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Results of m-polar Fuzzy Graphs with Application

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Abstract

An m-polar fuzzy model is useful for multi-polar information, multi-agent, multi-attribute and multi-object network models which gives more precision, flexibility, and comparability to the system as compared to the classical, fuzzy and bipolar fuzzy models. In this paper, m-polar fuzzy sets are used to introduce the notion of m-polar psi-morphism on product m-polar fuzzy graph (mFG). The action of this morphism is studied and established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in product mFG are defined and studied their properties. A real life situation has been modeled as an application of product mFG.

Keywords: product m-polar fuzzy graphs, degree, regularity, ψ-morphism

1 Introduction

After the introduction of fuzzy sets by Zadeh [30] in 1965, fuzzy set theory is included as large research fields. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences engineering, statistic, graph theory, artificial intelligence, signal processing, multi agent systems, decision making and automata theory. In a fuzzy set, each element is associated with a membership value selected from the interval [0,1]. Instead of using particular membership value as in fuzzy sets, m-polar fuzzy set can be used to represent the vagueness of a set more perfectly. In 2014, Chen et al. [6] introduced the notion of m-polar fuzzy set as a generalization of fuzzy set theory. The membership values in m-polar fuzzy sets is more expressive in capturing uncertainty of data.

The idea behind this is that "multipolar information" exists because data of real world problems are sometimes come from multiple agents. m-polar fuzzy sets allow more graphical representation of vague data, which facilitates significantly better analysis in data relationships, incompleteness, and similarity measures.

Graph theory besides being a well developed branch of Mathematics, it is an important tool for mathematical modeling. Realizing the importance, Rosenfeld [22] introduced the concept of fuzzy graphs, Mordeson and Nair [15] discussed about the properties of fuzzy graphs and hypergraphs. After that, the operation of union, join, Cartesian product and composition on two fuzzy graphs was defined by Mordeson and Peng [16]. Sunitha and Kumar [27] further studied the other properties of fuzzy graphs. The concept of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in [4]. After that several researchers are working on fuzzy graphs like in [3, 5, 13, 14, 17, 18, 23, 25, 26].

In 2011, using the concepts of bipolar fuzzy sets, Akram [1] introduced the bipolar fuzzy graphs and defined different operations on it. Rashmanlou et al. [19, 20, 21] studied bipolar fuzzy graphs, bipolar fuzzy graphs with categorical properties, product of bipolar fuzzy graphs and their degrees, etc. Some work on bipolar fuzzy graphs may be found on [12, 24, 28, 29].

Chen et al. [6] first introduced the concept of mFGs. Ghorai and Pal studied many properties of generalized mFGs [8], defined operations, density of mFGs [7], introduced the concept of m-polar fuzzy planar graphs [10, 11] and studied isomorphic properties of them [9].

This paper is organized in the following manner. In Section 1, introduction is given and the literature review is illustrated. Section 2 represents a brief study of some graph theoretic concept used in this paper.

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In Section 3, the notion of m-polar ψ-morphism is introduced on product mFG as a generalization of our usual homomorphism. The action of this morphism is studied and established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in product mFGs are defined and studied their properties. In Section 4, a real life situation have been modeled. Section 5 represents the conclusion of the paper.

2 Preliminaries

A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of all edges of G^*. Two vertices x and y in an undirected graph G^* are said to be adjacent in G^* if xy is an edge of G^*. A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices. A subgraph of a graph $G^* = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. The degree of a vertex in G is the number of edges incident with the vertex.

A fuzzy graph with V as the underlying set is a triplet $G = (V, \sigma, \mu)$, where $\sigma : V \to [0, 1]$ is a fuzzy subset of V and $\mu : V \times V \to [0, 1]$ is fuzzy relation on σ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.

Here, $[0, 1]^m$ (m-power of [0,1]) is considered to be a poset with point-wise order $\leq$, where m is an natural number. "$\leq$" is defined by $x \leq y \Leftrightarrow$ for each $i = 1, 2, \ldots, m$; $p_i(x) \leq p_i(y)$ where $x, y \in [0, 1]^m$ and $p_i : [0, 1]^m \to [0, 1]$ is the i-th projection mapping.

Definition 2.1. [6] *An m-polar fuzzy set $\big($or a $[0, 1]^m$-set$\big)$ on X is a mapping $A : X \to [0, 1]^m$. The set of all m-polar fuzzy sets on X is denoted by $m(X)$.*

Throughout the paper, G^* represents a crisp graph and $G = (V, A, B)$ represents a product mFG of G^*.

Definition 2.2. [7] *A product m-polar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$ where $A : V \to [0, 1]^m$ is an m-polar fuzzy set in V and $B : \widetilde{V^2} \to [0, 1]^m$ is an m-polar fuzzy set in $\widetilde{V^2}$ such that $p_i \circ B(xy) \leq p_i \circ A(x) \times p_i \circ A(y)$ for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$ and $B(xy) = \mathbf{0}$ for all $xy \in \widetilde{V^2} - E$, $\big(\mathbf{0} = (0, 0, \ldots, 0)$ is the smallest element in $[0, 1]^m\big)$.*

Definition 2.3. [8] *G is called strong if $p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y)$ for all $xy \in E$, $i = 1, 2, \ldots, m$.*

G is called complete if $p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y)$ for all $x, y \in V$, $i = 1, 2, \ldots, m$.

The complement of G is a product mFG $\overline{G} = (V, \overline{A}, \overline{B})$ where $\overline{A} = A$ and $\overline{B}$ is defined by $p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy)$, $xy \in \widetilde{V^2}$ and $i = 1, 2, \ldots, m$.

Definition 2.4. [8] *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product mFGs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively.*

A weak isomorphism between G_1 and G_2 is a bijective mapping $\phi : V_1 \to V_2$ such that ϕ is a homomorphism and $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ for all $x_1 \in V_1$ for $i = 1, 2, \ldots, m$.

A co-weak isomorphism between G_1 and G_2 is a bijective mapping $\phi : V_1 \to V_2$ such that ϕ is a homomorphism and $p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1y_1))$ for all $x_1y_1 \in \widetilde{V_1^2}$ for $i = 1, 2, \ldots, m$.

Definition 2.5. *Let $G = (V, A, B)$ be a product m-polar fuzzy graph of $G^* = (V, E)$.*

(i) The neighborhood degree of a vertex v is defined as $d_N(v) = \big(d_N^1(v), d_N^2(v), \cdots, d_N^m(v)\big)$ where $d_N^i(v) = \sum\limits_{u \in N(v)} p_i \circ A(u)$, $i = 1, 2, \ldots, m$.

(ii) The degree of a vertex v in G is defined by $d_G(v) = \big(d_G^1(v), d_G^2(v), \cdots, d_G^m(v)\big)$, where $d_G^i(v) = \sum\limits_{\substack{u \neq v \\ uv \in E}} p_i \circ B(uv)$, $i = 1, 2, \ldots, m$. If all the vertices of G have same degree, then G is called regular product mFG.

(iii) The closed degree of a vertex v is defined by $d_G[v] = \big(d_G^1[v], d_G^2[v], \cdots, d_G^m[v]\big)$, where $d_G^i[v] = d_G^i(v) + p_i \circ A(v)$, $i = 1, 2, \ldots, m$. If each vertex of G has same closed degree, then G is called totally regular product mFG.

3 Regularity and Isomorphism on mFGs

Regular graphs are the most widely studied classes. For example, regular fuzzy graphs play a key role in designing reliable communication networks. Here, the notion of m-polar ψ-morphism is introduced in product mFG. Also, d_2-degree, total d_2-degree, $(2,\overline{k})$-regularity and totally $(2,\overline{k})$-regularity are defined in product mFG and studied some important properties of them.

Definition 3.1. *Let G be a product mFG. The d_2- degree of a vertex u in G is $d_2(u) = (d_2^1(u), d_2^2(u), \cdots, d_2^m(u))$ where $d_2^i(u) = \sum p_i \circ B^2(uv)$ is such that $p_i \circ B^2(uv) = sup\{p_i \circ B(uu_1) \wedge p_i \circ B(u_1v)\}$.*

The minimum d_2-degree of G is denoted as $\delta_2(G) = (\delta_2^1(G), \delta_2^2(G), \cdots, \delta_2^m(G))$ where $\delta_2^i(G) = \wedge\{d_2^i(u) : u \in V\}$. The maximum d_2-degree of G is denoted as $\triangle_2(G) = (\triangle_2^1(G), \triangle_2^2(G), \cdots, \triangle_2^m(G))$ where $\triangle_2^i(G) = \vee\{d_2^i(u) : u \in V\}$.

Example 3.2. *Let G be a product 3-polar fuzzy graph where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$ (see Fig. 1). By routine computations we have*

$d_2^1(u_1) = \{0.3 \wedge 0.5\} + \{0.3 \wedge 0.4\} = 0.6$, $d_2^2(u_1) = \{0.4 \wedge 0.4\} + \{0.4 \wedge 0.2\} = 0.6$,
$d_2^3(u_1) = \{0.2 \wedge 0.1\} + \{0.3 \wedge 0.2\} = 0.3$, $d_2^1(u_2) = \{0.5 \wedge 0.4\} + \{0.3 \wedge 0.3\} = 0.7$,
$d_2^2(u_2) = \{0.4 \wedge 0.2\} + \{0.4 \wedge 0.4\} = 0.6$, $d_2^3(u_2) = \{0.1 \wedge 0.1\} + \{0.2 \wedge 0.3\} = 0.3$,
$d_2^1(u_3) = \{0.4 \wedge 0.4\} + \{0.5 \wedge 0.3\} = 0.7$, $d_2^2(u_3) = \{0.2 \wedge 0.2\} + \{0.4 \wedge 0.4\} = 0.6$,
$d_2^3(u_3) = \{0.1 \wedge 0.2\} + \{0.1 \wedge 0.2\} = 0.2$.

Hence, $d_2(u_1) = (0.6, 0.6, 0.3)$, $d_2(u_2) = (0.7, 0.6, 0.3)$, $d_2(u_3) = (0.7, 0.6, 0.2)$.

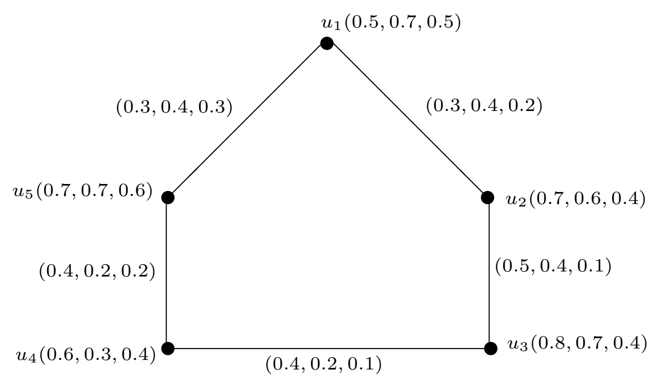

Figure 1: Product 3-polar fuzzy graph G

Definition 3.3. *If $d_2(u) = \overline{k}$ for all $u \in V$ then G is said to be $(2,\overline{k})$- regular product mFG.*

Example 3.4. *Consider the product 3-polar fuzzy graph as in Fig. 2. Here, $d_2(u_1) = d_2(u_2) = d_2(u_3) = d_2(u_4) = (0.2, 0.2, 0.2)$. So, G is $\big(2, (0.2, 0.2, 0.2)\big)$-regular product 3-polar fuzzy graph.*

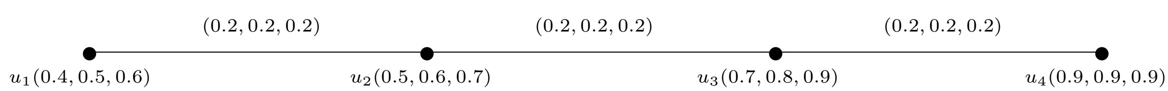

Figure 2: $\big(2, (0.2, 0.2, 0.2)\big)$-regular product 3-polar fuzzy graph G

Definition 3.5. *The total d_2- degree of a vertex $u \in V$ is defined as $td_2(u) = (td_2^1(u), td_2^2(u), \cdots, td_2^m(u))$, where $td_2^i(u) = \sum p_i \circ B^2(uv) + p_i \circ A(u)$, $i = 1, 2, \ldots, m$.*

Note 3.6. *If each vertex of G has the same total d_2-degree $\overline{l}$, then G is said to be totally $(2,\overline{l})$-regular product mFG.*

Example 3.7. *Consider the product 3-polar fuzzy graph G with the vertex set $V = \{u_1, u_2, u_3, u_4, u_5\}$ and edge set $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$ (see Fig. 3).*

We see that $d_2(u_1) = (1, 0.3, 0.6)$, $d_2(u_2) = (0.8, 0.2, 0.4)$, $d_2(u_3) = (0.8, 0.3, 0.6)$, $d_2(u_4) = (0.8, 0.2, 0.4)$, $d_2(u_5) = (0.8, 0.2, 0.4)$ and $td_2(u_1) = td_2(u_2) = td_2(u_3) = td_2(u_4) = td_2(u_5) = (1.7, 0.9, 1.2)$. Since each vertex has the same total d_2-degree, therefore G is totally $(2, (1.7, 0.9, 1.2))$-regular product 3-polar fuzzy graph. Although G is not $(2,\overline{k})$-regular.

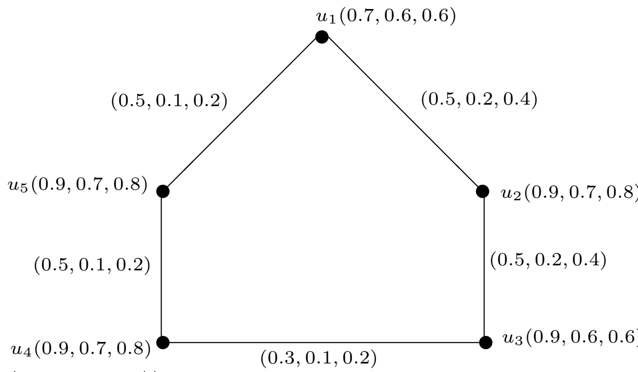

Figure 3: Totally $(2,(1.7,0.9,1.2))$-totally regular product 3-polar fuzzy graph G

Theorem 3.8. *Let $G = (V, A, B)$ be a product mFG. Then $A(u) = \overline{c} = (c_1, c_2, \cdots, c_m)$ for all $u \in V$ if and only the following are equivalent:*

(i) G is a $(2, \overline{k})$-regular product mFG,

(ii) G is a totally $(2, \overline{k} + \overline{c})$-regular product mFG.

Proof. Suppose that $A(u) = \overline{c} = (c_1, c_2, \cdots, c_m)$ for all $u \in V$. We will show that the statements (i) and (ii) are equivalent.

$(i) \Rightarrow (ii)$: Let G be a $(2, \overline{k})$-regular product mFG. Therefore, $d_2(u) = \overline{k}$ for all $u \in V$. Now, $td_2(u) = \overline{k} + \overline{c}$ for all $u \in V$. So, G is totally $(2, \overline{k} + \overline{c})$- regular.

$(ii) \Rightarrow (i)$: Now, suppose that G is totally $(2, \overline{k} + \overline{c})$-regular. Then $td_2(u) = \overline{k} + \overline{c}$ for all $u \in V$, i.e., $d_2(u) + A(u) = \overline{k} + \overline{c}$ for all $u \in V$, i.e., $d_2(u) = \overline{k}$ for all $u \in V$, i.e., G is $\overline{k}$-regular.

Conversely, let (i) and (ii) are equivalent. Let G be both totally $(2, \overline{k}+\overline{c})$-regular and $(2, \overline{k})$-regular. Then we have, $td_2(u) = \overline{k} + \overline{c}$ and $d_2(u) = \overline{k}$ for all $u \in V$, i.e., $d_2(u) + A(u) = \overline{k} + \overline{c}$ and $d_2(u) = \overline{k}$ for all $u \in V$. So, $A(u) = \overline{c}$ for all $u \in V$. Hence the result. □

Definition 3.9. *Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product mFGs. Then a bijective function $\psi : V_1 \to V_2$ is called an m-polar morphism or m-polar ψ-morphism if there exist positive real numbers k_1, k_2 such that for $i = 1, 2, \ldots, m$*

(i) $p_i \circ A_2(\psi(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$.

(ii) $p_i \circ B_2(\psi(u)\psi(v)) = k_2 p_i \circ B_1(uv)$ for all $uv \in \widetilde{V_1^2}$.

In this case, ψ is called (k_1, k_2) m-polar ψ-morphism from G_1 to G_2. If $k_1 = k_2 = k$, then we call ψ an m-polar k-morphism. When $k = 1$, we obtain usual m-polar morphism.

Note 3.10. *Let $G_1 = (V_1, A_1, B_1)$, $G_2 = (V_2, A_2, B_2)$ and $G_3 = (V_3, A_3, B_3)$ be three product mFGs of the graphs $G_1^* = (V_1, E_1)$, $G_2^* = (V_2, E_2)$ and $G_3^* = (V_3, E_3)$ respectively. A_1, A_2 and A_3 denote the membership function of the vertices in G_1, G_2, G_3 respectively; B_1, B_2, B_3 denote the membership function of the edges in G_1, G_2, G_3 respectively.*

Theorem 3.11. *The relation ψ-morphism is an equivalence relation in the collection of all product mFGs.*

Proof. Let $\mathcal{G}$ be the collection of all product mFGs. Define a relation '$\sim$' on $\mathcal{G} \times \mathcal{G}$ as follows: for $G_1, G_2 \in \mathcal{G}$, we say $G_1 \sim G_2$ if and only if there exist a (k_1, k_2) m-polar ψ-morphism from G_1 to G_2 for some non-zero k_1 and k_2.

We show that $\sim$ is an equivalence relation. First we see that $\sim$ is reflexive by simply taking the identity mapping from G_1 onto itself.

Let $G_1, G_2 \in \mathcal{G}$ and $G_1 \sim G_2$. Then there exists a (k_1, k_2) ψ-morphism from G_1 to G_2 for some non-zero k_1 and k_2. Therefore $p_i \circ A_2(\psi(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$ and $p_i \circ B_2(\psi(u)\psi(v)) = k_2 p_i \circ B_1(uv)$ for all $uv \in \widetilde{V_1^2}$. Now consider the function $\psi^{-1} : V_2 \to V_1$. Let $x, y \in V_2$. Since ψ is bijective, therefore there exist $u, v \in V_1$ such that $\psi(u) = x$ and $\psi(v) = y$. Then,

$$p_i \circ A_1(\psi^{-1}(x)) = p_i \circ A_1(u) = \frac{1}{k_1} p_i \circ A_2(\psi(u))$$

and

$$p_i \circ B_1(\psi^{-1}(x)\psi^{-1}(y)) = p_i \circ B_1(uv) = \frac{1}{k_2} p_i \circ B_2(\psi(u)\psi(v)) = \frac{1}{k_2} p_i \circ B_2(xy)$$

for $i = 1, 2, \ldots, m$. Thus, ψ^{-1} is a $(\frac{1}{k_1}, \frac{1}{k_2})$ m-polar morphism from G_2 to G_1. Hence $G_2 \sim G_1$. So, $\sim$ is symmetric.

Again let $G_1, G_2, G_3 \in \mathcal{G}$, $G_1 \sim G_2$ and $G_2 \sim G_3$. Then there exist a (k_1, k_2) m-polar ψ_1 morphism from G_1 to G_2 and a (k_3, k_4) m-polar ψ_2 morphism from G_2 to G_3 for some non-zero real numbers k_1, k_2, k_3 and k_4. Then, $p_i \circ A_2(\psi_1(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$, $p_i \circ B_2(\psi_1(u)\psi_1(v)) = k_2 p_i \circ B_1(uv)$ for all $uv \in \widetilde{V_1^2}$ and $p_i \circ A_3(\psi_2(u)) = k_3 p_i \circ A_2(u)$ for all $u \in V_2$ and $p_i \circ B_3(\psi_2(u)\psi_2(v)) = k_4 p_i \circ B_2(uv)$ for all $uv \in \widetilde{V_2^2}$, $i = 1, 2, \ldots, m$.

Let $\psi_3 = \psi_2 \circ \psi_1 : V_1 \to V_3$. Now,

$$p_i \circ A_3(\psi_3(u)) = p_i \circ A_3(\psi_2 \circ \psi_1(u)) = p_i \circ A_3(\psi_2(\psi_1(u))) = k_3 p_i \circ A_2(\psi_1(u)) = k_3 k_1 p_i \circ A_1(u)$$

and

$$\begin{aligned} & p_i \circ B_3(\psi_3(u)\psi_3(v)) = p_i \circ B_3(\psi_2 \circ \psi_1(u)\psi_2 \circ \psi_1(v)) = p_i \circ B_3(\psi_2(\psi_1(u))\psi_2(\psi_1(v))) \\ = \; & k_4 p_i \circ B_2(\psi_1(u)\psi_1(v)) = k_4 k_2 p_i \circ B_2(uv), \quad i = 1, 2, \ldots, m. \end{aligned}$$

Thus, ψ_3 is a $(k_3 k_1, k_4 k_2)$ m-polar morphism from G_1 to G_3. Therefore, $G_1 \sim G_3$ and hence $\sim$ is transitive. So, the relation $\sim$ is an equivalence relation in the collection of all mFGs. □

Theorem 3.12. *Let G_1 and G_2 be two product mFGs and ψ be a (k_1, k_2) m-polar fuzzy morphism from G_1 to G_2 for some non-zero k_1 and k_2. Then the image of strong edges in G_1 are also strong edges in G_2 if and only if $k_1 = k_2$.*

Proof. Let $u_1 v_1$ be a strong edge in G_1 and $k_1 = k_2$. Since ψ is a (k_1, k_2) m-polar fuzzy morphism from G_1 to G_2, therefore for $i = 1, 2, \ldots, m$

$$\begin{aligned} p_i \circ B_2(\psi(u_1)\psi(v_1)) &= k_2 p_i \circ B_1(u_1 v_1) \\ &= k_2 \{p_i \circ A_1(u_1) \times p_i \circ A_1(v_1)\} \\ &= k_2 p_i \circ A_1(u_1) \times k_2 p_i \circ A_1(v_1) \\ &= k_1 p_i \circ A_1(u_1) \times k_1 p_i \circ A_1(v_1) \\ &= p_i \circ A_2(u_1) \times p_i \circ A_2(v_1). \end{aligned}$$

So, the edge $\psi(u_1)\psi(v_1)$ in G_2 is strong.

Conversely, let $u_1 v_1$ be a strong edge in G_1 and its corresponding image $\psi(u_1)\psi(v_1)$ in G_2 is also strong. Then we have,

$$\begin{aligned} k_2 p_i \circ B_1(u_1 v_1) &= p_i \circ B_2(\psi(u_1)\psi(v_1)) \\ &= p_i \circ A_2(\psi(u_1)) \times p_i \circ A_2(\psi(v_1)) \\ &= k_1 p_i \circ A_1(u_1) \times k_1 p_i \circ A_1(v_1) \\ &= k_1 \{p_i \circ A_1(u_1) \times p_i \circ A_1(v_1)\} \\ &= k_1 p_i \circ B_1(u_1 v_1), \ i = 1, 2, \ldots, m. \end{aligned}$$

This implies that $k_1 = k_2$. Hence the result. □

Corollary 3.12.1. *Let G_1 and G_2 be two product mFGs and G_1 be a (k_1, k_2) m-polar fuzzy morphism to G_2. If G_1 is strong, then G_2 is strong if and only if $k_1 = k_2$.*

Theorem 3.13. *If the product mFG G_1 is co-weak isomorphic to the product mFG G_2 and G_1 is regular, then G_2 is regular also.*

Proof. Since G_1 is co-weak isomorphic to G_2, there exists a co-weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that $p_i \circ A_1(u) \leq p_i \circ A_2(\phi(u))$ and $p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $u, v \in V_1$, $i = 1, 2, \ldots, m$.

Since G_1 is regular, we have $d_{G_1}(u) = (c_1, c_2, \ldots, c_m)$ for all $u \in V_1$.

Now,

$$\begin{aligned} d^i_{G_2}(\phi(u)) &= \sum_{\substack{\phi(u)\neq\phi(v)\\ \phi(u)\phi(v)\in E_2}} p_i\circ B_2(\phi(u)\phi(v)) \\ &= \sum_{\substack{u\neq v\\ uv\in E_1}} p_i\circ B_1(uv) \\ &= c_i, \text{ for all } u\in V_1,\ i=1,2,\ldots,m. \end{aligned}$$

Hence, G_2 is regular. □

Remark 3.14. *If the product mFG G_1 is co-weak isomorphic to G_2 and G_1 is strong, then G_2 need not be strong.*

Theorem 3.15. *Let G_1 and G_2 be two product mFGs. If G_1 is weak isomorphic to G_2 and G_1 is strong, then G_2 is also strong.*

Proof. Since G_1 is weak isomorphic to G_2, there exists a weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that $p_i\circ A_1(u) = p_i\circ A_2(\phi(u))$ for all $u\in V_1$ and $p_i\circ B_1(uv) \le p_i\circ B_2(\phi(u)\phi(v))$ for all $uv\in \widetilde{V_1^2}$, $i=1,2,\ldots,m$.

As G_1 is strong, $p_i\circ B_1(uv) = p_i\circ A_1(u)\times p_i\circ A_1(v)$ for all $uv\in E_1$, $i=1,2,\ldots,m$. Now,

$$\begin{aligned} p_i\circ B_2(\phi(u)\phi(v)) &\ge p_i\circ B_1(uv) \\ &= p_i\circ A_1(u)\times p_i\circ A_1(v) \\ &= p_i\circ A_2(\phi(u))\times p_i\circ A_2(\phi(v)) \end{aligned}$$

and by definition of product mFG G_2

$$p_i\circ B_2(\phi(u)\phi(v)) \le p_i\circ A_2(\phi(u))\times p_i\circ A_2(\phi(v)) \text{ for } \phi(u)\phi(v)\in E_2, i=1,2,\ldots,m.$$

From the above, it follows that

$$p_i\circ B_2(\phi(u)\phi(v)) = p_i\circ A_2(\phi(u))\times p_i\circ A_2(\phi(v)) \text{ for } \phi(u)\phi(v)\in E_2, i=1,2,\ldots,m.$$

Hence, G_2 is strong. □

Corollary 3.15.1. *Let G_1 and G_2 be two product mFGs. If G_1 is weak isomorphic to G_2 and G_1 is regular, then G_2 need not be regular.*

Theorem 3.16. *If the product mFG G_1 is co-weak isomorphic with a strong regular product mFG G_2, then G_1 is strong regular product mFG.*

Proof. Since G_1 is co-weak isomorphic to G_2 there exists a co-weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that $p_i\circ A_1(u) \le p_i\circ A_2(\phi(u))$ for all $u\in V_1$ and $p_i\circ B_1(uv) = p_i\circ B_2(\phi(u)\phi(v))$ for all $uv\in \widetilde{V_1^2}$, $i=1,2,\ldots,m$. Now,

$$\begin{aligned} p_i\circ B_1(uv) &= p_i\circ B_2(\phi(u)\phi(v)) \\ &= p_i\circ A_2(\phi(u))\times p_i\circ A_2(\phi(v)) \text{ (Since } G_2 \text{ is strong)} \\ &\ge p_i\circ A_1(u)\times p_i\circ A_1(v). \end{aligned}$$

But by definition of mFG G_1,

$$p_i\circ B_1(uv) \le p_i\circ A_1(u)\times p_i\circ A_1(v) \text{ for all } uv\in \widetilde{V_1^2}.$$

So, from the above we have $p_i\circ B_1(uv) = p_i\circ A_1(u)\times p_i\circ A_1(v)$ for all $uv\in E_1$, $i=1,2,\ldots,m$. Hence, G_1 is strong.

Also, for $i = 1, 2, \ldots, m$ and $u \in V_1$,

$$\begin{aligned}\sum_{\substack{u \neq v \\ uv \in E_1}} p_i \circ B_1(uv) &= \sum_{\substack{\phi(u) \neq \phi(v) \\ \phi(u)\phi(v) \in E_2}} p_i \circ B_2(\phi(u)\phi(v)) \\ &= \text{constant (Since } G_2 \text{ is regular).}\end{aligned}$$

Hence, G_1 is regular. □

Theorem 3.17. *Let G_1 and G_2 be two isomorphic product mFGs. Then G_1 is strong regular if and only if G_2 is strong regular.*

Proof. As G_1 is isomorphic to G_2, there exists an isomorphism $\phi : V_1 \to V_2$ which is bijective and satisfies the relation $p_i \circ A_1(u) = p_i \circ A_2(\phi(u))$ for all $u \in V_1$ and $p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v))$ for all $uv \in \widetilde{V_1^2}$, $i = 1, 2, \ldots, m$. Now,

$$\begin{aligned} G_1 \text{ is strong} &\Leftrightarrow p_i \circ B_1(uv) = p_i \circ A_1(u) \times p_i \circ A_1(v) \text{for all} uv \in E_1, i = 1, 2, \ldots, m \\ &\Leftrightarrow p_i \circ B_2(\phi(u)\phi(v)) = p_i \circ A_2(\phi(u)) \times p_i \circ A_2(\phi(v)) \text{for all} \phi(u)\phi(v) \in E_2, i = 1, 2, \ldots, m \\ &\Leftrightarrow G_2 \text{ is strong.}\end{aligned}$$

Again,

$$\begin{aligned} G_1 \text{ is regular} &\Leftrightarrow \sum_{\substack{u \neq v \\ uv \in E_1}} p_i \circ B_1(uv) = \text{constant} \text{for all} u \in V_1 \\ &\Leftrightarrow \sum_{\substack{\phi(u) \neq \phi(v) \\ \phi(u)\phi(v) \in E_2}} p_i \circ B_2(\phi(u)\phi(v)) = \text{constant} \text{for all} \phi(u) \in V_2 \\ &\Leftrightarrow G_2 \text{ is regular.}\end{aligned}$$

□

Theorem 3.18. *A strong mFG G is strong regular if and only if its complement $\overline{G}$ is strong regular.*

Proof. From [7], we have if $G = (V, A, B)$ is a strong product mFG, then $\overline{G} = (V, \overline{A}, \overline{B})$ is also a strong product mFG where $\overline{A} = A$ and $\overline{B}$ is defined by

$$p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy)$$

for all $xy \in \widetilde{V^2}$, $i = 1, 2, \ldots, m$. Now,

$$\begin{aligned} & G \text{ is strong regular} \\ \Leftrightarrow\ & p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y) \\ \Leftrightarrow\ & p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy) = p_i \circ B(xy) - p_i \circ B(xy) = 0 \\ \Leftrightarrow\ & \sum p_i \circ \overline{B}(xy) = 0 \\ \Leftrightarrow\ & \overline{G} \text{ is strong regular.}\end{aligned}$$

□

4 Modeling of Products Design in a Company as Product mFG

Here, we model a real life situation of a company where a group of people decides which product design to manufacture. This type of network is an ideal example of product mFGs. It is very important for a company to decide which product design to manufacture so that they can make profit as much as possible. A very good product design is gladly acceptable to the peoples if it is also cheap in price. The determination of which product design to manufacture is called the decision making problem. By taking the very good decision (very good product design), one company can spread their product all over the world keeping in mind that the product design is very good, demandable, cheap, easily accessible etc. Before manufacturing a product design, engineers and manufacturers test several important things in a product. Suppose a company has to decide to manufacture a product design among five products, say D_1, D_2, D_3, D_4 and D_5. A product design is manufactured by a company keeping in mind its market demand, price, time taken to manufacture and accessibility.

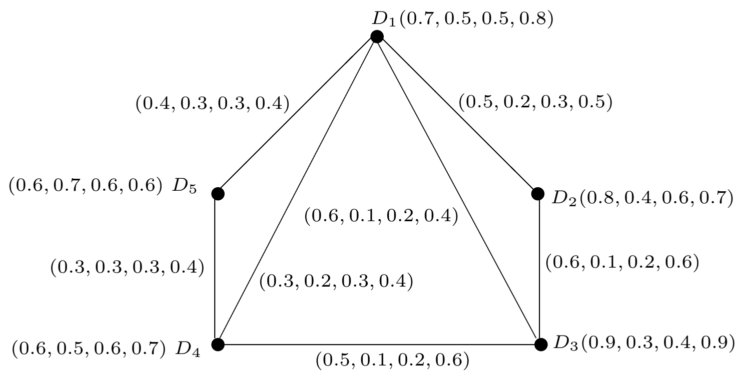

Figure 4: Modeling of products design in a company as a product 4-polar fuzzy graph G

We consider the above as a set, say $M = \{\text{demand, price, time, accessibility}\}$ and the set of product designs as $D = \{D_1, D_2, D_3, D_4, D_5\}$. Since all the above characteristics of a product design according to the different company are uncertain in real life, therefore we consider a 4-polar fuzzy subset A of the set D. This situation can be represented as a product 4-polar fuzzy graph by considering the different product design as the nodes and edges between them represent the relationship between two product designs (see Fig. 4). The membership value of each node represents the degree of demand, price, time taken to manufacture and accessibility to people in global market. Edge membership values which represent the relationship between the product design can be calculated by using the relation

$$p_i \circ B(uv) \leq p_i \circ A(u) \times p_i \circ A(v)$$

for all $u, v \in D, i = 1, 2, \ldots, 4$.

The edge between two product designs represents the degree of using common power equipments, raw materials, engineer employs and agencies involved for both products.

From the Fig. 4, we see that $G = (D, A, B)$ is a product 4-polar fuzzy graph and the product design D_3 has maximum demand, minimum price, minimum time to manufacture and has maximum accessibility compared to all others product designs.

5 Conclusions

The fuzzy graph theory is one of the most developing area of research, which has a variety of applications in different fields including computer science, communication networks, biological sciences, social networks, decision-making and optimization problems. A product mFG can be used to represent real world problems which involve multi-case of information and uncertainty. No other graphs can be used to model such types of problems. This is how m-polar fuzzy graph is a generalized version of all other graphs. Actually, if $m = 1$, then the 1-polar fuzzy graph becomes our usual fuzzy graph. A product mFG plays a vital role in many research domains and gives more precision, flexibility and comparability to the system as compared to the fuzzy and bipolar fuzzy models. In this paper, we have introduced the notion of m-polar ψ-morphism on product mFGs. The action of m-polar ψ-morphism on product mFGs is studied and established some results on weak and co-weak isomorphism. d_2-degree and total d_2-degree of a vertex in product mFGs are defined and studied $(2, \overline{k})$-regularity and totally $(2, \overline{k})$- regularity. A real life situation has been modeled as an application of product mFG.

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