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Optimal and Adaptive Control of a New Hyper-Chaotic System about Its Unstable Equilibrium Points

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Abstract

In this manuscript, a novel hyper-chaotic system with complex dynamics having one real equilibrium point has been proposed. Fundamental dynamical analysis for a novel hyper chaotic system such as dissipation, equilibrium points, time series, phase portraits, Lyapunov exponents, Kaplan-Yorke dimension, bifurcation and Poincaré maps are investigated theoretically as well as numerically. The new hyper-chaotic system has five equilibrium points, out of which only one equilibrium point lie in real plane whereas, all other equilibrium points lies in imaginary plane. Beauty of the system is that despite of having five equilibrium points only one equilibria is acceptable. A hyper-chaotic systems with complex equilibrium points are very rare in the literature. Such system can act as powerful models in many engineering applications, especially in chaos based cryptology and coding information. Further, we studied the optimal control for the novel hyper-chaotic system which is based on the Pontryagin minimum principle (PMP). Also, we apply Lyapunov stability theory for adaptive control approach and a parameter estimation update law is given for the novel hyper-chaotic system with completely unknown parameters. Finally, to demonstrate the effectiveness of the proposed method we use MATLAB bvp4c and ode45 for numerical simulation which illustrates the stabilized behaviour of states and control functions of equilibrium point. The plots displaying the time history of states functions and the parameters estimates have been drawn for the equilibrium point.

Keywords: hyper-chaos, Poincaré maps, bifurcation, optimal control, Pontryagin minimum principle, adaptive control, Lyapunov stability theory

1 Introduction

Chaos can be defined as confusion or disorder that occurs in systems so that the systems appear erratic and unpredictable. Chaos is an interesting as well as important phenomenon due to its emergent properties or surprises. An emergent property is something that emerges from a system that is unexpected which gives rise to surprise. We can't predict it, and we can't even judge what surprise will come. We can't even know if a surprise will happen. We can only make a guess. Whereas, chaotic behavior is a common feature of nonlinear dynamics, as well as hyper-chaos in high-dimensional systems. Due to wide applications of chaotic systems in the field of mathematics, chemistry, biology, physics, engineering, computer science and many more. Now a days research is on the peak to introduce the new chaotic and hyper chaotic system with more complicated topological structures [1, 3, 11, 12, 13, 21, 31, 37, 39]. Yet, it is not easy task to create to new chaotic system or hyper-chaotic system with a more topological structure, with more wings, with more scrolls, with less equilibria and which also fulfills the compromise between lyapunov exponent and dissipation. The first classical hyper-chaotic system is the well known hyper-chaotic Rössler system [30]. After that, many hyper-chaotic systems have been developed and the applications of these models have been enhanced recently. Over the past decades, many other hyper-chaotic systems have been introduced,such as hyper-chaotic Chen system [4], hyper-chaotic Chen system [18], hyper-chaotic Lü system [6], hyper-chaotic Nikolov system [25], hyper-chaotic Lorenz system [34], and hyper-chaotic Lorenz system [2].

Hyper-chaotic systems not only have all the features and properties of chaotic systems, but also have more complicated nonlinear dynamic characteristics because they have more than one positive Lyapunov exponents and they are expanded in more than one direction. Some hyper-chaotic systems being applied in the practical engineering, are preferable for those applications that require the complexity of dynamics, such as the network security and the data

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encryption. From both theoretical and practical points of view, it is always desirable to generate an attractor with both multiple wings and hyper-chaos possessing both complicated topological structures and complex dynamics such as rich bifurcations and wider frequency bandwidths. There is no doubt that such a four-wing hyper-chaos system can help us to make chaos more important in both engineering applications and theoretical research. Nearly all the 4D hyper-chaotic systems, reported up to now, have double-wing hyper-chaotic attractors with three or five equilibrium points [14, 15, 18, 19, 22, 23, 26, 28, 32, 33, 35]. Generating a hyper-chaotic attractor with one equilibrium point from a dynamical system is a very difficult and rare phenomenon [8, 20, 22]. Also, a hyper-chaotic system having multi-scroll or multi-wing attractor with complex equilibrium points are very rare to be found. Such systems can act as powerful models in many engineering applications, especially in chaos based cryptology and coding information and in many other areas.

In present manuscript, we have first constructed the new hyper-chaotic system with multi wings and only one real equilibrium point and four complex equilibrium points but here, we have not bothered about the existence of complex equilibrium points because all the realizable models have real signals or states which means that the states will never reach to these equilibrium points as they are not real. Further, we have discussed the basic dynamical characters of the new hyper-chaotic system investigated by means of dissipation, equilibria, stability analysis, time series, phase portraits, Lyapunov exponents, Poincaré maps and bifurcation analysis. As we are aware that chaos control of chaotic systems has also received a great attention due to their potential applications in physics, chemical reactor, biological networks, artificial neural networks, telecommunications etc. Basically, chaos controlling is the stabilization of an unstable periodic orbits or equilibria by means of tiny perturbations of the system. Ott, Grebogi, and Yorke [27] firstly proposed the method of chaos control and after that many useful and powerful methods have been developed for sustained development of humanity. These may include optimal control [9, 10], synchronization [7], adaptive control [24], state-feedback control [36], sliding mode control [29], time-delayed feedback control [5], etc. So, in this paper sophisticated computational strategy has been proposed for hyper-chaos and optimal control. To obtain the optimal controllers for the new hyper-chaotic system Pontryagins minimum principle (PMP) [17] has been applied. Furthermore, an adaptive control law is introduced to stabilize the new 4-D Hyper-chaos system with unknown parameters. The adaptive control results derived in this paper are established by using the Lyapunov stability theory [16]. The numerical simulation results are strong enough to show the efficiency and accuracy of the proposed technique.

The paper is organised as: Section 2 contains the formulation of a new 4-D Hyper-chaotic system. Section 3 is about the analysis of the presented new hyper-chaotic system. In Section 4, Hyper-chaos and optimal control law are formulated for new 4-D Hyper-chaos system along with the simulations. In Section 5, an adaptive control law is devised to stabilize the new 4-D Hyper-chaotic system. The computational studies of the unknown parameters have also been preformed in this section. Finally, in the last section conclusions are drawn.

2 Description and Formulation of a Novel Hyper-Chaotic System

Our new hyper-chaotic system is based on chaotic system presented by Zhang, et al. [38], which can be given as:

$$\begin{aligned} \dot{x} &= -ax+byz, \\ \dot{y} &= -cy^3+dxz, \\ \dot{z} &= ez-fxy. \end{aligned} \tag{2.1}$$

where x, y and z are the state variables and a,b,c,d,e,f are the positive constants. The above system shows chaotic behaviour when parameters are chosen as $a=2, b=10, c=6, d=3, e=3, f=1$. To generate hyper-chaos from the dissipative autonomous systems, the state equation must satisfy the following two basic conditions:

- The dimension and order of the state equations should to be at least 4 and 2 respectively.
- The system must have at least two positive Lyapunov exponents with the condition that the sum of all Lyapunov exponents is less than zero.

Based on system (2.1) and keeping in mind the above two basic conditions, we construct novel hyper-chaotic system defined by:

$$\begin{aligned} \dot{x} &= -ax+byz, \\ \dot{y} &= -cy^3+dxz+xw, \\ \dot{z} &= dz-xy, \\ \dot{w} &= kw+z, \end{aligned} \tag{2.2}$$

where $[x,y,z,w]^T \in R^4$is the state vector, and a, b,c,d and k are positive constant parameters of the system.

More precisely, we analyzed some complicated dynamics in detail by using phase portraits, time series, Lyapunov exponents, bifurcation diagrams and Poincaré maps. We have shown that the system has two positive Lyapunov exponents, so system orbits extensively expand in some directions but rapidly shrink in some other directions, which significantly increase the system's orbital degree of disorder and randomness. We have discussed the detailed bifurcation analysis, which illustrates the evolution processes of the system among sinks, periodic orbits, chaos and hyper chaos. Also, by using phase portraits we have observed that the four-wing transient chaos occurs in the system. Moreover, we have performed Poincaré map analysis, which shows that the system has extremely rich dynamics. The basic properties of equilibrium points of the our new system have are also been discussed in detail.

3 Dynamical Analysis of the Novel Hyper-Chaotic System

3.1 Lyapunov Exponent, Phase Portraits and Time Responses.

In this section, we use the fourth order Runge-kutta method in Matlab to solve system (2.2) for the parameters and initial conditions are $a = 2.6, b = 10, c = 7, d = 3, k = .05$ and $(.4, -.5, -.1, .7)$ respectively. The Lyapunov exponents are calculated as $LE_1 = .46058, LE_2 = .15104, LE_3 = 0.0006415$ and $LE_4 = -18.1521$. It is clear that $LE_3 = 0.0006415$ is very close to 0. As, for a hyper-chaotic character we need at least two positive Lyapunov exponents, one null Lyapunov exponent along the flow and one negative Lyapunov exponent. To ensure the boundness of the solution and the minimal dimension for a (continuous) hyper-chaotic system must be 4. Thus for the values of chosen parameters, the system exhibits hyper-chaos. Lyapunov exponents are shown in Figure 1. Also, the complex dynamic behaviour of system (2.2) via plotting phase portraits and time series are displayed in Figure 2 and Figure 3 respectively.

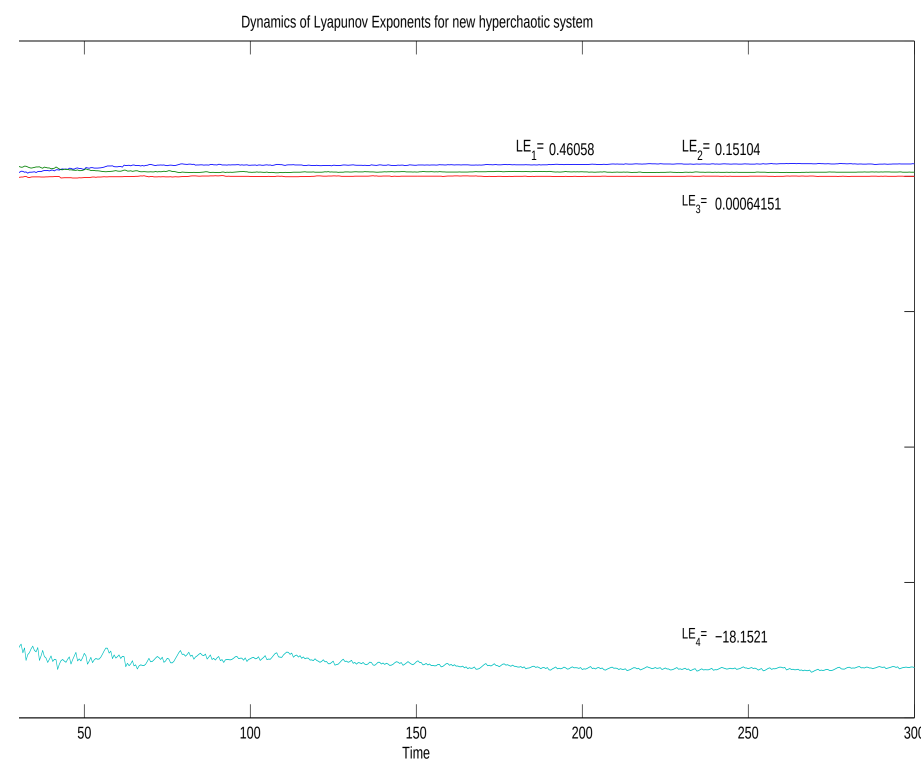


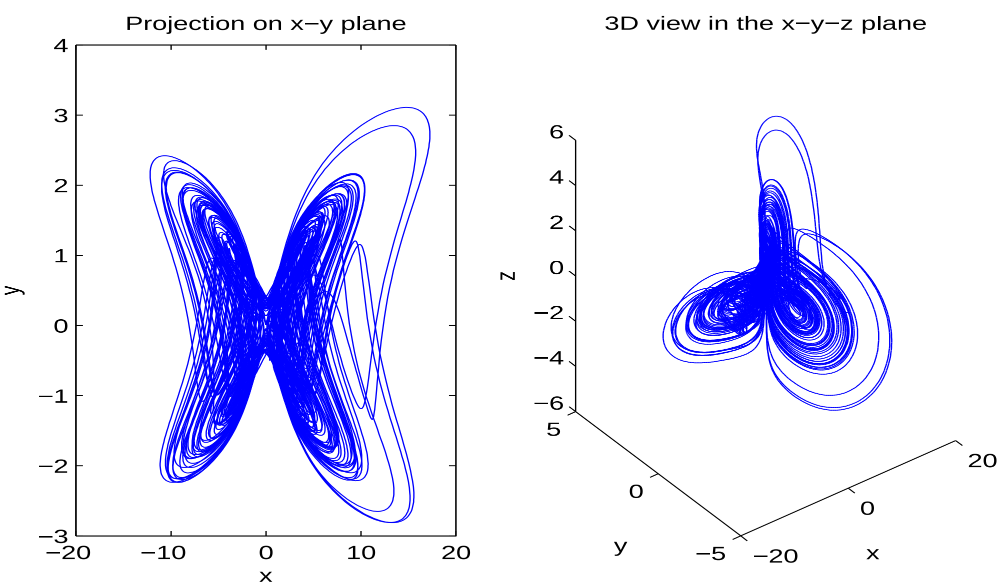
Figure 1: Lyapunov exponents graph for new 4-D Hyper-chaos systems

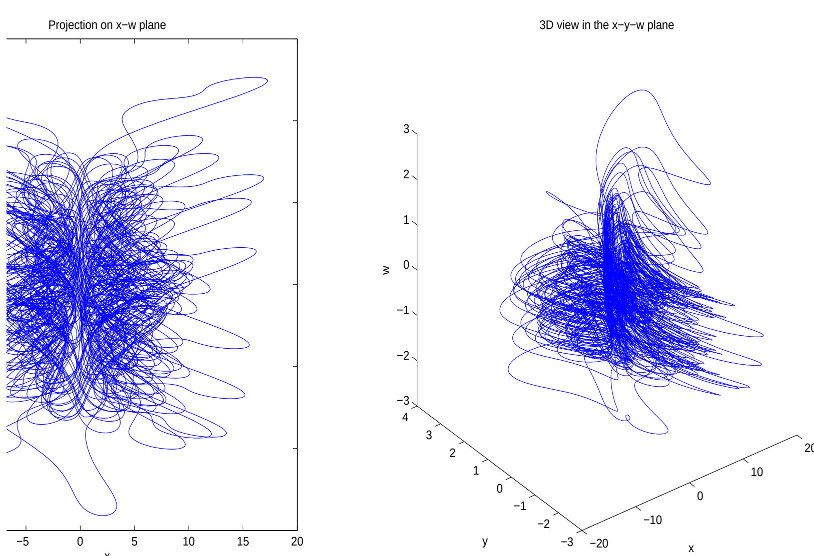
3.2 Kaplan-Yorke Dimension

The Kaplan-Yorke dimension of a chaotic system is defined as

$$D_{KY} = j + \sum_{i=1}^{j} \frac{LE_i}{|LE_{j+1}|}, \tag{3.1}$$

where j is the largest integer such that the sum of the j largest Lyapunov exponents is still non-negative. D_{KY} represents an upper bound for the information dimension of the system. The Kaplan Yorke dimension of system (2.2) is $D_{KY} = 3.03372951$.





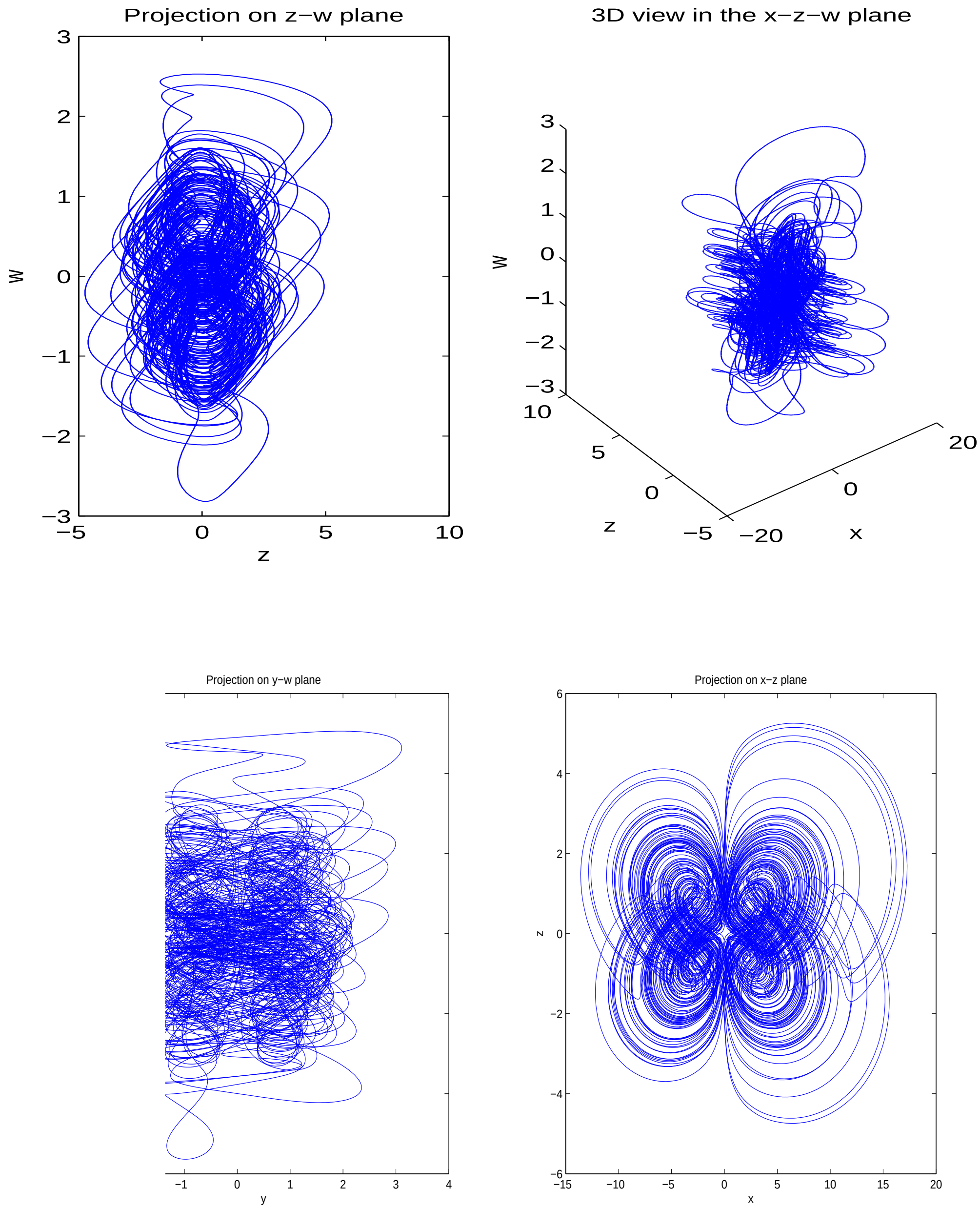


Figure 2: Phase portrait of a novel hyper-chaotic systems

3.3 Dissipation

The divergence of a vector field **F** of the system (2.2) can be obtained as:

$$\bigtriangledown.\mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w} = -a - 3cy^2 + d + k, \tag{3.2}$$

where

$$\mathbf{F} = (f_1, f_2, f_3, f_4) = (-ax + byz, -cy^3 + dxz + xw, dz - xy, kw + z). \tag{3.3}$$

So, system (2.2)would be dissipative $\bigtriangledown.\mathbf{F} < 0$ i.e when $-a - 3cy^2 + d + k < 0$ and will converges to subset of measure zero volume according to $\frac{dF}{dt} = e^{-a-3cy^2+d+k}$. This implies that all system orbits will ultimately be confined to a specific subset of zero volume and the asymptotic motion dies onto an attractor. It proves the existence of an attractor which is witnessed in Figure 2. Since, the sum of all the Lyapunov exponents of the new hyper-chaotic system is also negative, thus we conclude that the new hyper-chaotic system is dissipative.

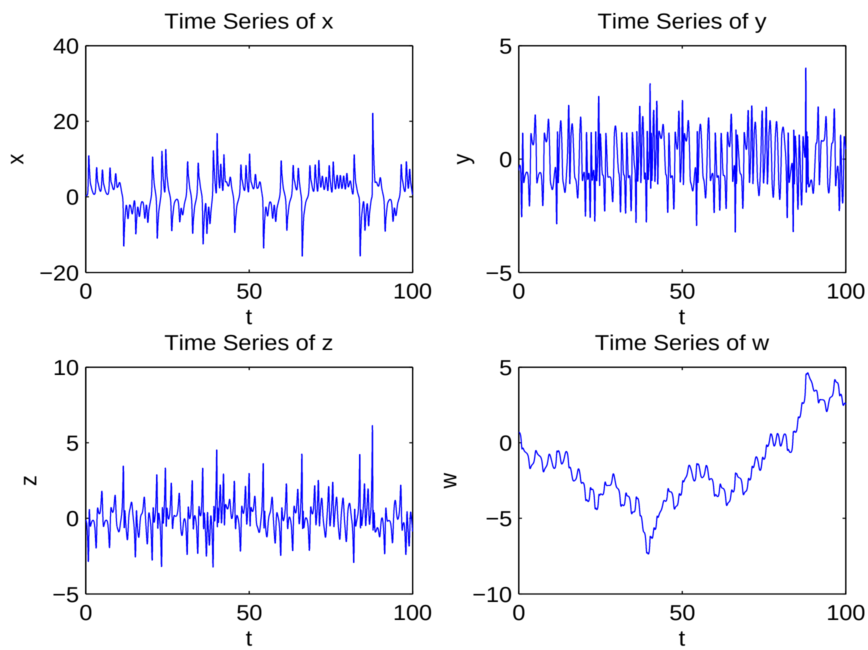


Figure 3: Time series of a novel hyper-chaotic systems

3.4 Symmetry and Invariance

The novel hyper-chaotic system (2.2) is invariant under the coordinate transformation $(x, y, z, w) \to (x, -y, z, w)$ and $(x, -y, -z, -w)$. Thereby the system (2.2) is symmetrical for any choice of the values of system parameters.

3.5 Equilibrium Points and Stability Analysis

The equilibria of new 4-D system (2.2) can be calculated by solving the following equations :

$$\begin{aligned} -ax + byz &= 0, \\ -cy^3 + dxz &= 0, \\ dz - xy &= 0, \\ kw + z &= 0. \end{aligned} \tag{3.4}$$

On solving above equation with the chosen parameters as $a = 2.6, b = 10, c = 7, d = 3$ and $k = .05$, we get five Equilibrium points for new 4-D hyper-chaos system which are given by:

$$\begin{aligned} E_1 &= (0, 0, 0, 0), \\ E_2 &= (0. - 0.981595i, 0.883176, 0. - 0.288974i, 0. + 5.77948i), \\ E_3 &= (0. + 0.981595i, 0.883176, 0. + 0.288974i, 0. - 5.77948i), \\ E_4 &= (0. + 0.981595i, -0.883176, 0. - 0.288974i, 0. + 5.77948i), \\ E_5 &= (0. - 0.981595i, -0.883176, 0. + 0.288974i, 0. - 5.77948i). \end{aligned} \tag{3.5}$$

It is very interesting to note that out of these five equilibrium points four equilibrium points i.e E_2, E_3, E_4 and E_5 lie in the imaginary plane and one equilibrium point E_1 is only one point which lies in the real plane. But here, we neglect the existence of complex equilibrium points because all the realizable models have real signals or states which indicates that the states will never reach to these equilibrium points as they are not real. Also, from the definition of the equilibrium point: it is the point that if the states are that point they will stay forever, so the states must be complex and this is impossible in practice. Thus, keeping it in mind we neglect all the complex equilibrium points and consider only one equilibrium point i.e $E_1 = (0,0,0,0)$. It is the beauty of the system that in spite of being five equilibrium point we only consider only one equilibrium points and investigate the complex dynamic behavior.

Proposition 1: The equilibrium point E_1 of new Hyper-chaotic system (2.2) is unstable for chosen parameter values $a = 2.6, b = 10, c = 7, d = 3$ and $k = .05$.

Proof: The Jacobian matrix for the Hyper-chaos system (2.2) is given by

$$\mathbf{J} = \begin{bmatrix} -a & bz & by & 0 \\ dz+w & -3cy^2 & dx & x \\ -y & -x & d & 0 \\ 0 & 0 & 1 & k \end{bmatrix} \tag{3.6}$$

with the parameter values $a = 2.6, b = 10, c = 7, d = 3$ and $k = .05$, the system (2.2) has E_1, E_2, E_3, E_4 and E_5 five equilibrium points given by (3.5). Out of which only equilibrium point E_1 is real. So we discuss the stability only about E_1.

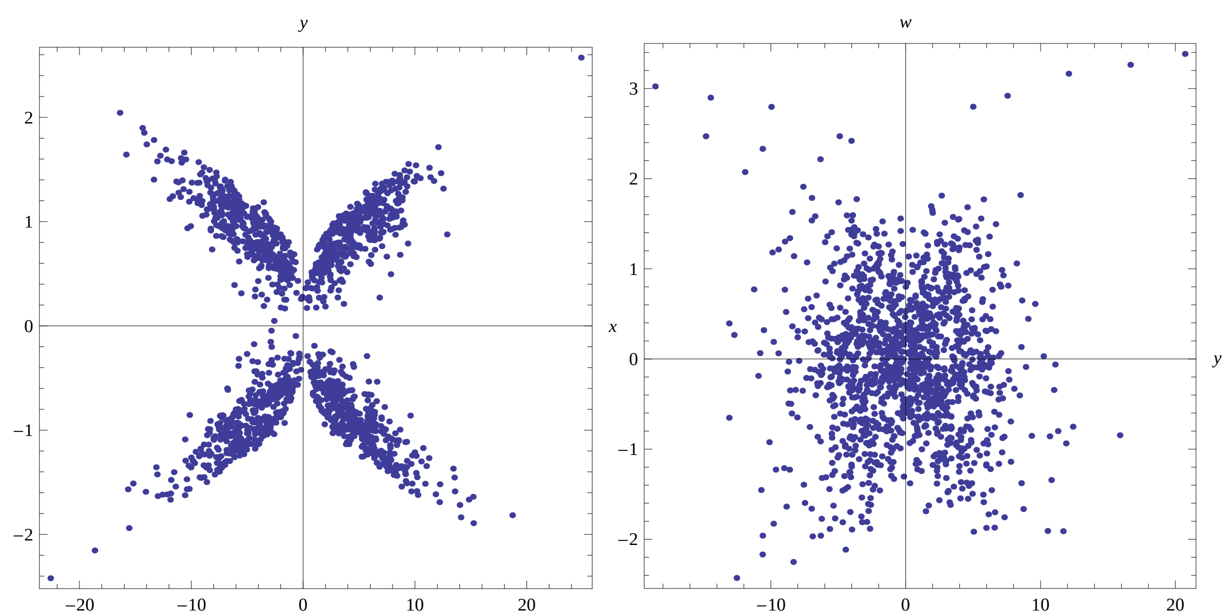
The eigenvalues of the jacobian matrix $\mathbf{J}$ at the equilibrium point $E_1 = (0,0,0,0)$ are given by: $\lambda_1 = 3, \lambda_2 = -2.6, \lambda_3 = 0.05, \lambda_4 = 0$. It is observed that two eigenvalues $\lambda_1$and λ_3 are positive. Thus, according to Lyapunov stability theory, the equilibrium point E_1 is unstable. Hence, it is established that the equilibrium point E_1 of new Hyper-chaotic system (2.2) is unstable for the chosen parameters $a = 2.6, b = 10, c = 7, d = 3$ and $k = 0.05$.

3.6 Poincaré Mapping

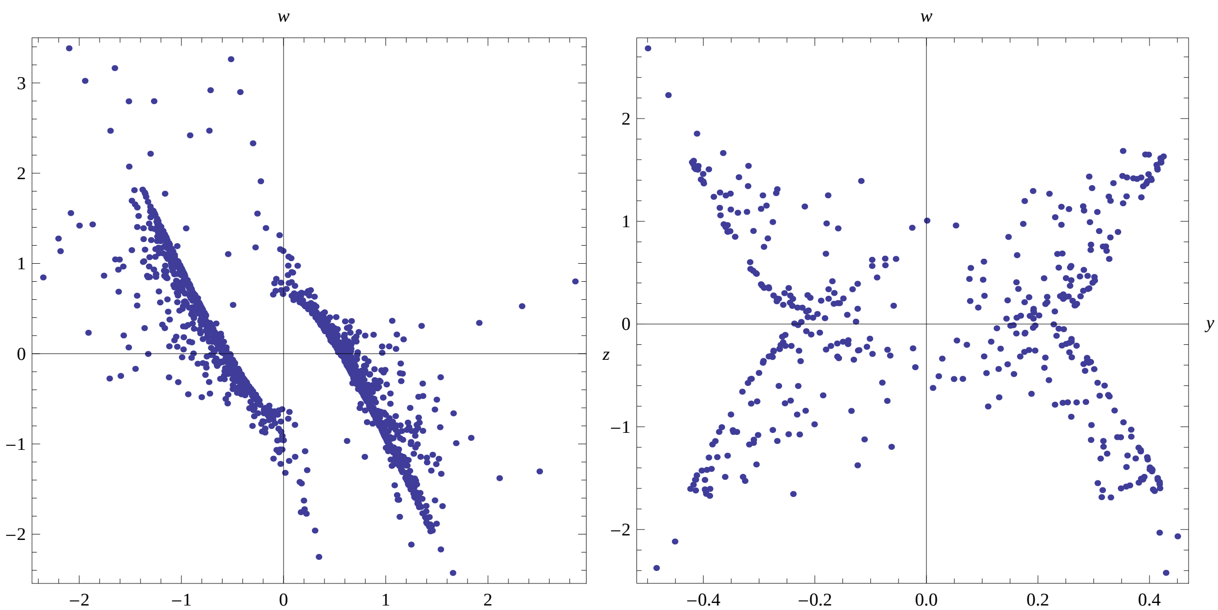
The Poincaré map is one such computational technique which helps to visualise the folding properties of chaos. This also provides the idea of the bifurcation. When $a = 2.6, b = 10, c = 7, d = 3$ and $k = 0.05$ and on taking the different crossing planes such as $z = 0, y = 0, x = 0$. The corresponding Poincaré maps on the $x-y, x-w, z-w$ and $y-w$ planes are displayed in Figure 4. Figure 4 illustrates that, system (2.2) has a self-similar structure and some sheets are folded and wing type structure is visualized.

3.7 Bifurcation

When the parameters $b = 10, c = 7, d = 3$ and $k = .05$ are fixed while parameter 'a' is varied, the corresponding bifurcation diagram of state y with respect to 'a' is obtained as shown in Figure 5. it is easy to see the chaotic behavior of new hyper-chaotic system when the parameter 'a'$\in [0,4]$.



(a) (b)



(c) (d)

Figure 4: Poincaré section for new hyper-chaotic systems (2.2) with the parameters $a = 2.6, b = 10, c = 7, d = 3$ and $k = .05$. (a) Projection on $x - y$ plane with $z = 0$; (b) Projection on $x - W$ plane with $y = 0$; (c) Projection on $z - w$ plane with $y = 0$; (d) Projection on $y - w$ plane with $x = 0$

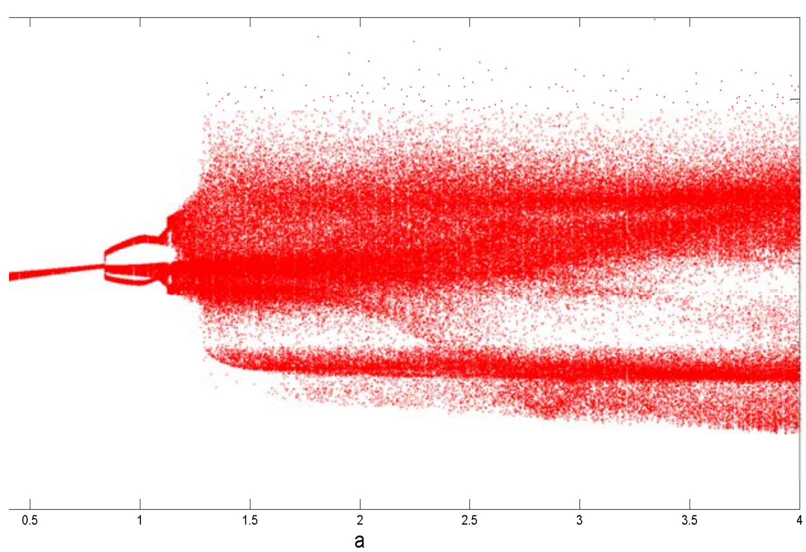


Figure 5: Bifurcation diagram of new hyper chaotic system (2.2) versus the parameter 'a'$\in [0, 4]$ when $b = 10, c = 7, d = 3, k = .05$

4 Hyper-Chaos and Optimal Control of a New 4-d Hyper-chaotic System

In this section, we apply Pontryagin minimum principle (PMP) to achieve optimal hyper-chaos control of the new hyper-chaotic system (2.2) at the equilibrium point $E_1 = (0, 0, 0, 0)$.

4.1 Methodology

For the purpose of optimal hyper-chaos control, we represent new 4-D Hyper-chaotic system (2.2) as:

$$\begin{aligned}\dot{x} &= -ax+byz+U_1,\\ \dot{y} &= -cy^3+dxz+xw+U_2,\\ \dot{z} &= dz-xy+U_3,\\ \dot{w} &= kw+z+U_4.\end{aligned}\tag{4.1}$$

where, U_1,U_2,U_3 and U_4 are the control inputs which should be satisfied by the optimal conditions at the equilibrium point $E_1=(0,0,0,0)$ obtained by PMP with respect to the cost function $\boldsymbol{J}$. The main strategy to control the system is to design the optimal control inputs U_1,U_2,U_3 and U_4 such that the state trajectories tends to the unstable equilibrium point $E_1=(0,0,0,0)$ in a given finite time interval $[0,t_f]$. Thus, the boundary conditions are:

$$\begin{aligned}x(0)=x_0, &\quad x(t_f)=\overline{x},\\ y(0)=y_0, &\quad y(t_f)=\overline{y},\\ z(0)=z_0, &\quad z(t_f)=\overline{z},\\ w(0)=w_0, &\quad w(t_f)=\overline{w}\end{aligned}\tag{4.2}$$

where $\overline{x},\overline{y},\overline{z}$ and $\overline{w}$ denote the coordinates of the equilibrium points. The objective functional to be minimized is defined as:

$$\boldsymbol{J}=1/2\int_0^{t_f}(\alpha_1(x-\overline{x})^2+\alpha_2(y-\overline{y})^2+\alpha_3(z-\overline{z})^2+\alpha_4(w-\overline{w})^2+\beta_1U_1^2+\beta_2U_2^2+\beta_3U_3^2+\beta_4U_4^2)dt,\tag{4.3}$$

where α_i and $\beta_i(i=1,2,3,4)$ are positive constants. Now, we will derive the optimality conditions as a nonlinear two point boundary value problem (TPBVP) arising in the Pontryagin minimum principle (PMP). The corresponding Hamiltonian function H will be:

$$\begin{aligned}H=&-1/2[\alpha_1(x-\overline{x})^2+\alpha_2(y-\overline{y})^2+\alpha_3(z-\overline{z})^2+\alpha_4(w-\overline{w})^2)+\beta_1U_1^2+\beta_2U_2^2+\beta_3U_3^2+\beta_4U_4^2]\\ &+\lambda_1[-ax+byz+U_1]+\lambda_2[-cy^3+dxz+xw+U_2]+\lambda_3[dz-xy+U_3]+\lambda_4[kw+z+U_4],\end{aligned}\tag{4.4}$$

where $\lambda_1,\lambda_2,\lambda_3$ and λ_4 are the costate variables. On applying the Pontryagin minimum principle (PMP), we obtain the Hamiltonian equations:

$$\begin{aligned}\dot{\lambda_1} &= -\frac{\partial H}{\partial x},\\ \dot{\lambda_2} &= -\frac{\partial H}{\partial y},\\ \dot{\lambda_3} &= -\frac{\partial H}{\partial z},\\ \dot{\lambda_4} &= -\frac{\partial H}{\partial w}.\end{aligned}\tag{4.5}$$

From (4.4) and (4.5), we have:

$$\begin{aligned}\dot{\lambda_1} &= \alpha_1(x-\overline{x})+a\lambda_1-(dz+w)\lambda_2+y\lambda_3,\\ \dot{\lambda_2} &= \alpha_2(y-\overline{y})-bz\lambda_1+3cy^2\lambda_2+x\lambda_3,\\ \dot{\lambda_3} &= \alpha_3(z-\overline{z})-by\lambda_1-dx\lambda_2-d\lambda_3-\lambda_4,\\ \dot{\lambda_4} &= \alpha_4(w-\overline{w})-x\lambda_2-k\lambda_4.\end{aligned}\tag{4.6}$$

The optimal control functions that have to be used are determined from the condition $\frac{\partial H}{\partial U_i}=0(i=1,2,3,4)$. Hence, we get

$$U_i^*=\frac{\lambda_i}{\beta_i}\quad(i=1,2,3,4).\tag{4.7}$$

After substituting the value from (4.7) into (4.2), we get the controlled non linear state equations:

$$\begin{aligned} \dot{x} &= -ax+byz+\frac{\lambda_1}{\beta_1}, \\ \dot{y} &= -cy^3+dxz+xw+\frac{\lambda_2}{\beta_2}, \\ \dot{z} &= dz-xy+\frac{\lambda_3}{\beta_3}, \\ \dot{w} &= kw+z+\frac{\lambda_4}{\beta_4}. \end{aligned} \tag{4.8}$$

The above system of nonlinear ordinary differential equations together with the (4.6) forms a complete system for solving the optimal control of the new hyper-chaos system. The boundary conditions for this system have been given in (4.2). On solving the above nonlinear two point boundary value problem, we can obtain the optimal control law and the optimal state trajectories.

4.2 Numerical Simulations and Discussions

In this section we demonstrate the effectiveness and feasibility of the proposed optimal control scheme by using the MATLAB's bvp4c in-built solver. We solve the system (4.8) along with (4.6) and using the boundary conditions given in (4.2). For solving we choose the finite time interval as: [0,7], initial values and system parameters for the new 4-d hyper-chaotic system are chosen as: $x(0) = .4, y(0) = -.5, z(0) = -.1, w(0) = .7, a = 2.6, b = 10, c = 7, d = 3$ and $k = 0.05$. Also, the positive constants in objective function $\boldsymbol{J}$ for the equilibrium point E_1 are chosen as: $\alpha_i = 5$ and $\beta_i = 2$, for $i = 1,2,3$ and 4.

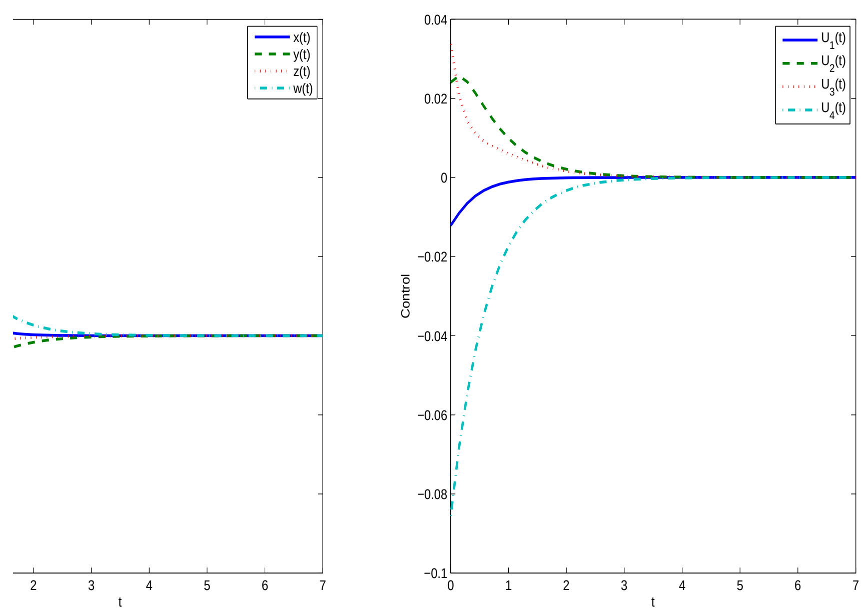


Figure 6: The stabilized behaviour of states and control functions for the equilibrium points E_1

Figures 6 exhibit the controlled behaviour of the states variables (x,y,z,w) as all the states trajectories converge to the equilibrium point $E_1 = (0,0,0,0)$ with in the chosen time frame [0,7]. It also displays the controlled behaviour of controllers (U_1,U_2,U_3,U_4) for the equilibrium point E_1 of the controlled new hyper-chaotic system.

5 Adaptive Control Approach for New 4-D Hyper-Chaos System with Unknown Parameters

This section aims to find an adaptive control law along with a parameter estimation update law for the new 4-D hyper-chaos system (2.2), such that all the state variables x,y,z and w converges to its equilibrium points as 't' approaches to infinity.

5.1 Design of the Adaptive Controllers

Let's assume that the controlled system can be written in the form:

$$\begin{aligned} \dot{x} &= -ax+byz+V_1, \\ \dot{y} &= -cy^3+dxz+xw+V_2, \\ \dot{z} &= dz-xy+V_3, \\ \dot{w} &= kw+z+V_4, \end{aligned} \tag{5.1}$$

where x,y,z and w are the states of the system and a,b,c,d and k are the unknown parameters of the system and V_1,V_2,V_3,V_4 are the adaptive controllers to be designed.

Theorem 5.1 The new hyper-chaotic system (2.2) with unknown parameters is asymptotically and globally stabilized for all initial values of states $(x(0),y(0),z(0),w(0))\in R^4$ by the following adaptive control law:

$$\begin{aligned} V_1 &= a_1x-b_1yz-\ell_1(x-\overline{x}), \\ V_2 &= c_1y^3-dxz-xw-\ell_2(y-\overline{y}), \\ V_3 &= -d_1z+xy-\ell_3(z-\overline{z}), \\ V_4 &= -k_1w-z-\ell_4(w-\overline{w}), \end{aligned} \tag{5.2}$$

and the parameter estimation update law:

$$\begin{aligned} \dot{a_1} &= -(x-\overline{x})x+\ell_5(a-a_1), \\ \dot{b_1} &= (x-\overline{x})yz+\ell_6(b-b_1), \\ \dot{c_1} &= -(y-\overline{y})y^3+\ell_7(c-c_1), \\ \dot{d_1} &= (z-\overline{z})z+(y-\overline{y})xz+\ell_8(d-d_1), \\ \dot{k_1} &= (w-\overline{w})w+\ell_9(k-k_1), \end{aligned} \tag{5.3}$$

where a_1,b_1,c_1,d_1,k_1 are estimated values of uncertain parameters a,b,c,d,k and $\ell_i(i=1,...,9)$ are the positive constants.

Proof. After substituting (5.2) into (5.1), we get the closed-loop system as:

$$\begin{aligned} \dot{x} &= -(a-a_1)x+(b-b_1)yz-\ell_1(x-\overline{x}), \\ \dot{y} &= -(c-c_1)y^3+(d-d_1)xz-\ell_2(y-\overline{y}), \\ \dot{z} &= (d-d_1)z-\ell_3(z-\overline{z}), \\ \dot{w} &= (k-k_1)w-\ell_4(w-\overline{w}). \end{aligned} \tag{5.4}$$

For the derivation of the update law for adjusting the parameter estimates, the lyapunov approach is used. We define the lyapunov function as:

$$\boldsymbol{V}(x,y,z,w,\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{k},\tilde{h}) = 1/2((x-\overline{x})^2+(y-\overline{y})^2+(z-\overline{z})^2+(w-\overline{w})^2+\tilde{a}^2+\tilde{b}^2+\tilde{c}^2+\tilde{d}^2+\tilde{k}^2) \tag{5.5}$$

where $\tilde{a}=a-a_1, \tilde{b}=b-b_1, \tilde{c}=c-c_1, \tilde{d}=d-d_1$ and $\tilde{k}=k-k_1$.

On taking the time derivative of the lyapunov function $\boldsymbol{V}$, we obtain

$$\dot{\boldsymbol{V}} = (x-\overline{x})\dot{x}+(y-\overline{y})\dot{y}+(z-\overline{z})\dot{z}+(w-\overline{w})\dot{w}+\tilde{a}\dot{\tilde{a}}+\tilde{b}\dot{\tilde{b}}+\tilde{c}\dot{\tilde{c}}+\tilde{d}\dot{\tilde{d}}+\tilde{k}\dot{\tilde{k}}. \tag{5.6}$$

On substituting (5.3)and (5.4) in (5.6), the derivative of lyapunov function with respect to time becomes:

$$\begin{aligned}\dot{V} = &-\ell_1(x-\bar{x})^2 - \ell_2(y-\bar{y})^2 - \ell_3(z-\bar{z})^2 - \ell_4(w-\overline{w})^2 \\ &-\ell_5(a-a_1)^2 - \ell_6(b-b_1)^2 - \ell_7(c-c_1)^2 - \ell_8(d-d_1)^2 - \ell_9(k-k_1)^2.\end{aligned} \tag{5.7}$$

Since the lyapunov function V is a positive definite function on R^9 and clearly its derivatives $\dot{V}$ on R^9 is negative definite function, then by using lyapunov stability theory, the controlled system (5.1) converge asymptotically and globally for all initial values to its equilibrium points with the adaptive control law (5.2) and the parameter update law (5.3). This completes the proof.

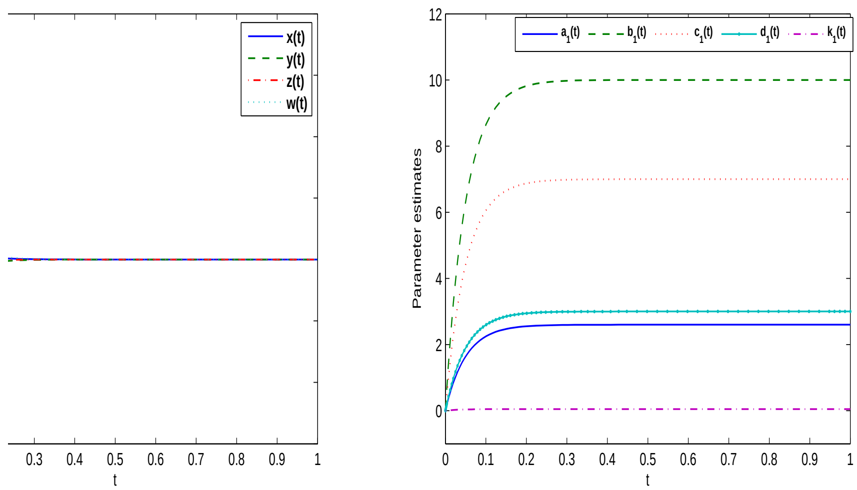


Figure 7: Time history of states function and parameter estimates for the equilibrium points E_1

5.2 Numerical Simulations and Discussions

Numerical results are presented to demonstrate the effectiveness of the proposed adaptive control technique. For simulation we solve the controlled hyper-chaotic system (5.1) with the adaptive control law (5.2) and the parameter update law (5.3) by using MATLAB's ode45 in-built solver. The initial and parameters values are respectively chosen as: $x(0) = .4, y(0) = -.5, z(0) = -.1, w(0) = .7$, $a = 2.6, b = 10, c = 7, d = 3$ and $k = .05$ for the controlled new 4d hyper-chaos system. Also, for adaptive and update laws,we take $\ell_1 = 10, \ell_2 = 10, \ell_3 = 10, \ell_4 = 10, \ell_5 = 10, \ell_6 = 10, \ell_7 = 10, \ell_8 = 10$ and $\ell_9 = 10$. Further, the initial values for parameter estimates are chosen as: $a_1 = 0, b_1 = 0, c_1 = 0, d_1 = 0$ and $k_1 = 0$. From Figure 7, it is clear that the trajectories of the controlled new hyper-chaotic system (5.1) converges asymptotically to $E_1 = (0,0,0,0)$ with time 't' and this figure also shows that the parameter estimates $a_1(t), b_1(t), c_1(t), d_1(t)$ and $k_1(t)$ actually converge to the system parameter values $a = 2.6, b = 10, c = 7, d = 3$ and $k = .05$ asymptotically with time. Thus, our desired goal of controlling the hyper-chaos and estimating the unknown parameters has been achieved successfully.

6 Conclusion

In present manuscript, we have introduced a new hyper-chaotic system with more complicated dynamical properties which have been successfully validated analytically and numerically. Dynamical properties of the new hyper chaotic system are analysed by means of time series, lyapunov exponent, equilibrium points. Also, poincaré and bifurcation analysis have been executed for the chosen parameters. An optimal control law has been formulated for the new 4-D hyper-chaos system, which is based on the PMP. Furthermore, an adaptive control law has been devised to stabilize the new 4-D hyper-chaos system with unknown parameters. Effectiveness and feasibility of results are validated via numerical simulations which are performed by using MATLAB'S bvp4c and ode-45 in-built solver. Remarkably, our analytic and computational results are in an excellent agreement.

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