Results of \textit{m}-polar Fuzzy Graphs with Application

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Abstract

An \textit{m}-polar fuzzy model is useful for multi-polar information, multi-agent, multi-attribute and multi-object network models which gives more precision, flexibility, and comparability to the system as compared to the classical, fuzzy and bipolar fuzzy models. In this paper, \textit{m}-polar fuzzy sets are used to introduce the notion of \textit{m}-polar psi-morphism on product \textit{m}-polar fuzzy graph (mFG). The action of this morphism is studied and established some results on weak and co-weak isomorphism. $d_2$-degree and total $d_2$-degree of a vertex in product mFG are defined and studied their properties. A real life situation has been modeled as an application of product mFG.

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1 Introduction

After the introduction of fuzzy sets by Zadeh \cite{30} in 1965, fuzzy set theory is included as large research fields. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences engineering, statistic, graph theory, artificial intelligence, signal processing, multi agent systems, decision making and automata theory. In a fuzzy set, each element is associated with a membership value selected from the interval $[0,1]$. Instead of using particular membership value as in fuzzy sets, \textit{m}-polar fuzzy set can be used to represent the vagueness of a set more perfectly. In 2014, Chen et al. \cite{6} introduced the notion of $m$-polar fuzzy set as a generalization of fuzzy set theory. The membership values in $m$-polar fuzzy sets is more expressive in capturing uncertainty of data.

The idea behind this is that “multipolar information” exists because data of real world problems are sometimes come from multiple agents. $m$-polar fuzzy sets allow more graphical representation of vague data, which facilitates significantly better analysis in data relationships, incompleteness, and similarity measures.

Graph theory besides being a well developed branch of Mathematics, it is an important tool for mathematical modeling. Realizing the importance, Rosenfeld \cite{22} introduced the concept of fuzzy graphs, Mordeson and Nair \cite{15} discussed about the properties of fuzzy graphs and hypergraphs. After that, the operation of union, join, Cartesian product and composition on two fuzzy graphs was defined by Mordeson and Peng \cite{16}. Sunitha and Kumar \cite{27} further studied the other properties of fuzzy graphs. The concept of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in \cite{4}. After that several researchers are working on fuzzy graphs like in \cite{3, 5, 13, 14, 17, 18, 23, 25, 26}.

In 2011, using the concepts of bipolar fuzzy sets, Akram \cite{1} introduced the bipolar fuzzy graphs and defined different operations on it. Rashmanlou et al. \cite{19, 20, 21} studied bipolar fuzzy graphs, bipolar fuzzy graphs with categorical properties, product of bipolar fuzzy graphs and their degrees, etc. Some work on bipolar fuzzy graphs may be found on \cite{12, 24, 25, 29}.

Chen et al. \cite{8} first introduced the concept of mFGs. Ghorai and Pal studied many properties of generalized mFGs \cite{7}, defined operations, density of mFGs \cite{7}, introduced the concept of \textit{m}-polar fuzzy planar graphs \cite{10, 11} and studied isomorphic properties of them \cite{9}.

This paper is organized in the following manner. In Section \textsuperscript{1} introduction is given and the literature review is illustrated. Section \textsuperscript{2} represents a brief study of some graph theoretic concept used in this paper.

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In Section 3, the notion of m-polar $\psi$-morphism is introduced on product mFG as a generalization of our usual homomorphism. The action of this morphism is studied and established some results on weak and co-weak isomorphism. $d_2$-degree and total $d_2$-degree of a vertex in product mFGs are defined and studied their properties. In Section 4, a real life situation have been modeled. Section 5 represents the conclusion of the paper.

2 Preliminaries

A graph is an ordered pair $G^* = (V, E)$, where $V$ is the set of vertices of $G^*$ and $E$ is the set of all edges of $G^*$. Two vertices $x$ and $y$ in an undirected graph $G^*$ are said to be adjacent in $G^*$ if $xy$ is an edge of $G^*$. A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices.

A subgraph of a graph $G^* = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. The degree of a vertex in $G$ is the number of edges incident with the vertex.

A fuzzy graph with $V$ as the underlying set is a triplet $G = (V, \sigma, \mu)$, where $\sigma : V \rightarrow [0, 1]$ is a fuzzy subset of $V$ and $\mu : V \times V \rightarrow [0, 1]$ is fuzzy relation on $\sigma$ such that $\mu(x, y) \leq \sigma(x) \land \sigma(y)$ for all $x, y \in V$.

Throughout the paper, $G^*$ represents a crisp graph and $G = (V, A, B)$ represents a product mFG of $G^*$.

Definition 2.1. [6] An m-polar fuzzy set (or a $[0,1]^m$-set) on $X$ is a mapping $A : X \rightarrow [0,1]^m$. The set of all m-polar fuzzy sets on $X$ is denoted by $m(X)$.

Definition 2.2. [7] A product m-polar fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (V, A, B)$ where $A : V \rightarrow [0,1]^m$ is an m-polar fuzzy set in $V$ and $B : V^2 \rightarrow [0,1]^m$ is an m-polar fuzzy set in $V^2$ such that $p_i \circ B(xy) \leq p_i \circ A(x) \times p_i \circ A(y)$ for all $xy \in V^2$, $i = 1, 2, \ldots, m$ and $B(xy) = 0$ for all $xy \in V^2 - E$, $(0 = (0,0,\ldots,0)$ is the smallest element in $[0,1]^m$).

Definition 2.3. [8] $G$ is called strong if $p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y)$ for all $xy \in E$, $i = 1, 2, \ldots, m$.

$G$ is called complete if $p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y)$ for all $x, y \in V$, $i = 1, 2, \ldots, m$.

The complement of $G$ is a product mFG $\overline{G} = (V, \overline{A}, \overline{B})$ where $\overline{A} = A$ and $\overline{B}$ is defined by $p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy)$, $xy \in V^2$ and $i = 1, 2, \ldots, m$.

Definition 2.4. [8] Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product mFGs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively.

A weak isomorphism between $G_1$ and $G_2$ is a bijective mapping $\phi : V_1 \rightarrow V_2$ such that $\phi$ is a homomorphism and $p_i \circ A_1(x_1) = p_i \circ A_2(\phi(x_1))$ for all $x_1 \in V_1$ for $i = 1, 2, \ldots, m$.

A co-weak isomorphism between $G_1$ and $G_2$ is a bijective mapping $\phi : V_1 \rightarrow V_2$ such that $\phi$ is a homomorphism and $p_i \circ B_1(x_1y_1) = p_i \circ B_2(\phi(x_1y_1))$ for all $x_1y_1 \in V_1^2$ for $i = 1, 2, \ldots, m$.

Definition 2.5. Let $G = (V, A, B)$ be a product m-polar fuzzy graph of $G^* = (V, E)$.

(i) The neighborhood degree of a vertex $v$ is defined as $d_N(v) = (d_N^1(v), d_N^2(v), \ldots, d_N^m(v))$ where $d_N^i(v) = \sum_{u \in N(v)} p_i \circ A(u)$, $i = 1, 2, \ldots, m$.

(ii) The degree of a vertex $v$ in $G$ is defined by $d_G(v) = (d_G^1(v), d_G^2(v), \ldots, d_G^m(v))$, where $d_G^i(v) = \sum_{u \neq v} p_i \circ B(uv)$, $i = 1, 2, \ldots, m$. If all the vertices of $G$ have same degree, then $G$ is called regular product mFG.

(iii) The closed degree of a vertex $v$ is defined by $d_C[v] = (d_C^1[v], d_C^2[v], \ldots, d_C^m[v])$, where $d_C^i(v) = d_G^i(v) + p_i \circ A(v)$, $i = 1, 2, \ldots, m$. If each vertex of $G$ has same closed degree, then $G$ is called totally regular product mFG.
\section{Regularity and Isomorphism on mFGs}

Regular graphs are the most widely studied classes. For example, regular fuzzy graphs play a key role in designing reliable communication networks. Here, the notion of $m$-polar $\psi$-morphism is introduced in product mFG. Also, $d_2$-degree, total $d_2$-degree, $(2, k)$-regularity and totally $(2, k)$-regularity are defined in product mFG and studied some important properties of them.

\begin{definition}
Let $G$ be a product mFG. The $d_2$- degree of a vertex $u$ in $G$ is $d_2(u) = (d_2^1(u), d_2^2(u), \ldots, d_2^m(u))$ where $d_2^i(u) = \sum p_i \circ B^2(uv)$ is such that $p_i \circ B^2(uv) = \sup\{p_i \circ B(uwv) : u, v \in V\}$. The minimum $d_2$-degree of $G$ is denoted as $d_2(G) = (d_2^1(G), d_2^2(G), \ldots, d_2^m(G))$ where $d_2^i(G) = \min\{d_2^i(u) : u \in V\}$. The maximum $d_2$-degree of $G$ is denoted as $\Delta_{d_2}(G) = (\Delta_{d_2}^1(G), \Delta_{d_2}^2(G), \ldots, \Delta_{d_2}^m(G))$ where $\Delta_{d_2}^i(G) = \max\{d_2^i(u) : u \in V\}$.
\end{definition}

\begin{example}
Let $G$ be a product 3-polar fuzzy graph where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$ (see Fig. \ref{fig:3polarfg}). By routine computations we have
\begin{align*}
d_2^1(u_1) &= (0.3 \land 0.5) + (0.3 \land 0.4) = 0.6, \quad d_2^2(u_1) = (0.4 \land 0.4) + (0.4 \land 0.2) = 0.6, \\
d_2^1(u_2) &= (0.2 \land 0.1) + (0.3 \land 0.2) = 0.3, \quad d_2^2(u_2) = (0.5 \land 0.4) + (0.3 \land 0.3) = 0.7, \\
d_2^1(u_3) &= (0.4 \land 0.2) + (0.4 \land 0.4) = 0.6, \quad d_2^2(u_3) = (0.1 \land 0.1) + (0.2 \land 0.3) = 0.3, \\
d_2^1(u_4) &= (0.4 \land 0.4) + (0.5 \land 0.3) = 0.7, \quad d_2^2(u_4) = (0.2 \land 0.2) + (0.4 \land 0.4) = 0.6, \\
d_2^1(u_5) &= (0.1 \land 0.2) + (0.1 \land 0.2) = 0.2.
\end{align*}
Hence, $d_2(u_1) = (0.6, 0.6, 0.3)$, $d_2(u_2) = (0.7, 0.6, 0.3)$, $d_2(u_3) = (0.7, 0.6, 0.2)$.
\end{example}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3polarfg.png}
\caption{Product 3-polar fuzzy graph $G$}
\end{figure}

\begin{definition}
If $d_2(u) = \overline{k}$ for all $u \in V$ then $G$ is said to be $(2, \overline{k})$-regular product mFG.
\end{definition}

\begin{example}
Consider the product 3-polar fuzzy graph as in Fig. \ref{fig:2kregular}. Here, $d_2(u_1) = d_2(u_2) = d_2(u_3) = d_2(u_4) = (0.2, 0.2, 0.2)$. So, $G$ is $(2, (0.2, 0.2, 0.2))$-regular product 3-polar fuzzy graph.
\end{example}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2kregular.png}
\caption{$(2, (0.2, 0.2, 0.2))$-regular product 3-polar fuzzy graph $G$}
\end{figure}

\begin{definition}
The total $d_2$- degree of a vertex $u \in V$ is defined as $td_2(u) = (td_2^1(u), td_2^2(u), \ldots, td_2^m(u))$, where $td_2^i(u) = \sum p_i \circ B^2(uv) + p_i \circ A(u)$, $i = 1, 2, \ldots, m$.
\end{definition}

\begin{note}
If each vertex of $G$ has the same total $d_2$-degree $\overline{l}$, then $G$ is said to be totally $(2, \overline{l})$-regular product mFG.
\end{note}

\begin{example}
Consider the product 3-polar fuzzy graph $G$ with the vertex set $V = \{u_1, u_2, u_3, u_4, u_5\}$ and edge set $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$ (see Fig. \ref{fig:2klregular}).
We see that $d_2(u_1) = (1.0, 0.3, 0.6)$, $d_2(u_2) = (0.8, 0.2, 0.4)$, $d_2(u_3) = (0.8, 0.3, 0.6)$, $d_2(u_4) = (0.8, 0.2, 0.4)$, $d_2(u_5) = (0.8, 0.2, 0.4)$ and $td_2(u_1) = td_2(u_2) = td_2(u_3) = td_2(u_4) = td_2(u_5) = (1.7, 0.9, 1.2)$. Since each vertex has the same total $d_2$-degree, therefore $G$ is totally $(2, (1.7, 0.9, 1.2))$-regular product 3-polar fuzzy graph. Although $G$ is not $(2, \overline{k})$-regular.
\end{example}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2klregular.png}
\caption{totally $(2, (1.7, 0.9, 1.2))$-regular product 3-polar fuzzy graph $G$}
\end{figure}
Theorem 3.8. Let \( G = (V, A, B) \) be a product mFG. Then \( A(u) = \bar{c} = (c_1, c_2, \cdots, c_m) \) for all \( u \in V \) if and only the following are equivalent:

(i) \( G \) is a \((2, \bar{k})\)-regular product mFG,

(ii) \( G \) is a totally \((2, \bar{k} + \bar{\tau})\)-regular product mFG.

Proof. Suppose that \( A(u) = \bar{c} = (c_1, c_2, \cdots, c_m) \) for all \( u \in V \). We will show that the statements (i) and (ii) are equivalent.

(i) \( \Rightarrow \) (ii) : Let \( G \) be a \((2, \bar{k})\)-regular product mFG. Therefore, \( d_2(u) = \bar{k} \) for all \( u \in V \). Now, \( td_2(u) = \bar{k} + \bar{\tau} \) for all \( u \in V \). So, \( G \) is totally \((2, \bar{k} + \bar{\tau})\)-regular.

(ii) \( \Rightarrow \) (i) : Now, suppose that \( G \) is totally \((2, \bar{k} + \bar{\tau})\)-regular. Then \( td_2(u) = \bar{k} + \bar{\tau} \) for all \( u \in V \), i.e., \( d_2(u) + A(u) = \bar{k} + \bar{\tau} \) for all \( u \in V \), i.e., \( d_2(u) = \bar{k} \) for all \( u \in V \). So, \( G(u) = \bar{c} \) for all \( u \in V \). Hence the result.

Definition 3.9. Let \( G_1 = (V_1, A_1, B_1) \) and \( G_2 = (V_2, A_2, B_2) \) be two product mFGs. Then a bijective function \( \psi : V_1 \rightarrow V_2 \) is called an m-polar morphism or m-polar \( \psi \)-morphism if there exist positive real numbers \( k_1, k_2 \) such that for \( i = 1, 2, \ldots, m \)

(i) \( p_i \circ A_2(\psi(u)) = k_1p_i \circ A_1(u) \) for all \( u \in V_1 \).

(ii) \( p_i \circ B_2(\psi(u)(v)) = k_2p_i \circ B_1(uv) \) for all \( uv \in \bar{V}_2^2 \).

In this case, \( \psi \) is called the \((k_1, k_2)\) m-polar \( \psi \)-morphism from \( G_1 \) to \( G_2 \). If \( k_1 = k_2 = k \), then we call \( \psi \) an \( m \)-polar \( k \)-morphism. When \( k = 1 \), we obtain usual m-polar morphism.

Note 3.10. Let \( G_1 = (V_1, A_1, B_1) \), \( G_2 = (V_2, A_2, B_2) \) and \( G_3 = (V_3, A_3, B_3) \) be three product mFGs of the graphs \( G_1^1 = (V_1, E_1) \), \( G_2^2 = (V_2, E_2) \) and \( G_3^3 = (V_3, E_3) \) respectively. \( A_1, A_2 \) and \( A_3 \) denote the membership function of the vertices in \( G_1, G_2, G_3 \) respectively; \( B_1, B_2, B_3 \) denote the membership function of the edges in \( G_1, G_2, G_3 \) respectively.

Theorem 3.11. The relation \( \psi \)-morphism is an equivalence relation in the collection of all product mFGs.

Proof. Let \( \mathcal{G} \) be the collection of all product mFGs. Define a relation \( \sim \) on \( \mathcal{G} \times \mathcal{G} \) as follows: for \( G_1, G_2 \in \mathcal{G} \), we say \( G_1 \sim G_2 \) if and only if there exist a \((k_1, k_2)\) m-polar \( \psi \)-morphism from \( G_1 \) to \( G_2 \) for some non-zero \( k_1 \) and \( k_2 \).

We show that \( \sim \) is an equivalence relation. First we see that \( \sim \) is reflexive by simply taking the identity mapping from \( G_1 \) onto itself.

Let \( G_1, G_2 \in \mathcal{G} \) and \( G_1 \sim G_2 \). Then there exists a \((k_1, k_2)\) \( \psi \)-morphism from \( G_1 \) to \( G_2 \) for some non-zero \( k_1 \) and \( k_2 \). Therefore \( p_i \circ A_2(\psi(u)) = k_1p_i \circ A_1(u) \) for all \( u \in V_1 \) and \( p_i \circ B_2(\psi(u)(v)) = k_2p_i \circ B_1(uv) \) for all \( uv \in \bar{V}_2^2 \). Now consider the function \( \psi^{-1} : V_2 \rightarrow V_1 \). Let \( x, y \in V_2 \). Since \( \psi \) is bijective, therefore there exist \( u, v \in V_1 \) such that \( \psi(u) = x \) and \( \psi(v) = y \). Then,

\[ p_i \circ A_1(\psi^{-1}(x)) = p_i \circ A_1(u) = \frac{1}{k_1}k_1p_i \circ A_2(\psi(u)) \]

Figure 3: Totally \((2, (1.7, 0.9, 1.2))\)-totally regular product 3-polar fuzzy graph \( G \)
and 
\[ p_i \circ B_1(\psi^{-1}(x)\psi^{-1}(y)) = p_i \circ B_1(uv) = \frac{1}{k_2} p_i \circ B_2(\psi(u)\psi(v)) = \frac{1}{k_2} p_i \circ B_2(xy) \]
for \( i = 1, 2, \ldots, m \). Thus, \( \psi^{-1} \) is a \( \left( \frac{1}{k_1}, \frac{1}{k_2} \right) \)-m-polar morphism from \( G_2 \) to \( G_1 \). Hence \( G_2 \sim G_1 \). So, \( \sim \) is symmetric.

Again let \( G_1, G_2, G_3 \in \mathcal{G} \), \( G_1 \sim G_2 \) and \( G_2 \sim G_3 \). Then there exist a \( (k_1, k_2) \)-m-polar \( \psi_1 \) morphism from \( G_1 \) to \( G_2 \) and a \( (k_3, k_4) \)-m-polar \( \psi_2 \) morphism from \( G_2 \) to \( G_3 \) for some non-zero real numbers \( k_1, k_2, k_3 \) and \( k_4 \). Then, \( p_i \circ A_2(\psi_1(u)) = k_1 p_i \circ A_1(u) \) for all \( u \in V_1 \), \( p_i \circ B_2(\psi_1(u)\psi_1(v)) = k_2 p_i \circ B_1(uv) \) for all \( uv \in V_1^2 \), and \( p_i \circ A_3(\psi_2(u)) = k_3 p_i \circ A_2(u) \) for all \( u \in V_2 \) and \( p_i \circ B_3(\psi_2(u)\psi_2(v)) = k_4 p_i \circ B_2(uv) \) for all \( uv \in V_2^2 \), \( i = 1, 2, \ldots, m \).

Let \( \psi_3 = \psi_2 \circ \psi_1 : V_1 \to V_3 \). Now,
\[ p_i \circ A_3(\psi_3(u)) = p_i \circ A_3(\psi_2(\psi_1(u))) = k_3 p_i \circ A_2(\psi_1(u)) = k_3 k_1 p_i \circ A_1(u) \]
and
\[ p_i \circ B_3(\psi_3(u)\psi_3(v)) = p_i \circ B_3(\psi_2(\psi_1(u)\psi_2(\psi_1(v))) = p_i \circ B_3(\psi_2(\psi_1(u))\psi_2(\psi_1(v))) = k_4 k_2 p_i \circ B_2(uv), \quad i = 1, 2, \ldots, m. \]
Thus, \( \psi_3 \) is a \( (k_3 k_1, k_3 k_2) \)-m-polar morphism from \( G_1 \) to \( G_3 \). Therefore, \( G_1 \sim G_3 \) and hence \( \sim \) is transitive. So, the relation \( \sim \) is an equivalence relation in the collection of all mFGs.

**Theorem 3.12.** Let \( G_1 \) and \( G_2 \) be two product mFGs and \( \psi \) be a \( (k_1, k_2) \)-m-polar fuzzy morphism from \( G_1 \) to \( G_2 \) for some non-zero \( k_1 \) and \( k_2 \). Then the image of strong edges in \( G_1 \) are also strong edges in \( G_2 \) if and only if \( k_1 = k_2 \).

**Proof.** Let \( u_1 v_1 \) be a strong edge in \( G_1 \) and \( k_1 = k_2 \). Since \( \psi \) is a \( (k_1, k_2) \)-m-polar fuzzy morphism from \( G_1 \) to \( G_2 \), therefore for \( i = 1, 2, \ldots, m \)
\[ p_i \circ B_2(\psi(u_1)\psi(v_1)) = k_2 p_i \circ B_1(u_1 v_1) \]
\[ = k_2 \{ p_i \circ A_1(u_1) \times p_i \circ A_1(v_1) \} \]
\[ = k_2 p_i \circ A_1(u_1) \times k_2 p_i \circ A_1(v_1) \]
\[ = k_1 p_i \circ A_1(u_1) \times k_1 p_i \circ A_1(v_1) \]
\[ = p_i \circ A_2(u_1) \times p_i \circ A_2(v_1). \]
So, the edge \( \psi(u_1)\psi(v_1) \) in \( G_2 \) is strong.

Conversely, let \( u_1 v_1 \) be a strong edge in \( G_1 \) and its corresponding image \( \psi(u_1)\psi(v_1) \) in \( G_2 \) is also strong. Then we have,
\[ k_2 p_i \circ B_1(u_1 v_1) = p_i \circ B_2(\psi(u_1)\psi(v_1)) \]
\[ = p_i \circ A_2(\psi(u_1)) \times p_i \circ A_2(\psi(v_1)) \]
\[ = k_2 p_i \circ A_1(u_1) \times k_1 p_i \circ A_1(v_1) \]
\[ = k_1 \{ p_i \circ A_1(u_1) \times p_i \circ A_1(v_1) \} \]
\[ = k_1 p_i \circ B_1(u_1 v_1), \quad i = 1, 2, \ldots, m. \]
Thus implies that \( k_1 = k_2 \). Hence the result.

**Corollary 3.12.1.** Let \( G_1 \) and \( G_2 \) be two product mFGs and \( G_1 \) be a \( (k_1, k_2) \)-m-polar fuzzy morphism to \( G_2 \). If \( G_1 \) is strong, then \( G_2 \) is strong if and only if \( k_1 = k_2 \).

**Theorem 3.13.** If the product mFG \( G_1 \) is co-weak isomorphic to the product mFG \( G_2 \) and \( G_1 \) is regular, then \( G_2 \) is regular also.

**Proof.** Since \( G_1 \) is co-weak isomorphic to \( G_2 \), there exists a co-weak isomorphism \( \phi : V_1 \to V_2 \) which is bijective such that \( p_i \circ A_1(u) \leq p_i \circ A_2(\phi(u)) \) and \( p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v)) \) for all \( u, v \in V_1, i = 1, 2, \ldots, m \).

Since \( G_1 \) is regular, we have \( d_{G_1}(u) = (c_1, c_2, \ldots, c_m) \) for all \( u \in V_1 \).
Hence, \( G_2 \) is regular. \( \square \)

**Remark 3.14.** If the product mFG \( G_1 \) is co-weak isomorphic to \( G_2 \) and \( G_1 \) is strong, then \( G_2 \) need not be strong.

**Theorem 3.15.** Let \( G_1 \) and \( G_2 \) be two product mFGs. If \( G_1 \) is weak isomorphic to \( G_2 \) and \( G_1 \) is strong, then \( G_2 \) is also strong.

**Proof.** Since \( G_1 \) is weak isomorphic to \( G_2 \), there exists a weak isomorphism \( \phi : V_1 \to V_2 \) which is bijective such that \( p_i \circ A_1(u) = p_i \circ A_2(\phi(u)) \) for all \( u \in V_1 \) and \( p_i \circ B_1(uv) \leq p_i \circ B_2(\phi(u)\phi(v)) \) for all \( uv \in \tilde{V}_1^2 \), \( i = 1, 2, \ldots, m \).

As \( G_1 \) is strong, \( p_i \circ B_1(uv) = p_i \circ A_1(u) \times p_i \circ A_1(v) \) for all \( uv \in E_1 \), \( i = 1, 2, \ldots, m \). Now,

\[
p_i \circ B_2(\phi(u)\phi(v)) \geq p_i \circ B_1(uv) = p_i \circ A_1(u) \times p_i \circ A_1(v) = p_i \circ A_2(\phi(u)) \times p_i \circ A_2(\phi(v))
\]

and by definition of product mFG \( G_2 \)

\[
p_i \circ B_2(\phi(u)\phi(v)) \leq p_i \circ A_2(\phi(u)) \times p_i \circ A_2(\phi(v)) \text{ for } \phi(u)\phi(v) \in E_2, i = 1, 2, \ldots, m.
\]

From the above, it follows that

\[
p_i \circ B_2(\phi(u)\phi(v)) = p_i \circ A_2(\phi(u)) \times p_i \circ A_2(\phi(v)) \text{ for } \phi(u)\phi(v) \in E_2, i = 1, 2, \ldots, m.
\]

Hence, \( G_2 \) is strong. \( \square \)

**Corollary 3.15.1.** Let \( G_1 \) and \( G_2 \) be two product mFGs. If \( G_1 \) is weak isomorphic to \( G_2 \) and \( G_1 \) is regular, then \( G_2 \) need not be regular.

**Theorem 3.16.** If the product mFG \( G_1 \) is co-weak isomorphic with a strong regular product mFG \( G_2 \), then \( G_1 \) is strong regular product mFG.

**Proof.** Since \( G_1 \) is co-weak isomorphic to \( G_2 \) there exists a co-weak isomorphism \( \phi : V_1 \to V_2 \) which is bijective such that \( p_i \circ A_1(u) \leq p_i \circ A_2(\phi(u)) \) for all \( u \in V_1 \) and \( p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v)) \) for all \( uv \in \tilde{V}_1^2 \), \( i = 1, 2, \ldots, m \). Now,

\[
p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v)) = p_i \circ A_2(\phi(u)) \times p_i \circ A_2(\phi(v)) \text{ (Since } G_2 \text{ is strong)}
\]

\[
\geq p_i \circ A_1(u) \times p_i \circ A_1(v).
\]

But by definition of mFG \( G_1 \),

\[
p_i \circ B_1(uv) \leq p_i \circ A_1(u) \times p_i \circ A_1(v) \text{ for all } uv \in \tilde{V}_1^2.
\]

So, from the above we have \( p_i \circ B_1(uv) = p_i \circ A_1(u) \times p_i \circ A_1(v) \) for all \( uv \in E_1 \), \( i = 1, 2, \ldots, m \). Hence, \( G_1 \) is strong.
Also, for \( i = 1, 2, \ldots, m \) and \( u \in V_1 \),

\[
\sum_{u \neq v, \ uv \in E_1} p_i \circ B_1(uv) = \sum_{\phi(u) \neq \phi(v)} p_i \circ B_2(\phi(u)\phi(v)) = \text{constant (Since } G_2 \text{ is regular).}
\]

Hence, \( G_1 \) is regular.

\[\text{Theorem 3.17. Let } G_1 \text{ and } G_2 \text{ be two isomorphic product mFGs. Then } G_1 \text{ is strong regular if and only if } G_2 \text{ is strong regular.}\]

\[\begin{align*}
\text{Proof. As } G_1 \text{ is isomorphic to } G_2, \text{ there exists an isomorphism } \phi : V_1 & \to V_2 \text{ which is bijective and satisfies} \\
\text{the relation } p_i \circ A_1(u) = p_i \circ A_2(\phi(u)) \text{ for all } u \in V_1 \text{ and } p_i \circ B_1(uv) = p_i \circ B_2(\phi(u)\phi(v)) \text{ for all } uv \in E_1, i = 1, 2, \ldots, m. \text{ Now,} \\
G_1 \text{ is strong } \iff & \quad p_i \circ B_1(uv) = p_i \circ A_1(u) \times p_i \circ A_1(v) \text{ for all } uv \in E_1, i = 1, 2, \ldots, m \\
\iff & \quad p_i \circ B_2(\phi(u)\phi(v)) = p_i \circ A_2(\phi(u)) \times p_i \circ A_2(\phi(v)) \text{ for all } \phi(u)\phi(v) \in E_2, i = 1, 2, \ldots, m \\
\iff & \quad G_2 \text{ is strong.}
\end{align*}\]

Again,

\[\begin{align*}
G_1 \text{ is regular } \iff & \quad \sum_{u \neq v, \ uv \in E_1} p_i \circ B_1(uv) = \text{constant for all } u \in V_1 \\
\iff & \quad \sum_{\phi(u) \neq \phi(v)} p_i \circ B_2(\phi(u)\phi(v)) = \text{constant for all } \phi(u) \in V_2 \\
\iff & \quad G_2 \text{ is regular.}
\end{align*}\]

\[\text{Theorem 3.18. A strong mFG } G \text{ is strong regular if and only if its complement } \overline{G} \text{ is strong regular.}\]

\[\begin{align*}
\text{Proof. From } \overline{G}, \text{ we have if } G = (V, A, B) \text{ is a strong product mFG, then } \overline{G} = (V, \overline{A}, \overline{B}) \text{ is also a strong product mFG where } \overline{A} = A \text{ and } \overline{B} \text{ is defined by} \\
p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy) \text{ for all } xy \in \overline{E}_2, i = 1, 2, \ldots, m. \text{ Now,} \\
G \text{ is strong regular } \iff & \quad p_i \circ B(xy) = p_i \circ A(x) \times p_i \circ A(y) \\
\iff & \quad p_i \circ \overline{B}(xy) = p_i \circ A(x) \times p_i \circ A(y) - p_i \circ B(xy) = p_i \circ B(xy) - p_i \circ B(xy) = 0 \\
\iff & \quad \sum p_i \circ \overline{B}(xy) = 0 \\
\iff & \quad \overline{G} \text{ is strong regular.}
\end{align*}\]

4 Modeling of Products Design in a Company as Product mFG

Here, we model a real life situation of a company where a group of people decides which product design to manufacture. This type of network is an ideal example of product mFGs. It is very important for a company to decide which product design to manufacture so that they can make profit as much as possible. A very good product design is gladly acceptable to the peoples if it is also cheap in price. The determination of which product design to manufacture is called the decision making problem. By taking the very good decision
(very good product design), one company can spread their product all over the world keeping in mind that
the product design is very good, demandable, cheap, easily accessible etc. Before manufacturing a product
design, engineers and manufacturers test several important things in a product. Suppose a company has to
decide to manufacture a product design among five products, say $D_1, D_2, D_3, D_4$ and $D_5$. A product design
is manufactured by a company keeping in mind its market demand, price, time taken to manufacture and
accessibility.

![Figure 4: Modeling of products design in a company as a product 4-polar fuzzy graph $G$](image)

We consider the above as a set, say $M = \{\text{demand, price, time, accessibility}\}$ and the set of product
designs as $D = \{D_1, D_2, D_3, D_4, D_5\}$. Since all the above characteristics of a product design according to the
different company are uncertain in real life, therefore we consider a 4-polar fuzzy subset $A$ of the set $D$. This
situation can be represented as a product 4-polar fuzzy graph by considering the different product design
as the nodes and edges between them represent the relationship between two product designs (see Fig. 4).
The membership value of each node represents the degree of demand, price, time taken to manufacture and
accessibility to people in global market. Edge membership values which represent the relationship between
the product design can be calculated by using the relation

$$p_i \circ B(uv) \leq p_i \circ A(u) \times p_i \circ A(v)$$

for all $u, v \in D, i = 1, 2, \ldots, 4$.

The edge between two product designs represents the degree of using common power equipments, raw
materials, engineer employs and agencies involved for both products.

From the Fig. 4, we see that $G = (D, A, B)$ is a product 4-polar fuzzy graph and the product design
$D_3$ has maximum demand, minimum price, minimum time to manufacture and has maximum accessibility
compared to all others product designs.

5 Conclusions

The fuzzy graph theory is one of the most developing area of research, which has a variety of applications
in different fields including computer science, communication networks, biological sciences, social networks,
decision-making and optimization problems. A product mFG can be used to represent real world problems
which involve multi-case of information and uncertainty. No other graphs can be used to model such types of
problems. This is how $m$-polar fuzzy graph is a generalized version of all other graphs. Actually, if $m = 1$,
then the 1-polar fuzzy graph becomes our usual fuzzy graph. A product mFG plays a vital role in many
research domains and gives more precision, flexibility and comparability to the system as compared to the
fuzzy and bipolar fuzzy models. In this paper, we have introduced the notion of $m$-polar $\psi$-morphism on
product mFGs. The action of $m$-polar $\psi$-morphism on product mFGs is studied and established some results
on weak and co-weak isomorphism. $d_2$-degree and total $d_2$-degree of a vertex in product mFGs are defined and
studied $(2, k)$-regularity and totally $(2, k)$- regularity. A real life situation has been modeled as an application
of product mFG.

References


