1) \preceq_{ch} (\succeq_{ch}) f(\xi_2). \tag{3.1}$$

Proof: Since $f(\cdot)$ is an increasing (decreasing) function, for all $t \in \mathbb{R}$, we have

$$\mathrm{Ch}(f(\xi_1) > t) = \mathrm{Ch}(\xi_1 > f^{-1}(t)) \le (\ge) \mathrm{Ch}(\xi_2 > f^{-1}(t)) = \mathrm{Ch}(f(\xi_2) > t). \tag{3.2}$$

This completes the proof.

Proposition 3.2 *The uncertain random variable ξ_1 with the chance distribiution function $\Phi_{\xi_1}(\cdot)$ is said to be smaller than the uncertain random variable ξ_2 with the chance distribiution function $\Phi_{\xi_2}(\cdot)$ in the chance ordering if and only if*

$$\Phi_{\xi_1}(t) \ge \Phi_{\xi_2}(t) \ \textit{for any} \ t \in \mathbb{R}. \tag{3.3}$$

Proof: It follows from the definition of the chance order which for each $t \in \mathbb{R}$

$$\begin{aligned} \mathrm{Ch}(\xi_1 > t) \leq \mathrm{Ch}(\xi_2 > t) &\iff \mathrm{Ch}(\xi_1 \leq t) \geq \mathrm{Ch}(\xi_2 \leq t) \\ &\iff \Phi_{\xi_1}(t) \geq \Phi_{\xi_2}(t) \end{aligned}$$

Thus, this completes the proof.

Corollary 3.1 *If $\xi_1 \preceq_{ch} \xi_2$, and $\xi_2 \preceq_{ch} \xi_3$, then*

$$\xi_1 \preceq_{ch} \xi_3. \tag{3.4}$$

Example 3.1 *Let η_1, η_2 be two random variables with continuous distribution functions $U(0,1)$ and $U(-1,0)$, and let τ_1, τ_2 be two uncertain variables with continuous uncertainty distribution functions $L(0,1)$ and $L(2,3)$, respectively. Then the chance distribution functions $\xi_1 = \eta_1 + \tau_1$ and $\xi_2 = \eta_2 + \tau_2$ are*

$$\Phi_{\xi_1}(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t, & 0 \leq t < 1 \\ -\frac{1}{2}t^2 + 2t - 1, & 1 \leq t < 2 \\ 1, & t \geq 2 \end{cases}$$

and

$$\Phi_{\xi_2}(t) = \begin{cases} 0, & t < 1 \\ \frac{1}{2}t^2 - t + \frac{1}{2}, & 1 \leq t < 2 \\ -\frac{1}{2}t^2 + 3t - \frac{7}{2}, & 2 \leq t < 3 \\ 1, & t \geq 3. \end{cases}$$

Since for any $t \in \mathbb{R}$

$$\Phi_{\xi_2}(t) \leq \Phi_{\xi_1}(t),$$

we conclude that $\xi_1 \preccurlyeq_{ch} \xi_2$.

Example 3.2 *Let τ_1, τ_2 be independent uncertain variables with uncertainty distribiution functions $\Upsilon_{\tau_1}(\cdot)$, $\Upsilon_{\tau_2}(\cdot)$, and $\tau_1^{'}, \tau_2^{'}$ be independent uncertain variables with uncertainty distribiution functions $\Upsilon_{\tau_1^{'}}(\cdot)$, $\Upsilon_{\tau_2^{'}}(\cdot)$, respectively. Then the uncertain random variable*

$$\xi = \left\{ \begin{array}{ll} \tau_1 & \text{with probability } p; \\ \tau_2 & \text{with probability } q, \end{array} \right. \tag{3.5}$$

is smaller than the uncertain random variable

$$\xi^{'} = \left\{ \begin{array}{ll} \tau_1^{'} & \text{with probability } p; \\ \tau_2^{'} & \text{with probability } q, \end{array} \right. \tag{3.6}$$

if and only if for any $t \in \mathbb{R}$,

$$\Phi_{\xi}(t) = p\Upsilon_{\tau_1}(t) + q\Upsilon_{\tau_2}(t) \geq p\Upsilon_{\tau_1^{'}}(t) + q\Upsilon_{\tau_2^{'}}(t) = \Phi_{\xi'}(t). \tag{3.7}$$

Especially, if for all $t \in \mathbb{R}$, $\Upsilon_{\tau_i}(t) \geq \Upsilon_{\tau_i^{'}}(t)$, $i = 1, 2$, then $\Phi_{\xi}(t) \geq \Phi_{\xi'}(t)$.

Remark 3.2 *Since any uncertain random variable is a function of uncertain variables and random variables, in the following, we derive the chance order of two uncertain random variables from the stochastic order between their random components and the uncertain dominance between their uncertain components.*

Let η_1, η_2 be random variables and τ_1, τ_2 be uncertain variables, and if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is strictly monotone function component-wise, then $f(\eta_1, \tau_1)$, $f(\eta_2, \tau_2)$ are uncertain random variables. In the following, we derive the chance order of $\xi_1 = f(\eta_1, \tau_1)$, $\xi_2 = f(\eta_2, \tau_2)$.

Theorem 3.3 *Let η_1, η_2 be random variables with probability distribiution functions $\Psi_{\eta_1}(\cdot)$, $\Psi_{\eta_2}(\cdot)$, and τ_1, τ_2 be uncertain variables with uncertainty distribution functions $\Upsilon_1(\cdot)$, $\Upsilon_2(\cdot)$, respectively. Then $\eta_1 \preccurlyeq_{st} \eta_2$ and $\tau_1 \preccurlyeq_{un} \tau_2$ imply that*

$$\xi_1 \preccurlyeq_{ch} \xi_2 \tag{3.8}$$

provided that $f(x, y)$ is a component-wise strictly increasing function.

Proof: Since $f(\cdot, \cdot)$ is a component-wise strictly increasing function, hence for all $y_1, y_2 \in \mathbb{R}$ using Theorem 2.6, the uncertainty distribution function of $f(y_i, \tau_i)$ becomes

$$F_i(x, y_i) = \sup_{f(y_i,x_i)=x} \Upsilon_i(x_i). \tag{3.9}$$

Also, $\tau_1 \preccurlyeq_{un} \tau_2$ and (2.15) imply that

$$\Upsilon_1(t) \geq \Upsilon_2(t) \ for\ any\ t \in \mathbb{R}. \tag{3.10}$$

Therefore, for $y = \min\{y_1, y_2\}$, by (3.9) and (3.10) we have

$$F_1(t, y) = \sup_{f(y,x_1)=t} \Upsilon_1(x_1) \geq \sup_{f(y,x_2)=t} \Upsilon_2(x_2) = F_2(t, y). \tag{3.11}$$

On the other hand, from Theorem 2.10 by the chance distribution function of the uncertain random variables $\xi_i = f(\eta_i, \tau_i)$, $i = 1, 2$, and $\eta_1 \preccurlyeq_{st} \eta_2$, we have

$$\begin{aligned} \Phi_{\xi_1}(t) &= \int_{\mathbb{R}} F_1(t, y)\mathrm{d}\Psi_{\eta_1}(y) \\ &\geq \int_{\mathbb{R}} F_2(t, y)\mathrm{d}\Psi_{\eta_1}(y) \\ &\geq \int_{\mathbb{R}} F_2(t, y)\mathrm{d}\Psi_{\eta_2}(y) \\ &= \Phi_{\xi_2}(t), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Since $F_2(t, y)$ is a decreasing function with respect to y, the second inequality follows from Theorem 2.2 immediately. Thus, the proof is complete.

Similarly, let $\eta_1, \eta_2, \ldots, \eta_m$ and $\eta_1^{'}$, $\eta_2^{'}$, …, $\eta_m^{'}$ be two sets of independent random variables, $\tau_1, \tau_2, \ldots, \tau_n$ and $\tau_1^{'}$, $\tau_2^{'}$, …, $\tau_n^{'}$ be two sets of independent uncertain variables, and if $f : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ is a component-wise strictly monotone function, then $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ and $f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_1^{'}, \cdots, \tau_n^{'})$ are uncertain random variables. In the next Theorem, we derive the chance order of two functions:

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \ \text{ and } \ \xi^{'} = f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_1^{'}, \cdots, \tau_n^{'}).$$

Theorem 3.4 *Let η_1, η_2, …, η_m and $\eta_1^{'}$, $\eta_2^{'}$, …, $\eta_m^{'}$ be two sets of independent random variables with probability distribution functions $\Psi_{\eta_1}(\cdot)$, $\Psi_{\eta_2}(\cdot)$, …, $\Psi_{\eta_m}(\cdot)$ and $\Psi_{\eta_1^{'}}(\cdot)$, $\Psi_{\eta_2^{'}}(\cdot)$, …, $\Psi_{\eta_m^{'}}(\cdot)$, and let τ_1, τ_2,…, τ_n and $\tau_1^{'}$, $\tau_2^{'}$, …, $\tau_n^{'}$ be two sets of independent uncertain variables with uncertainty distribution functions $\Upsilon_{\tau_1}(\cdot)$, $\Upsilon_{\tau_2}(\cdot)$, …, $\Upsilon_{\tau_n}(\cdot)$, and $\Upsilon_{\tau_1^{'}}(\cdot)$, $\Upsilon_{\tau_2^{'}}(\cdot)$, …, $\Upsilon_{\tau_n^{'}}(\cdot)$, respectively. Then $\eta_i \preccurlyeq_{st} \eta_i^{'}$, $i = 1, 2, \ldots, m$, and $\tau_j \preccurlyeq_{un} \tau_j^{'}$, $j = 1, 2, \ldots, n$ imply that*

$$\xi \preccurlyeq_{ch} \xi^{'} \tag{3.12}$$

provided that $f(y_1, y_2, \cdots, y_m, x_1, x_2, \cdots, x_n)$ is a component-wise strictly increasing function.

Proof: Since $f(\cdot, \cdots, \cdot)$ is a component-wise strictly increasing function, hence for all $y_1, \ldots, y_n, y_1^{'}, \ldots, y_n^{'} \in \mathbb{R}$ using Theorem 2.6, the uncertainty distribution functions of

$$f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$$

and

$$f(y_1^{'}, y_2^{'}, \cdots, y_m^{'}, \tau_1^{'}, \tau_2^{'}, \cdots, \tau_n^{'})$$

become

$$F_1(x, y_1, y_2, \cdots, y_m) = \sup_{f(y_1,y_2,\cdots,y_m,z_1,z_2,\cdots,z_n)=x} \min_{1\le j\le n} (\Upsilon_{\tau_j}(z_j)), \tag{3.13}$$

and

$$F_2(x, y_1^{'}, y_2^{'}, \cdots, y_m^{'}) = \sup_{f(y_1^{'},y_2^{'},\cdots,y_m^{'},z_1^{'},z_2^{'},\cdots,z_n^{'})=x} \min_{1\le j\le n} (\Upsilon_{\tau_j^{'}}(z_j^{'})), \tag{3.14}$$

respectively. Also, $\tau_j \preccurlyeq_{un} \tau_j^{'}$, $j = 1, 2, \ldots, n$, and (2.15) imply that

$$\Upsilon_{\tau_j}(t) \ge \Upsilon_{\tau_j^{'}}(t), \qquad \forall t \in \mathbb{R}. \tag{3.15}$$

Therefore, for $s_i = \min\{y_i, y_i^{'}\}$, $i = 1, 2, \ldots, m$, by (3.13), (3.14), and (3.15) we have

$$\sup_{f(s_1,s_2,\cdots,s_m,z_1,z_2,\cdots,z_n)=t} \min_{1\le j\le n} (\Upsilon_{\tau_j}(z_j)) \ge \sup_{f(s_1,s_2,\cdots,s_m,z_1^{'},z_2^{'},\cdots,z_n^{'})=t} \min_{1\le j\le n} (\Upsilon_{\tau_j^{'}}(z_j^{'})).$$

The last inequality is equivalent to

$$F_1(t, s_1, s_2, \cdots, s_m) \ge F_2(t, s_1, s_2, \cdots, s_m), \qquad \forall t \in \mathbb{R}. \tag{3.16}$$

On the other hand, from Theorem 2.10 by the chance distribution function of uncertain random variables ξ and $\xi^{'}$, and also $\eta_i \preccurlyeq_{st} \eta_i^{'}$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \Phi_\xi(t) &= \int_{\mathbb{R}^m} F_1(t, s_1, s_2, \cdots, s_m) \mathrm{d}\Psi_{\eta_1}(s_1) \cdots \Psi_{\eta_m}(s_m) \\ &\ge \int_{\mathbb{R}^m} F_2(t, s_1, s_2, \cdots, s_m) \mathrm{d}\Psi_{\eta_1}(s_1) \cdots \Psi_{\eta_m}(s_m) \\ &\ge \int_{\mathbb{R}^m} F_2(t, s_1, s_2, \cdots, s_m) \mathrm{d}\Psi_{\eta_1^{'}}(s_1) \cdots \Psi_{\eta_m^{'}}(s_m) \\ &= \Phi_{\xi^{'}}(t), \quad \textit{for any } t \in \mathbb{R}. \end{aligned}$$

Hence, the proof is complete.

Example 3.3 *Suppose $\eta_1, \ldots, \eta_m$, $\eta_1^{'}, \ldots, \eta_m^{'}$, $\tau_1, \ldots, \tau_n$, and $\tau_1^{'}, \ldots, \tau_n^{'}$ are the same as in Theorem 3.4.*

(i) If $f(y_1, \cdots, y_m, x_1, \cdots, x_n) = \sum_{i=1}^m y_i + \sum_{j=1}^n x_j$, where $y_i, x_j \in \mathbb{R}, i = 1, \ldots, m, j = 1, \ldots, n$, then

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \preccurlyeq_{ch} f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_1^{'}, \cdots, \tau_n^{'}) = \xi^{'} \tag{3.17}$$

because of

$$\begin{aligned} \Phi_\xi(t) &= \int_{-\infty}^{\infty} \Upsilon_\tau(t-y) \mathrm{d}\Psi_\eta(y) \\ &\ge \int_{-\infty}^{\infty} \Upsilon_{\tau^{'}}(t-y) \mathrm{d}\Psi_{\eta^{'}}(y) \\ &= \Phi_{\xi^{'}}(t) \ \textit{for any } t \in \mathbb{R}, \end{aligned}$$

where $\Psi_\eta(\cdot)$ and $\Psi_{\eta^{'}}(\cdot)$ are the probability distribution functions of $\eta = \sum_{i=1}^m \eta_i$ and $\eta^{'} = \sum_{i=1}^m \eta_i^{'}$, and $\Upsilon_\tau(\cdot)$ and $\Upsilon_{\tau^{'}}(\cdot)$ are the uncertainty distribution functions of $\tau = \sum_{j=1}^n \tau_j$ and $\tau^{'} = \sum_{j=1}^n \tau_j^{'}$, respectively.

(ii) If $f(y_1, \cdots, y_m, x_1, \cdots, x_n) = \prod_{i=1}^m y_i \times \prod_{j=1}^n x_j$, where $y_i, x_j \in \mathbb{R}^+, i = 1, \ldots, m, j = 1, \ldots, n$, then

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \preccurlyeq_{ch} f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_1^{'}, \cdots, \tau_n^{'}) = \xi^{'} \tag{3.18}$$

because of

$$\begin{aligned}\Phi_\xi(t) &= \int_{-\infty}^{\infty} \Upsilon_\tau(t/y)\mathrm{d}\Psi_\eta(y)\\ &\geq \int_{-\infty}^{\infty} \Upsilon_{\tau'}(t/y)\mathrm{d}\Psi_{\eta'}(y) = \Phi_{\xi'}(t) \text{ for any } t\in\mathbb{R},\end{aligned}$$

where $\Psi_\eta(\cdot)$ *and* $\Psi_{\eta'}(\cdot)$ *are the probability distribution functions of* $\eta=\prod_{i=1}^m \eta_i$ *and* $\eta^{'}=\prod_{i=1}^m \eta_i^{'}$, *and* $\Upsilon_\tau(\cdot)$ *and* $\Upsilon_{\tau'}(\cdot)$ *are the uncertainty distribution functions of* $\tau=\prod_{j=1}^n \tau_j$ *and* $\tau^{'}=\prod_{j=1}^n \tau_j^{'}$, *respectively.*

(iii) If $f(y_1,\cdots,y_m,x_1,\cdots,x_n)=\{\bigwedge_{i=1}^m y_i\}\wedge\{\bigwedge_{j=1}^n x_j\}$, *where* $y_i,x_j\in\mathbb{R}, i=1,\ldots,m, j=1,\ldots,n$, *then*

$$\xi=f(\eta_1,\cdots,\eta_m,\tau_1,\cdots,\tau_n)\preccurlyeq_{ch} f(\eta_1^{'},\cdots,\eta_m^{'},\tau_1^{'},\cdots,\tau_n^{'})=\xi^{'} \tag{3.19}$$

because of

$$\begin{aligned}\Phi_\xi(t) &= \Psi_\eta(t)+\Upsilon_\tau(t)-\Psi_\eta(t)\Upsilon_\tau(t)\\ &\geq \Psi_{\eta'}(t)+\Upsilon_{\tau'}(t)-\Psi_{\eta'}(t)\Upsilon_{\tau'}(t)=\Phi_{\xi'}(t) \text{ for any } t\in\mathbb{R},\end{aligned}$$

where $\Psi_\eta(\cdot)$ *and* $\Psi_{\eta'}(\cdot)$ *are the probability distribution functions of* $\eta=\bigwedge_{i=1}^m \eta_i$ *and* $\eta^{'}=\bigwedge_{i=1}^m \eta_i^{'}$, *and* $\Upsilon_\tau(\cdot)$ *and* $\Upsilon_{\tau'}(\cdot)$ *are the uncertainty distribution functions of* $\tau=\bigwedge_{j=1}^n \tau_j$ *and* $\tau^{'}=\bigwedge_{j=1}^n \tau_j^{'}$, *respectively.*

(iv) If $f(y_1,\cdots,y_m,x_1,\cdots,x_n)=\{\bigvee_{i=1}^m y_i\}\vee\{\bigvee_{j=1}^n x_j\}$, *where* $y_i,x_j\in\mathbb{R}, i=1,\ldots,m, j=1,\ldots,n$, *then*

$$\xi=f(\eta_1,\cdots,\eta_m,\tau_1,\cdots,\tau_n)\preccurlyeq_{ch} f(\eta_1^{'},\cdots,\eta_m^{'},\tau_1^{'},\cdots,\tau_n^{'})=\xi^{'} \tag{3.20}$$

because of

$$\begin{aligned}\Phi_\xi(t) &= \Psi_\eta(t)\Upsilon_\tau(t)\\ &\geq \Psi_{\eta'}(t)\Upsilon_{\tau'}(t)=\Phi_{\xi'}(t) \text{ for any } t\in\mathbb{R},\end{aligned}$$

where $\Psi_\eta(\cdot)$ *and* $\Psi_{\eta'}(\cdot)$ *are the probability distribution functions of* $\eta=\bigvee_{i=1}^m \eta_i$ *and* $\eta^{'}=\bigvee_{i=1}^m \eta_i^{'}$, *and* $\Upsilon_\tau(\cdot)$ *and* $\Upsilon_{\tau'}(\cdot)$ *are the uncertainty distribution functions of* $\tau=\bigvee_{j=1}^n \tau_j$ *and* $\tau^{'}=\bigvee_{j=1}^n \tau_j^{'}$, *respectively.*

Remark 3.3 *It is worth mentioning that even if some of the conditions of the Theorem 3.4 are not satisfied, the chance order can be established.*

Example 3.4 *Consider the Example 3.1. Then we have* $\xi_1\preccurlyeq_{ch}\xi_2$ *while* $\eta_2\preccurlyeq_{st}\eta_1$.

Definition 3.2 *Let* $(\xi_1,\xi_2,\cdots,\xi_n)$ *and* $(\xi_1^{'},\xi_2^{'},\cdots,\xi_n^{'})$ *be uncertain random vectors, and if*

$$\mathrm{Ch}(\xi_1\leq x_1,\cdots,\xi_n\leq x_n)=\mathrm{Ch}(\xi_1^{'}\leq x_1,\cdots,\xi_n^{'}\leq x_n) \tag{3.21}$$

for any real numbers $x_1,x_2,\ldots,x_n\in\mathbb{R}$, *then* $(\xi_1,\xi_2,\cdots,\xi_n)$ *is said to be identically distributed with* $(\xi_1^{'},\xi_2^{'},\cdots,\xi_n^{'})$, *denoted by* $(\xi_1,\xi_2,\cdots,\xi_n)=_D(\xi_1^{'},\xi_2^{'},\cdots,\xi_n^{'})$.

Theorem 3.5 *Let* η_1, η_2 *be two random variables, and* τ_1, τ_2 *be two uncertain variables. Then uncertain random variables* $\xi_1=f(\eta_1,\tau_1)$ *and* $\xi_2=f(\eta_2,\tau_2)$ *satisfy* $\xi_1\preccurlyeq_{ch}\xi_2$ *if there exist two random variables* $\tilde{\eta}_1$, $\tilde{\eta}_2$, *defined on the same probability space, and two uncertain variables* $\tilde{\tau}_1$, $\tilde{\tau}_2$, *defined on the same uncertainty space, such that* $\eta_i=_D\tilde{\eta}_i$, $\tau_i=_D\tilde{\tau}_i$, $i=1,2$, *and*

$$\mathrm{Ch}\left(f(\tilde{\eta}_1,\tilde{\tau}_1)\leq f(\tilde{\eta}_2,\tilde{\tau}_2)\right)=1 \tag{3.22}$$

provided that $f(x,y)$ *is a component-wise strictly increasing function.*

Proof: Let $\tilde{\xi}_1 = f(\tilde{\eta}_1, \tilde{\tau}_1)$ and $\tilde{\xi}_2 = f(\tilde{\eta}_2, \tilde{\tau}_2)$, and there exist two random variables $\tilde{\eta}_1$, $\tilde{\eta}_2$, defined on the same probability space, and two uncertain variables $\tilde{\tau}_1$, $\tilde{\tau}_2$, defined on the same uncertainty space, such that $\eta_i =_D \tilde{\eta}_i$, $\tau_i =_D \tilde{\tau}_i$, $i = 1, 2$, and (3.22) holds. Then $\eta_i =_D \tilde{\eta}_i$, and $\tau_i =_D \tilde{\tau}_i$, $i = 1, 2$, imply that $\tilde{\xi}_i = f(\tilde{\eta}_i, \tilde{\tau}_i) =_D f(\eta_i, \tau_i) = \xi_i$, $i = 1, 2$.

Using Theorem 2.6, for $i = 1, 2$, the uncertainty distribution functions of $f(y_i, \tau_i)$ and $f(\tilde{y}_i, \tilde{\tau}_i)$ are

$$F_i(x, y_i) = \sup_{f(y_i, x_i) = x} \Upsilon_{\tau_i}(x_i), \tag{3.23}$$

and

$$\tilde{F}_i(\tilde{x}, \tilde{y}_i) = \sup_{f(\tilde{y}_i, \tilde{x}_i) = \tilde{x}} \Upsilon_{\tilde{\tau}_i}(\tilde{x}_i). \tag{3.24}$$

Also, $\tau_i =_D \tilde{\tau}_i$ imply that

$$\Upsilon_{\tau_i}(x) = \Upsilon_{\tilde{\tau}_i}(x), \quad \forall x \in \mathbb{R}. \tag{3.25}$$

Then, for $s_i = \min\{y_i, \tilde{y}_i\}$, $i = 1, 2$, by (3.23), (3.24), and (3.25), we have

$$F_i(x, s_i) = \sup_{f(s_i, x_i) = x} \Upsilon_{\tau_i}(x_i) = \sup_{f(s_i, \tilde{x}_i) = x} \Upsilon_{\tilde{\tau}_i}(\tilde{x}_i) = \tilde{F}_i(x, s_i). \tag{3.26}$$

Also, using Theorem 2.10 for uncertain random variables $\xi_i = f(\eta_i, \tau_i)$, $\tilde{\xi}_i = f(\tilde{\eta}_i, \tilde{\tau}_i)$, $i = 1, 2$, and since $\eta_1 =_D \eta_2$, we obtain

$$\begin{aligned}
\Phi_{\xi_i}(x) &= \int_{\mathbb{R}} F_i(x, s_i) \mathrm{d}\Psi_{\eta_i}(s_i) \\
&= \int_{\mathbb{R}} \tilde{F}_i(x, s_i) \mathrm{d}\Psi_{\eta_i}(s_i) \\
&= \int_{\mathbb{R}} \tilde{F}_i(x, s_i) \mathrm{d}\Psi_{\tilde{\eta}_i}(y) \\
&= \Phi_{\tilde{\xi}_i}(x), \quad \forall x \in \mathbb{R}.
\end{aligned}$$

So $\tilde{\xi}_i =_D \xi_i$, $i = 1, 2$. Therefore, $\xi_1 \preccurlyeq_{ch} \xi_2$ follows from (3.22) immediately.

The next theorem is an extended version of Theorem 3.5.

Theorem 3.6 *Let η_1, η_2, ..., η_m, and $\eta_1^{'}$, $\eta_2^{'}$, ..., $\eta_m^{'}$ be two sets of random variables, and τ_1, τ_2, ..., τ_n, and $\tau_1^{'}$, $\tau_2^{'}$, ..., $\tau_n^{'}$ be two sets of uncertain variables, respectively. Then uncertain random variables*

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \ \ \text{and} \ \ \xi^{'} = f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_1^{'}, \cdots, \tau_n^{'}) \tag{3.27}$$

satisfy $\xi \preccurlyeq_{ch} \xi^{'}$ if there exist random variables $\tilde{\eta}_1$, $\tilde{\eta}_2$, ..., $\tilde{\eta}_m$, and $\tilde{\eta}_1^{'}$, $\tilde{\eta}_2^{'}$, ..., $\tilde{\eta}_m^{'}$, defined on the same probability space, and uncertain variables $\tilde{\tau}_1$, $\tilde{\tau}_2$, ..., $\tilde{\tau}_n$, and $\tilde{\tau}_1^{'}$, $\tilde{\tau}_2^{'}$, ..., $\tilde{\tau}_n^{'}$, defined on the same uncertainty space, such that $\eta_i =_D \tilde{\eta}_i$, $\eta_i^{'} =_D \tilde{\eta}_i^{'}$, $i = 1, 2, \ldots, m$, $\tau_j =_D \tilde{\tau}_j$, $\tau_j^{'} =_D \tilde{\tau}_j^{'}$, $j = 1, 2, \ldots, n$, and

$$\mathrm{Ch}[f(\tilde{\eta}_1, \cdots, \tilde{\eta}_m, \tilde{\tau}_1, \cdots, \tilde{\tau}_n) \le f(\tilde{\eta}_1^{'}, \cdots, \tilde{\eta}_m^{'}, \tilde{\tau}_1^{'}, \cdots, \tilde{\tau}_n^{'})] = 1 \tag{3.28}$$

provided that $f(x_1, \cdots, x_m, y_1, \cdots, y_n)$ is a component-wise strictly increasing function.

Proof: Let $\tilde{\xi} = f(\tilde{\eta}_1, \cdots, \tilde{\eta}_m, \tilde{\tau}_1, \cdots, \tilde{\tau}_n)$ and $\tilde{\xi}^{'} = f(\tilde{\eta}_1^{'}, \cdots, \tilde{\eta}_m^{'}, \tilde{\tau}_1^{'}, \cdots, \tilde{\tau}_n^{'})$, and there exist random variables $\tilde{\eta}_1$, $\tilde{\eta}_2$, ..., $\tilde{\eta}_m$ and $\tilde{\eta}_1^{'}$, $\tilde{\eta}_2^{'}$, ..., $\tilde{\eta}_m^{'}$, defined on the same probability space, and uncertain variables $\tilde{\tau}_1$, $\tilde{\tau}_2$, ..., $\tilde{\tau}_n$, and $\tilde{\tau}_1^{'}$, $\tilde{\tau}_2^{'}$, ..., $\tilde{\tau}_n^{'}$, defined on the same uncertainty space, such that $\eta_i =_D \tilde{\eta}_i$, $\eta_i^{'} =_D \tilde{\eta}_i^{'}$, $i = 1, 2, \ldots, m$, $\tau_j =_D \tilde{\tau}_j$, $\tau_j^{'} =_D \tilde{\tau}_j^{'}$, $j = 1, 2, \ldots, n$, and (3.28) holds. Then $\eta_i =_D \tilde{\eta}_i$, $\eta_i^{'} =_D \tilde{\eta}_i^{'}$, $i = 1, 2, \ldots, m$, and $\tau_j =_D \tilde{\tau}_j$, $\tau_j^{'} =_D \tilde{\tau}_j^{'}$, $j = 1, 2, \ldots, n$ imply that $\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) =_D f(\tilde{\eta}_1, \cdots, \tilde{\eta}_m, \tilde{\tau}_1, \cdots, \tilde{\tau}_n) = \tilde{\xi}$ and $\xi^{'} = f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_1^{'}, \cdots, \tau_n^{'}) =_D f(\tilde{\eta}_1^{'}, \cdots, \tilde{\eta}_m^{'}, \tilde{\tau}_1^{'}, \cdots, \tilde{\tau}_n^{'}) = \tilde{\xi}^{'}$.

Applying Theorem 2.6, for uncertaint random variables

$$f(y_1,y_2,\cdots,y_m,\tau_1,\tau_2,\cdots,\tau_n),\quad f(y_1',y_2',\cdots,y_m',\tau_1',\tau_2',\cdots,\tau_n')$$

and

$$f(\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_m,\tilde{\tau}_1,\tilde{\tau}_2,\cdots,\tilde{\tau}_n),\quad f(\tilde{y}_1',\tilde{y}_2',\cdots,\tilde{y}_m',\tilde{\tau}_1',\tilde{\tau}_2',\cdots,\tilde{\tau}_n'),$$

we have

$$F_1(x,y_1,y_2,\cdots,y_m) = \sup_{f(y_1,y_2,\cdots,y_m,z_1,z_2,\cdots,z_n)=x} \min_{1\le j\le n}(\Upsilon_{\tau_j}(z_j)), \tag{3.29}$$

$$F_2(x,y_1',y_2',\cdots,y_m') = \sup_{f(y_1',y_2',\cdots,y_m',z_1',z_2',\cdots,z_n')=x} \min_{1\le j\le n}(\Upsilon_{\tau_j'}(z_j')), \tag{3.30}$$

and

$$\tilde{F}_1(x,\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_m) = \sup_{f(\tilde{y}_1,\tilde{y}_2,\cdots,\tilde{y}_m,\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_n)=x} \min_{1\le j\le n}(\Upsilon_{\tilde{\tau}_j}(\tilde{z}_j)), \tag{3.31}$$

$$\tilde{F}_2(x,\tilde{y}_1',\tilde{y}_2',\cdots,\tilde{y}_m') = \sup_{f(\tilde{y}_1',\tilde{y}_2',\cdots,\tilde{y}_m',\tilde{z}_1',\tilde{z}_2',\cdots,\tilde{z}_n')=x} \min_{1\le j\le n}(\Upsilon_{\tilde{\tau}_j'}(\tilde{z}_j')), \tag{3.32}$$

respectively. Also, $\tau_j =_D \tilde{\tau}_j$, $\tau_j' =_D \tilde{\tau}_j'$, $j=1,2,\ldots,n$ imply that

$$\Upsilon_{\tau_j}(x)=\Upsilon_{\tilde{\tau}_j}(x),\ \Upsilon_{\tau_j'}(x)=\Upsilon_{\tilde{\tau}_j'}(x),\qquad \forall x\in\mathbb{R}. \tag{3.33}$$

Then, for $s_i=\min\{y_i,\tilde{y}_i\}$, $i=1,2,\ldots,m$, by (3.29) and (3.31), we have

$$\sup_{f(s_1,s_2,\cdots,s_m,z_1,z_2,\cdots,z_n)=x} \min_{1\le j\le n}(\Upsilon_{\tau_j}(z_j)) = \sup_{f(s_1,s_2,\cdots,s_m,\tilde{z}_1,\tilde{z}_2,\cdots,\tilde{z}_n)=x} \min_{1\le j\le n}(\Upsilon_{\tilde{\tau}_j}(\tilde{z}_j)).$$

That is equivalent to

$$F_1(x,s_1,s_2,\cdots,s_m)=\tilde{F}_1(x,s_1,s_2,\cdots,s_m),\qquad \forall x\in\mathbb{R}. \tag{3.34}$$

Similarly, for $s_i'=\min\{y_i',\tilde{y}_i'\}$, $i=1,2,\ldots,m$, by (3.30) and (3.32), we have

$$\sup_{f(s_1',s_2',\cdots,s_m',z_1',z_2',\cdots,z_n')=x} \min_{1\le j\le n}(\Upsilon_{\tau_j'}(z_j')) = \sup_{f(s_1',s_2',\cdots,s_m',\tilde{z}_1',\tilde{z}_2',\cdots,\tilde{z}_n')=x} \min_{1\le j\le n}(\Upsilon_{\tilde{\tau}_j'}(\tilde{z}_j')).$$

So

$$F_2(x,s_1',s_2',\cdots,s_m')=\tilde{F}_2(x,s_1',s_2',\cdots,s_m'),\qquad \forall x\in\mathbb{R}. \tag{3.35}$$

Also, using Theorem 2.10, for uncertain random variables ξ, $\tilde{\xi}$, ξ', $\tilde{\xi}'$, and since $\eta_i =_D \tilde{\eta}_i$, $\eta_i' =_D \tilde{\eta}_i'$, $i=1,2,\ldots,m$, we obtain

$$\begin{aligned}\Phi_\xi(x) &= \int_{\mathbb{R}^m} F_1(x,s_1,s_2,\cdots,s_m)\mathrm{d}\Psi_{\eta_1}(s_1)\cdots\Psi_{\eta_m}(s_m)\\ &= \int_{\mathbb{R}^m} \tilde{F}_1(x,s_1,s_2,\cdots,s_m)\mathrm{d}\Psi_{\eta_1}(s_1)\cdots\Psi_{\eta_m}(s_m)\\ &= \int_{\mathbb{R}^m} \tilde{F}_1(x,s_1,s_2,\cdots,s_m)\mathrm{d}\Psi_{\tilde{\eta}_1}(s_1)\cdots\Psi_{\tilde{\eta}_m}(s_m)=\Phi_{\tilde{\xi}}(x),\quad \forall x\in\mathbb{R}.\end{aligned}$$

and

$$\begin{aligned}\Phi_{\xi'}(x) &= \int_{\mathbb{R}^m} F_2(x,s_1',s_2',\cdots,s_m')\mathrm{d}\Psi_{\eta_1'}(s_1')\cdots\Psi_{\eta_m'}(s_m')\\ &= \int_{\mathbb{R}^m} \tilde{F}_2(x,s_1',s_2',\cdots,s_m')\mathrm{d}\Psi_{\eta_1'}(s_1')\cdots\Psi_{\eta_m'}(s_m')\\ &= \int_{\mathbb{R}^m} \tilde{F}_2(x,s_1',s_2',\cdots,s_m')\mathrm{d}\Psi_{\tilde{\eta}_1'}(s_1')\cdots\Psi_{\tilde{\eta}_m'}(s_m')=\Phi_{\tilde{\xi}'}(x),\quad \forall x\in\mathbb{R}.\end{aligned}$$

These imply that $\xi =_D \tilde{\xi}$ and $\xi^{'} =_D \tilde{\xi}^{'}$. Therefore, $\xi \preccurlyeq_{ch} \xi^{'}$ follows from (3.28) immediately.

Suppose we have two stores. For each store, consider that the number of customers entering a store on a given day is a random variable and the amount of money spent by a customer is an uncertain variable. On a given day, in which store will the amount of sales be more?

To model and answer such problems, we must consider the random sum of uncertain variables. In the next proposition, two random sums of uncertain variables will be compared.

Proposition 3.7 *Let $\{\tau_i,\ i = 1, 2, \ldots\}$ and $\{\tau_i^{'},\ i = 1, 2, \ldots\}$ be sequences of nonnegative independent and identically distributed uncertain variables, and let η be a nonnegative integer- valued random variable, which is independent of τ_i's, and $\eta^{'}$ be a nonnegative integer- valued random variable, which is independent of $\tau_i^{'}$'s. Then if $\tau_i \preceq_{un} \tau_i^{'}$, for $i = 1, 2, \ldots$, and $\eta \preceq_{st} \eta^{'}$, we have*

$$\sum_{i=1}^{\eta} \tau_i \preceq_{ch} \sum_{i=1}^{\eta^{'}} \tau_i^{'}. \tag{3.36}$$

Proof: Let $\tau(m) = \sum_{i=1}^{m} \tau_i$ and $\tau^{'}(m) = \sum_{i=1}^{m} \tau_i^{'}$. Then by applying Theorem 2.8, we have

$$\tau(m) \preceq_{un} \tau^{'}(m), \quad for\ all\ m \in \mathbb{N}. \tag{3.37}$$

Since $\tau(\eta)$ and $\tau^{'}(\eta)$ are uncertain random variables, so Theorem 3.4 implies that

$$\tau(\eta) \preceq_{ch} \tau^{'}(\eta). \tag{3.38}$$

Also, since $\eta \preceq_{st} \eta^{'}$, Theorem 3.4 yields

$$\tau^{'}(\eta) \preceq_{ch} \tau^{'}(\eta^{'}). \tag{3.39}$$

Thus, (3.38), (3.39), and the Proposition 3.1 complete the proof.

4 An Application in Reliability

Consider a reliability system consisting of n components $C = \{c_1, c_2, \ldots, c_n\}$ whose structure function is given by

$$\phi : \{0,1\}^n := \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n\ times} \longrightarrow \{0,1\}. \tag{4.1}$$

It is assumed that

(1) $\phi(0, \cdots, 0) = 0$;

(2) $\phi(1, \cdots, 1) = 1$;

(3) (Monotonicity) For $(u_1, \cdots, u_n), (v_1, \cdots, v_n) \in \{0,1\}^n$,

$$u_i \leq v_i, i = 1, \ldots, n \Longrightarrow \phi(u_1, \cdots, u_n) \leq \phi(v_1, \cdots, v_n). \tag{4.2}$$

In a complex system, some components may have enough samples to estimate their probability distributions and can be regarded as random variables, while some others may have no samples, and can only be evaluated by the experts and regarded as uncertain variables. In this case, the system can not be simply modeled by a stochastic system or an uncertain system [29]. Therefore, in order to compare the reliability of this type system, which is known as the uncertain random system, we employ an uncertain random variable to model the system. In the following, a description of the model is given.

For $i = 1, \ldots, m$, let $\{S_i^\eta(t);\ t \in \mathbb{R}_+\}$ be a decreasing, right-continuous, and $\{0,1\}$-valued stochastic process representing the state of component c_i at time t, that is,

$$S_i^\eta(t) = \begin{cases} 1, & \text{if component } c_i \text{ is functioning at time } t; \\ 0, & \text{if component } c_i \text{ is failed at time } t. \end{cases} \tag{4.3}$$

Similarly, for $i = m+1, \ldots, n$, let $\{S_i^\tau(t);\ t \in \mathbb{R}_+\}$ be a decreasing, right-continuous, and $\{0,1\}$-valued uncertain process representing the state of component c_i at time t, that is,

$$S_i^\tau(t) = \begin{cases} 1, & \text{if component } c_i \text{ is functioning at time } t; \\ 0, & \text{if component } c_i \text{ is failed at time } t. \end{cases} \tag{4.4}$$

Similarly, let $\{S^\xi(t);\ t \in \mathbb{R}_+\}$ be a decreasing, right-continuous, and $\{0,1\}$-valued uncertain random process representing the state of the system at time t, that is,

$$S^\xi(t) = \begin{cases} 1, & \text{if the system is functioning at time } t; \\ 0, & \text{if the system is failed at time } t. \end{cases} \tag{4.5}$$

Using the definition of structure function, we have

$$S^\xi(t) = \phi(S_1^\eta(t), \cdots, S_m^\eta(t), S_{m+1}^\tau(t), \cdots, S_n^\tau(t)), \quad t \in \mathbb{R}_+. \tag{4.6}$$

Now, let η_i, $i = 1, \ldots, m$ be nonnegative random variables representing the lifetime (or failure time) of component c_i. Since

$$\{\eta_i > t\} \Longleftrightarrow \{S_i^\eta(t) = 1\}, \tag{4.7}$$

we have

$$P(S_i^\eta(t) = 1) = P(\eta_i > t) = \bar{\Psi}_{\eta_i}(t) = 1 - \Psi_{\eta_i}(t), \quad t \in \mathbb{R}_+, \tag{4.8}$$

where $P(S_i^\eta(t) = 1)$ is defined as the probability measure that the component c_i is working at time t.

Similarly, let τ_i, $i = m+1, \ldots, n$ be nonnegative uncertain variables representing the lifetime (or failure time) of component c_i. Since

$$\{\tau_i > t\} \Longleftrightarrow \{S_i^\tau(t) = 1\}, \tag{4.9}$$

we have

$$\mathcal{M}(\{S_i^\tau(t) = 1) = \mathcal{M}(\tau_i > t) = \bar{\Upsilon}_{\tau_i}(t) = 1 - \Upsilon_{\tau_i}(t), \quad t \in \mathbb{R}_+, \tag{4.10}$$

where $\mathcal{M}(S_i^\tau(t) = 1)$ is defined as the uncertain measure that the component c_i is working at time t.

Similarly, if ξ be a nonnegative uncertain random variable representing the lifetime (or failure time) of the system, then since

$$\{\xi > t\} \Longleftrightarrow \{S^\xi(t) = 1\}, \tag{4.11}$$

we have

$$\mathrm{Ch}(S^\xi(t) = 1) = \mathrm{Ch}(\xi > t) = \bar{\Phi}_\xi(t) = 1 - \Phi_\xi(t), \quad t \in \mathbb{R}_+, \tag{4.12}$$

where $\mathrm{Ch}(S^\xi(t) = 1)$ is defined as the chance measure that the system is working at time t.

If we define a function

$$f : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+, \tag{4.13}$$

by

$$f(t_1,\cdots,t_n) \quad := \quad sup\ \{t\in\mathbb{R}_+:\ \phi(1_{[0,t_1)}(t),\cdots,1_{[0,t_n)}(t))=1\} \tag{4.14}$$
$$= \quad inf\ \{t\in\mathbb{R}_+:\ \phi(1_{[0,t_1)}(t),\cdots,1_{[0,t_n)}(t))=0\}, \tag{4.15}$$

then it determines the lifetime of the system by the lifetimes of the components, that is,

$$\xi = f(\eta_1,\cdots,\eta_m,\tau_{m+1},\cdots,\tau_n), \tag{4.16}$$

and we call f the system lifetime function of ϕ. It is noted that, since ϕ is an increasing function, f is also an increasing function.

An important example of a (monotone) reliability system is the k-out-of-n system ($k\in\{1,\ldots,n\}$), whose structure function $\phi_{k|n}:\{0,1\}^n\longrightarrow\{0,1\}$ is defined by

$$\phi_{k|n}(s_1,\cdots,s_n)=\begin{cases}1, & \text{if } s_1+s_2+\cdots+s_n\ge k\\ 0, & \text{if } s_1+s_2+\cdots+s_n< k,\end{cases} \tag{4.17}$$

where $(s_1,\cdots,s_n)\in\{0,1\}^n$.

Here we will consider a k-out-of-n system, and we focus on the uncertain random system, whose components contain both uncertain and random variables. Note that, in this situation, the structure function is defined by

$$\phi_{k|n}(y_1,\cdots,y_m,x_{m+1},\cdots,x_n)=\begin{cases}1, & \text{if } \sum_{i=1}^m y_i+\sum_{i=m+1}^n x_i\ge k\\ 0, & \text{if } \sum_{i=1}^m y_i+\sum_{i=m+1}^n x_i< k.\end{cases} \tag{4.18}$$

where $(y_1,\cdots,y_m,x_{m+1},\cdots,x_n)\in\{0,1\}^n$.

Both the parallel and the series systems are special cases of the k-out-of-n system. In other words, the n-out-of-n and 1-out-of-n systems are called the series system and the parallel system, respectively, and their structure functions $\phi_{n|n}$ and $\phi_{1|n}$ are given by

$$\phi_{n|n}(y_1,\cdots,y_m,x_{m+1},\cdots,x_n) \quad = \quad (\bigwedge_{i=1}^m y_i)\wedge(\bigwedge_{i=m+1}^n x_i),$$
$$\phi_{1|n}(y_1,\cdots,y_m,x_{m+1},\cdots,x_n) \quad = \quad (\bigvee_{i=1}^m y_i)\vee(\bigvee_{i=m+1}^n x_i),$$

where $(y_1,\cdots,y_m,x_{m+1},\cdots,x_n)\in\{0,1\}^n$.

The lifetime of most of the systems considered in reliability theory can be expressed as some function of component lifetimes. For example, the system lifetime function $f_{k|n}$ of k-out-of-n system $\phi_{k|n}$ is given by

$$f_{k|n}(t_1^y,\cdots,t_m^y,t_{m+1}^x,\cdots,t_n^x)=(t^yt^x)^{(k)},\ (t_1^y,\cdots,t_m^y,t_{m+1}^x,\cdots,t_n^x)\in\mathbb{R}_+^n,$$

where $(t^yt^x)^{(k)}$ is the kth order time of $t_1^y,\ldots,t_m^y,t_{m+1}^x,\ldots,t_n^x$. As particular cases, we have

$$f_{n|n}=(\bigwedge_{i=1}^m t_i^y)\wedge(\bigwedge_{i=m+1}^n t_i^x),\ (t_1^y,\cdots,t_m^y,t_{m+1}^x,\cdots,t_n^x)\in\mathbb{R}_+^n;$$

$$f_{1|n}=(\bigvee_{i=1}^m t_i^y)\vee(\bigvee_{i=m+1}^n t_i^x),\ (t_1^y,\cdots,t_m^y,t_{m+1}^x,\cdots,t_n^x)\in\mathbb{R}_+^n.$$

Since the system lifetime function f of any reliability system with a monotone structure function ϕ is increasing, we have:

Example 4.1 *Let $(\eta_i, \eta_i^{'})$, $i = 1, 2, \ldots, m$ be independent pairs of nonnegative random variables and $(\tau_i, \tau_i^{'})$, $i = m+1, m+2, \ldots, n$ be independent pairs of nonegative uncertain variables representing the lifetimes of component c_i, $i = 1, 2, \ldots, n$. Then by using Theorem 3.4*

(i) for any (monotone) reliability system with structure function ϕ,

$$\eta_i \preccurlyeq_{st} \eta_i^{'},\ i = 1, 2, 3, \ldots, m$$
$$\tau_i \preccurlyeq_{un} \tau_i^{'},\ i = m+1, \ldots, n$$

imply

$$f(\eta_1, \cdots, \eta_m, \tau_{m+1}, \cdots, \tau_n) \quad \preccurlyeq_{ch} \quad f(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_{m+1}^{'}, \cdots, \tau_n^{'}).$$

(ii)

$$\eta_i \preccurlyeq_{st} \eta_i^{'},\ i = 1, 2, 3, \ldots, m$$
$$\tau_i \preccurlyeq_{un} \tau_i^{'},\ i = m+1, \ldots, n$$

imply

$$f_{1|n}(\eta_1, \cdots, \eta_m, \tau_{m+1}, \cdots, \tau_n) \quad \preccurlyeq_{ch} \quad f_{1|n}(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_{m+1}^{'}, \cdots, \tau_n^{'}).$$

(iii)

$$\eta_i \preccurlyeq_{st} \eta_i^{'},\ i = 1, 2, 3, \ldots, m$$
$$\tau_i \preccurlyeq_{un} \tau_i^{'},\ i = m+1, \ldots, n$$

imply

$$f_{n|n}(\eta_1, \cdots, \eta_m, \tau_{m+1}, \cdots, \tau_n) \quad \preccurlyeq_{ch} \quad f_{n|n}(\eta_1^{'}, \cdots, \eta_m^{'}, \tau_{m+1}^{'}, \cdots, \tau_n^{'}).$$

5 Conclusions

This paper has proposed a new concept of order for uncertain random variables as an extension of corresponding results for random variables and uncertain variables. Moreover, some properties of this order have also been studied. Finally, as an application, we have applied the chance order of uncertain random variables to compare the reliability of the k-out-of-n systems.

References

[1] Ahmadzade, H., Sheng, Y., and F. Hassantabar Darzi, Some results of moments of uncertain random variables, *Iranian Journal of Fuzzy Systems*, vol.14, no.2, pp.1–21, 2017.

[2] Belzunce, F., Martinez Riquelme, C., and J. Mulero, *An Introduction to Stochastic Orders*, 1st Edition, Academic Press, Elsevier Ltd., 2016.

[3] Gao, R., and H. Ahmadzade, Further results of convergence of uncertain random sequences, *Iranian Journal of Fuzzy Systems*, in press.

[4] Gao, Y., Gao, R., and L. Yang, Analysis of order statistics of uncertain variables, *Journal of Uncertainty Analysis and Applications*, vol.3, no.1, 7 pages, 2015.

[5] Gao, Z., and H. Guo, Some properties of uncertain dominance, *International Information Institute*, vol.15, no.12, pp.168–173, 2012.

[6] Gao, R., and D.A. Ralescu, Convergence in distribution for uncertain random variables, *IEEE Transactions on Fuzzy Systems*, in press.

[7] Gao, R., and Y. Sheng, Law of large numbers for uncertain random variables with different chance distributions, *Journal of Intelligent & Fuzzy Systems*, vol.31, no.3, pp.1227–1234, 2016.

[8] Gao, R., Sun, Y., and D.A. Ralescu, Order statistics of uncertain random variables with application to k-out-of-n system, *Fuzzy Optimization and Decision Making*, vol.16, no.2, pp.159–181, 2017.

[9] Gao, R., and K. Yao, Importance index of components in uncertain random systems, *Knowledge-Based Systems*, 2016, doi: 10.1016/j.knosys.2016.07.006.

[10] Gao, J., and K. Yao, Some concepts and theorems of uncertain random process, *International Journal of Intelligent Systems*, vol.30, no.1, pp.52–65, 2015.

[11] Guo, H., and X. Wang, Variance of uncertain random variables, *Journal of Uncertainty Analysis and Applications*, vol.2, no.6, 7 pages, 2014.

[12] Gut, A., *Probability: A Graduate Course*, 2nd Edition, Springer, New York, 2013.

[13] Hou, Y., Subadditivity of chance measure, *Journal of Uncertainty Analysis and Applications*, vol.2, no.14, 8 pages, 2014.

[14] Ke, H., Su, T., and Y. Ni, Uncertain random multilevel programming with application to production control problem, *Soft Computing*, vol.19, no.6, pp.1739–1746, 2015.

[15] Liu, B., Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, vol.2, no.1, pp.3–16, 2008.

[16] Liu, B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[17] Liu, B., Some research problems in uncertainty theory, *Journal of Uncertain Systems*, vol.3, no.1, pp.3–10, 2009.

[18] Liu, B., *Theory and Practice of Uncertain Programming*, 2nd Edition, Springer-Verlag, Berlin, 2009.

[19] Liu, B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.

[20] Liu, B., Why is there a need for uncertainty theory?, *Journal of Uncertain Systems*, vol.6, no.1, pp.3–10, 2012.

[21] Liu, B., Uncertain random graph and uncertain random network, *Journal of Uncertain Systems*, vol.8, no.1, pp.3–12, 2014.

[22] Liu, B., *Uncertainty Theory*, 4nd Edition, Springer-Verlag, Berlin, 2015.

[23] Liu, Y., Uncertain random programming with applications, *Fuzzy Optimization and Decision Making*, vol.12, no.2, pp.153–169, 2013.

[24] Liu, Y., Uncertain random variables: a mixture of uncertainty and randomness, *Soft Computing*, vol.17, no.4, pp.625–634, 2013.

[25] Liu, Y., and D.A. Ralescu, Risk index in uncertain random risk analysis, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.22, no.4, pp.491–504, 2014.

[26] Liu, Y., and K. Yao, Uncertain random logic and uncertain random entailment, *Journal of Ambient Intelligence and Humanized Computing*, vol.8, no.5, pp.695–706, 2017.

[27] Shaked, M., and G. Shanthikumar, *Stochastic Orders*, Springer, New York, 2007.

[28] Sheng, Y., and K. Yao, Some formulas of variance of uncertain random variable, *Journal of Uncertainty Analysis and Applications*, vol.2, no.12, 10 pages, 2014.

[29] Wen, M., and R. Kang, Reliability analysis in uncertain random system, *Fuzzy Optimization and Decision Making*, vol.15, no.4, pp.491–506, 2016.

[30] Yao, K., and J. Gao, Law of large numbers for uncertain random variables, *IEEE Transactions on Fuzzy Systems*, vol.24, no.3, pp.615–621, 2015.

[31] Zuo, Y., and X. Ji, Theoretical foundation of uncertain dominance, *Proceedings of the Eighth International Conference on Information and Management Sciences*, pp.827–832, 2009.