Journal of Uncertain Systems
Vol.12, No.1, pp.36-46, 2018
Online at: www.jus.org.uk

Fuzzy Assessment Methods for the Acquisition of the van Hiele Levels in Geometry

Michael Gr. Voskoglou*
Department of Applied Mathematics, School of Technological Applications Graduate T. E. I. of Western Greece

Received 19 December 2016; Revised 17 December 2017

Abstract

Three different fuzzy methods are applied in this work for the evaluation of student group geometric reasoning in terms of the van Hiele level theory: The group's total possibilistic uncertainty and Triangular Fuzzy Numbers (TFNs) are used for assessing the group's mean performance, while a special form of the Centre of Gravity defuzzification technique, known as the Rectangular Fuzzy Assessment Model (RFAM), is utilized to assess its quality performance. The calculation of the total possibilistic uncertainty is laborious and can be used for comparing the performance of different groups only under the assumption that the existing in these groups uncertainty is the same before the experiment. On the contrary, the application of the TFNs provides clear outcomes, which can be easily interpreted. Concerning the group quality performance the RFAM is useful to be preferred in cases where two groups share the same value of the traditional GPA index, because in such cases the application of the GPA index could not lead to logically based conclusions. The data of a paradigm for evaluating the acquisition of the van Hiele levels in three–dimensional Geometry by three secondary student classes presented by Gutierrez et al. are reused here to illustrate our fuzzy methods and to compare their outcomes with those of two traditional assessment methods of the bi-valued logic, the calculation of the mean value and the Grade Point Average index.

Keywords: fuzzy sets, fuzzy logic, fuzzy system uncertainty, center of gravity defuzzification technique, rectangular fuzzy assessment model, triangular fuzzy numbers, grade point average index, van Hiele levels of geometric reasoning

1 Introduction

Situations appear frequently in our day to day life characterized by a degree of uncertainty and/or fuzziness. In Education for example, a teacher is frequently not sure about the degree of acquisition of the new knowledge by students, or for the proper mark to characterize a student's performance. There used to be a tradition in science and engineering of turning to probability theory when a problem was faced in which uncertainty plays an important role. Today this is no longer the rule. *Fuzzy Logic* (FL), due to its property of characterizing the ambiguous cases with multiple values, provides a rich and meaningful addition to standard logic enabling the modeling under conditions which are imprecisely defined, despite the concerns of classical logicians.

The present author, starting in 1999 with a model for the process of learning a subject matter in the classroom [20], has frequently utilized principles of FL in the past for describing and evaluating several human or machine (e.g. Case-Based Reasoning systems in computers [24]) activities. The assessment methods used in those works involve:

- The measurement of a fuzzy system's probabilistic or possibilistic uncertainty, e.g. [22, 23, 27], etc.
- An adaptation of the Center of Gravity (COG) defuzzification technique that was called the Rectangular Fuzzy Assessment Model (RFAM), e.g. [13, 23, 25, 27], etc.
- The Generalized Rectangular, the Triangular and the Trapezoidal Fuzzy Assessment Models (GRFAM, TFAM and TpFAM respectively), which are variations of RFAM, e.g. [14, 15, 27, 28], etc. It turns out [15] that all these models are equivalent to each other. Also, although not equivalent with RFAM in characterizing a group's performance, they always provide the same assessment outcomes with it when comparing the performances of two or more groups [28].

* Corresponding author.
Email: mvosk@hol.gr (M.G. Voskoglou).

- The Triangular (TFNs) and Trapezoidal (TpFNs) Fuzzy Numbers, e.g. [26, 27], etc.

In the present paper we use and compare to each other the fuzzy system's probabilistic uncertainty, the RFAM and the TFNs to evaluate the acquisition by students of the van Hiele (vH) levels for geometric reasoning. The main contribution of the paper to the state of the art is that it provides fuzzy assessment methods that can be used in cases where the traditional assessment methods of the bi-valued logic either cannot be used (e.g. calculation of the mean value of student scores which are given by qualitative expressions and not numerically) or they could not lead to logically based conclusions (e.g. equal values of the GPA index for two different student groups).

The rest of the paper is formulated as follows: In Section 2 we give a brief account of the vH level theory and of several previous researches on it connected to the present work. In Section 3 we deal with the main types of a fuzzy system's uncertainty. In Section 4 we give a brief account of the development of RFAM and we compare it with the traditional Grade Point Average (GPA) index, since both measure a system's quality performance. In Section 5 we present briefly the background on TFNs needed in this work. In Section 6 we give an example illustrating the applicability of the above fuzzy methods (possibilistic uncertainty, RFAM and TFNs) to evaluate the acquisition of the vH levels in Geometry by high school students. We close, in Section 7, with our conclusion and some hints for future research.

For general facts on *Fuzzy Sets (FS)* and the uncertainty connected to them we refer to the book [6].

2 The van Hiele Levels of Geometric Reasoning

The vH theory of geometric reasoning [18, 19] suggests that students can progress through five levels of increasing structural complexity. A higher level contains all knowledge of any lower level and some additional knowledge which is not explicit at the lower levels. Therefore, each level appears as a meta-theory of the previous one [2]. The five vH levels include:

- L1 (*Visualization*): Students perceive the geometric figures as entities according to their appearance, without explicit regard to their properties.
- L2 (*Analysis*): Students establish the properties of geometric figures be means of an informal analysis of their component parts and begin to recognize them by their properties.
- L3 (*Abstraction*): Students become able to relate the properties of figures, to distinguish between the necessity and sufficiency of a set of properties in determining a concept and to form abstract definitions.
- L4 (*Deduction*): Students reason formally within the context of a geometric system and they gasp the significance of deduction as means of developing geometric theory.
- L5 (*Rigor*): Students understand the foundations of geometry and can compare geometric systems based on different axioms.

Obviously the level L_5 is very difficult, if not impossible, to appear in secondary classrooms, while level L_4 also appears very rarely.

Although van Hiele [19] claimed that the above levels are discrete – which means that the transition from a level to the next one does not happen gradually but all at once – alternative researches by Burnes & Shaughnessy [1], Fuys et al. [3], Wilson [30], Guttierrez et al. [4], and by Perdikaris [9] suggest that the vH levels are continuous characterized by transitions between the adjacent levels. This means that from the teacher's point of view there exists fuzziness about the degree of student acquisition of each vH level. Therefore, principles of FL can be used for the assessment of student geometric reasoning skills.

3 Uncertainty in Fuzzy Systems

Uncertainty is the shortage of precise knowledge and of complete information on data, which describe together the state of the corresponding system. One of the key problems of artificial intelligence is the modelling of the uncertainty for solving real life problems and several models have been proposed for this purpose.

The amount of information obtained by an action can be measured by the reduction of the uncertainty resulting from this action. Therefore, a system's uncertainty is connected to its capacity for obtaining relevant information. Accordingly a measure of uncertainty could be adopted as a measure of a system's effectiveness in solving related problems. The greater is the decrease of the uncertainty resulting from the action (i.e. the difference of the existing uncertainty before and after the action), the better the system's performance with respect to the action.

In terms of the classical probability theory a system's uncertainty and the information connected to it are measured by the Shannon's formula, better known as the *Shannon's entropy*[*] [12]. For use in a fuzzy environment Shannon's formula has been adapted ([7], p.20) in the form:

$$H = -\frac{1}{\ln n}\sum_{s=1}^{n} m_s \ln m_s \ . \qquad (1)$$

If U is the universal set of the discourse, then m: U $\rightarrow$ [0, 1] in formula (1) is the membership function of the corresponding FS, $m_s = m(s)$ denotes the membership degree of the element s of U and n denotes the total number of the elements of U. The sum is divided by the natural logarithm of n in order to be normalized. Thus H takes values within the real interval [0, 1]. Formula (1) measures a fuzzy system's *probabilistic uncertainty*.

It is recalled that the *fuzzy probability* of an element s of U is defined in a way analogous to the crisp probability, i.e. by

$$P_s = \frac{m_s}{\sum_{s\in U} m_s} \ . \qquad (2)$$

However, according to Shackle [11] and many other researchers after him, human reasoning can be formulated more adequately by the possibility rather, than by the probability theory. The *possibility,* say r_s, of an element s of U is defined by

$$r_s = \frac{m_s}{\max\{m_s\}} \ , \qquad (3)$$

where max $\{m_s\}$ denotes the maximal value of m_s, for all s in U. In other words, the possibility of s expresses the relative membership degree of s with respect to max $\{m_s\}$.

Within the domain of possibility theory uncertainty consists of *strife* (*or discord*), which expresses conflicts among the various sets of alternatives, and of *non-specificity* (*or imprecision*), which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives ([7], p.28). For a better understanding of the above two types of uncertainty we give the following simple example:

EXAMPLE: Let U be the set of integers from 0 to 120 representing human ages and let Y = young, A = adult and O = old be fuzzy subsets of U defined by the membership functions m_Y, m_A and m_O respectively. People are considered as young, adult or old according to their outer appearance. Then, given x in U, there usually exists a degree of uncertainty about the values that the membership degrees $m_Y(x)$, $m_A(x)$ and $m_O(x)$ could take, resulting to a conflict among the fuzzy subsets Y, A and O of U. For instance, if x = 18, values like $m_Y(x)$ = 0.8 and $m_A(x)$ = 0.3 are acceptable, but they are not the only ones. In fact, the values $m_Y(x)$ = 1 and $m_A(x)$ = 0.5 are also acceptable, etc. The existing conflict becomes even greater if x =50. In fact, is it reasonable in this case to take $m_Y(x)$ =0? Probably not, because sometimes people being 50 years old look much younger than others aged 40 or even 30 years. But, there exist also people aged 50 who look older from others aged 70, or even 80 years! All the above are examples of the type of uncertainty that we have termed as strife. On the other hand, non - specificity is connected to the question: How many x in U should have non zero membership degrees in Y, A and O respectively? In other words, the existing in this case uncertainty creates a conflict among the cardinalities (sizes) of the fuzzy subsets of U. It is recalled that the *cardinality* of a fuzzy subset, say B, of U is defined to be the sum $\sum_{x\in U} m_B(x)$ of all membership degrees of the elements of U in B.

Strife is measured ([7]; p.28) by the function $ST(r)$ on the ordered possibility distribution r: $r_1=1\geq r_2 \geq\cdots\geq r_n \geq r_{n+1}$ of a group of a system's entities defined by

$$ST(r) = \frac{1}{\log 2}\Big[\sum_{i=2}^{m}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\Big] . \qquad (4)$$

Under the same conditions non-specificity is measured ([7]; p.28) by the function

$$N(r) = \frac{1}{\log 2}\Big[\sum_{i=2}^{m}(r_i - r_{i+1})\log i\Big] . \qquad (5)$$

The sum $T(r) = ST(r)+N(r)$ measures of the *total possibilistic uncertainty* for ordered possibility distributions.

---

[*] This name is due to the mathematical definition of the information I by I= $-\Delta(\log P)/\log 2$, where P is the probability of appearance of one of the possible cases of the evolution of the corresponding real situation. This is analogous to the well known from Physics formula $\Delta S = \Delta Q/T$, where ΔS is the increase of a physical system's entropy caused by an increase of the heat by ΔQ, when the absolute temperature T remains constant.

4 The COG Defuzzification Technique as an Assessment Method (RFAM)

It is recalled that the solution of a problem in terms of FL involves the following steps:

- Choice of the universal set U of the discourse.
- Fuzzification of the problem's data by defining the proper membership functions.
- Evaluation of the fuzzy data by applying rules and principles of FL to obtain a unique fuzzy set which determines the problem's solution.
- Defuzzification of the final outcomes in order to apply the solution found in terms of FL to the original, real world problem.

One of the most popular in fuzzy mathematics defuzzification methods is the *Centre of Gravity* (*COG*) technique. To apply the COG technique, let $A = \{(x, m(x)): x \in U\}$ be the fuzzy set determining the problem's solution. In order to design the graph of $y = m(x)$ one corresponds to each $x \in U$ an interval of values from a prefixed numerical distribution, which actually means that U is replaced by a set of real intervals. There is a commonly used in FL approach to represent the system's fuzzy data by the coordinates (x_c, y_c) of the COG, say F_c, of the area F contained between the graph of A and the X-axis [17]. The COG coordinates are calculated by the following well-known [29] from Mechanics formulas:

$$x_c = \frac{\iint_F x\,dxdy}{\iint_F dxdy},\ y_c = \frac{\iint_F y\,dxdy}{\iint_F dxdy} \tag{6}$$

Subbotin et al. [13], based on Voskoglou's [20] fuzzy model for the process of learning a subject matter adapted the COG technique for use as an assessment method of student learning skills. Since then, Subbotin and Voskoglou, working either jointly or individually improved and used the COG technique and several variations of it as assessment methods in many other human activities, e.g. [15, 23, 27], etc. For the needs of the present work this method is sketched below:

Let G be a group of individuals (or of any other objects; e.g. CBR systems [24]). We choose as set of the discourse the set U = {A, B, C, D, F} of the *fuzzy linguistic labels* (characterizations) of excellent (A), very good (B), good (C), fair (D) and unsatisfactory (F) performance respectively of the group's members. When a numerical value, say *y*, is assigned to a group's member (e.g. a mark in case of a student), then its performance is characterized by F, if $y \in [0, 1)$, by D, if $y \in [1, 2)$, by C, if $y \in [2, 3)$, by B if $y \in [3, 4)$ and by A if $y \in [4, 5]$ respectively. Consequently, we have that $y_1 = m(x) = m(F)$ for all x in [0, 1), $y_2 = m(x) = m(D)$ for all x in [1, 2), $y_3 = m(x) = m(C)$ for all x in [2, 3), $y_4 = m(x) = m(B)$ for all x in [3, 4) and $y_5 = m(x) = m(A)$ for all x in [4, 5]. Therefore, the graph of the membership function $y = m(x)$ takes in this case the form of Figure 1, where the area of the level's section F contained between the graph and the X-axis is equal to the sum of the areas of the rectangles S_i, i=1, 2, 3, 4, 5.

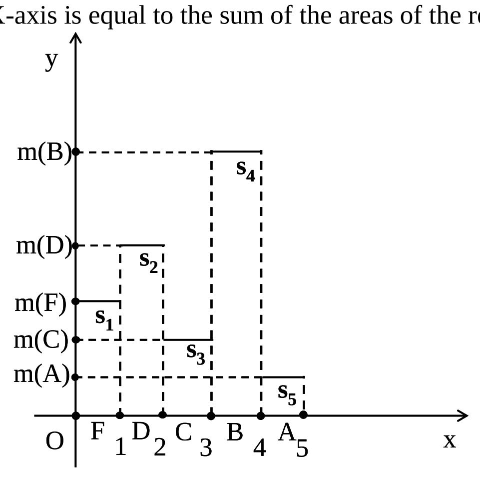

Figure 1: The graph of the COG method

Then, calculating the double integrals in formulas (1) ([23], Section 3), it is straightforward to check that

$$x_c = (y_1+3y_2+5y_3+7y_4+9y_5),\ \ y_c = \frac{1}{2}\ (y_1^2+y_2^2+y_3^2+y_4^2+y_5^2)\ , \tag{7}$$

with x_1=F, x_2=D, x_3=C, x_4=B, x_5=A and $y_i = m(x_i)\Big/\sum_{j=1}^{5} m(x_j)$, i = 1, 2, 3, 4, 5.

Note that the membership function $y = m(x)$ can be defined, as it happens with fuzzy sets in general, according to the user's choice any compatible to the common logic way In this work we define $y = m(x)$ in terms of the frequencies. For example, if n is the total number of the members of G and n_A is the number of the members of G whose performance was characterized by the label A, we set $y_1 = n_A/n$, etc. Then $\sum_{i=1}^{5} m(x_i) = 1$ (100%).

Using elementary algebraic inequalities one finds that the minimal value of the y-coordinate corresponds to the COG F_M $(5/2, 1/10)$ ([23], Section 3). Also, applying formulas (7) it is straightforward to observe that the group's ideal performance (y_5 =1 and y_i = 0 for $i \neq 5$) corresponds to the GOG $F_I(9/2, 1/2)$, while its worst performance corresponds to the GOG $F_W(1/2, 1/2)$. Therefore the COG lies in the area of the triangle $F_MF_IF_W$ of Figure 2.

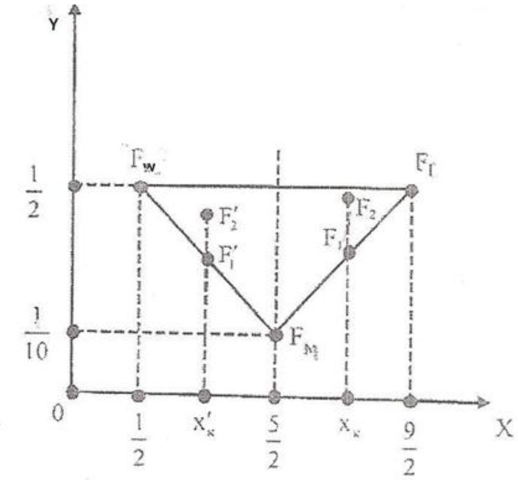

Figure 2: The area where the COG lies

Making elementary geometric observations on Figure 2 it is straightforward to obtain the following assessment criterion:

- *Between two groups, the group with the greater value of xc performs better.*
- *If two groups have the same xc 2.5, then the group with the greater yc performs better.*
- *If two groups have the same xc < 2.5, then the group with the smaller yc performs better.*

As it becomes evident from the above criterion, a group's performance depends mainly on the value of the x-coordinate of the COG, which is calculated by the first of formulas (7). In this formula, greater coefficients (weights) are assigned to the higher grades. Therefore, the COG method focuses on the group's *quality performance.*

Since in case of the ideal performance the first of formulas (7) gives that $x_c = 9/2$, values of x_c greater than half of the above value, i.e. greater than $9/4$ = 2.25, could be considered as demonstrating a more than satisfactory group's performance.

REMARK (*Comparison of the COG technique to the GPA index*): A very popular in the USA and other Western countries assessment method of the traditional logic is the calculation of the *Grade Point Average* (*GPA*) index. This index is a weighted average in which greater coefficients (weights) are assigned to the higher scores. Therefore, in an analogy to the COG technique, it is connected to a group's quality performance.

The GPA, is calculated by the formula

$$\text{GPA} = \frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n}, \tag{8}$$

where n is the total number of the group's members and n_A, n_B, n_C, n_D and n_F denote the numbers of the group's members that demonstrated a performance characterized by A, B, C, D and F respectively [16]. Formula (8) can be written in terms of the frequencies defined in Section 4 in the form

$$\text{GPA} = y_2 + 2y_3 + 3y_4 + 4y_5. \tag{9}$$

Therefore, in order to find the numerical relation between GPA and the x-coordinate of the COG technique, we write the first of formulas (7) as

$$x_c = \frac{1}{2}[2(y_2+2y_3+3y_4+8y_5)+(y_1+y_2+2y_3+y_4+y_5)]$$

$$= \frac{1}{2}(2\text{GPA}+1), \text{ or finally in the form } x_c = \text{GPA}+0.5. \quad (10)$$

Consider now two groups, say G_1 and G_2 with GPA values GPA_1 and GPA_2 and values of the x-coordinates of their COG x_{c1} and x_{c2} respectively. Then, if $GPA_1 > GPA_2$, equation (10) gives that $x_{c1} > x_{c2}$. Therefore, by the first case of the assessment criterion of Section 4, one concludes that the GPA index and the COG technique provide the same assessment outcomes. However, if $GPA_1=GPA_2$, equation (10) gives that $x_{c1}=x_{c1}$. Therefore, one of the last two cases of the assessment criterion of Section 4 is applicable now, which means that the GPA index and the COG technique provide different assessment outcomes in this case.

The following example presented in ([15], Section 4, paragraph vii) shows that *in case of the same GPA values the application of the GPA index could not lead to logically based conclusions*. Therefore, in such situations, the assessment criterion of Section 4 becomes useful due to its logical nature.

EXAMPLE*:* The student grades of two different classes are depicted in the following Table:

Table 1: Student grades

Grades	Class I	Class II
C	10	0
B	0	20
A	50	40

The GPA index for the two classes is

$$\frac{4*50+2*10}{60} = \frac{4*40+3*20}{60} \approx 3.67,$$

which means that the two Classes demonstrate the same quality performance. Further, equation (9) gives that $x_c \approx 4.17 > 2.5$ for both Classes. But

$$\sum_{i=1}^{5} y_i^2 = (\frac{1}{6})^2 + (\frac{5}{6})^2 = \frac{26}{36}$$

for the first and $\sum_{i=1}^{5} y_i^2 = 20/36$ for the second Class. Therefore, by the second case of the assessment criterion for RFAM, Class I performed better.

Now which one of the above two conclusions is closer to the reality? For this, let us consider the *quality of knowledge*, i.e. the ratio of the students received B or better to the total number of students, which is equal to $5/6$ for the first and 1 for the second Class. Therefore, from the common point of view, the situation in Class II is better.

Also, assigning, as it looks logical, to the grades A, B, C, D and F the numbers 5, 4, 3, 2, 1 respectively one finds the mean value

$$\overline{X} = \frac{3*10+5*50}{60} \approx 4.67 \quad \text{and} \quad \overline{X^2} = \frac{3*10^2+5*50^2}{60} \approx 213.33$$

for Classes I. Therefore the *variance* of X is equal to $213.33 - (4.67)^2 \approx 191.52$. In the same way one finds that the variance of X for Class II is equal to $160 - (4.67)^2 \approx 138.19 < 213.33$. Therefore the standard deviation for the second Class is definitely smaller, which means that, from the statistical point of view, the situation in Class II is also better.

However, some instructors could prefer the situation in Class I, which has more e3xcellent students. Everything is determined by the personal preference of the goals. The conclusion of the RFAM agrees with the second point of view, while the conclusion of the GPA looks as not having any logical basis.

5 Assessment of a Student Group Performance Using TFNs

For general facts on *Fuzzy Numbers* (*FNs*) we refer to the book [5] and to the article [26].

It is recalled here that a FN is a FS on the set ***R*** of the real numbers which is *normal* (i.e. there exists *x* in ***R*** such that the membership degree $m(x) = 1$) and *convex* (i.e. all its a-cuts with a in [0, 1] are closed real intervals), while its membership function $y = m(x)$ is *piecewise continuous*.

It is also recalled that a TFN (a, b, c), with a, b, c real numbers such that $a<c<b$, is a FN with membership function defined by

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

Let A (a, b, c) and B (a_1, b_1, c_1) be two TFNs and let k be a positive real number. Then the *sum* A + B = ($a+a_1$, $b+b_1$, $c+c_1$) and the *scalar product kA*=(ka, kb, kc) ([26], paragraph 10). Further, given the TFNs A_i, i = 1, 2,…, n , where n a non negative integer, $n \geq 2$, we define their *mean value* to be the TFN

$$A= \frac{1}{n} (A_1 + A_2 + \cdots + A_n). \tag{11}$$

Our method of using TFNs for the assessment of a student group performance involves first the numerical evaluation of each student's individual performance in a climax from 0 to 100. Then the student scores are attached to the fuzzy assessment grades of U defined in Section 4 as follows: A: 85-100, B: 75-84, C: 60-74, D: 50-59) and F: (0-49)[†]. Further, a TFN of the form A=(85, 92.5, 100), B=(75, 79.5, 84), C=(60, 67, 74), D=(50, 54.5, 59) and F=(0, 24,5, 48) respectively is attached to each of the above grades, denoted for reasons of simplicity of our notation with the same letter. Observe that the middle entry of the above TFNs is equal to the mean value of the student scores attached to the corresponding grade. Of course one could attach other kinds of FNs to each of the above grades, e.g. trapezoidal FNs. However the choice of TFNs was preferred because they are the simplest form of FNs, which means that the calculations needed become easier.

In this way a TFN can be assigned to each student assessing his/her individual performance. Therefore, it is logical to accept the use of the mean value M of all those TFNs as means for evaluating the student group overall performance. Moreover, the COG technique is used for the defuzzification of the TFNs. This leads to the representation of a TFN T=(a, b, c) by the x-coordinate, say x(T), of the COG of its graph. Since this graph is a triangle ABC with A(a, 0), B(1, b) and C(0,c) ([26], Figure 2),

$$x(T)=\frac{a+b+c}{3} \quad \text{([26], Proposition 1].} \tag{12}$$

In particular, if T is one of the TFNs A, B, C, D, F, then by their definition we have that b= $(a+c)/2$. Therefore, equality (12) gives that

$$x(T)= \frac{a+\frac{a+c}{2}+c}{3} = \frac{3(a+c)}{6} = b.$$

But M=k_1A+k_2B+k_3C+k_4D+k_5F, with k_i non negative real numbers, i = 1, 2, 3, 4, 5. Therefore, if M(a, b, c), then obviously

$$x(M) = k_1x(A)+k_2x(B)+k_3x(C)+k_4x(D)+k_5x(F) = b.$$

REMARK (*Comparison of x*(*M*) *with the Yager index*)*:* An alternative way for defuzzifying a TFN T=(a, b, c) is the use of the *Yager Index* Y(T), introduced in [31] in terms of the α -cuts of T with α in [0, 1] in order to help the ordering of FSs. It can be shown ([10], p. 62) that

$$Y(T)=\frac{2b+a+c}{4}.$$

Observe now that

$$x(T)=Y(T) \Leftrightarrow \frac{a+b+c}{3} = \frac{2b+a+c}{4} \Leftrightarrow 4(a+b+c)=3(2b+a+c) \Leftrightarrow a+c=2b.$$

The last equality is not true in general for $a<b<c$; e.g. take a=1, b=2.5 and c=3. In other words we have in general that x(T)$\neq$ Y(T). However, for the TFNs A, B, C, D, and F the above equality holds. Therefore, since the mean value M is a linear combination of those TFNs, it is straightforward to check that x(M)=Y(M). In other words, the COG technique and the Yager index provide the same outcomes when applied in the assessment method described in Section 5.

[†] The above correspondence of the student scores to the fuzzy assessment grades is not unique. For example, in a more strict assessment one could take A:90-100, B: 80-89, C: 70-79, D: 60-69, F: 0-59, etc.

6 Assessing the Acquisition of the van Hiele Levels

Gutierrez et al. [4] presented a paradigm for evaluating the acquisition of the vH levels in three–dimensional Geometry by three different groups, say G_1, G_2 and G_3, consisting of 20, 21 and 9 students respectively. Here, in order to illustrate the applicability of the fuzzy assessment methods described in Sections 3, 4 and 5 to the vH level theory, we shall use the data of this paradigm, which are depicted in Table 2.

Table 2: *Data of the paradigm*

[Degree of acquisition]

Group	vH level	F	D	C	B	A
G_1	L_1	0	0	0	0	20
G_1	L_2	1	0	3	6	10
G_1	L_3	2	3	6	6	3
G2	L_1	0	0	1	2	18
G2	L_2	0	3	4	13	1
G2	L_3	9	6	5	1	0
G3	L_1	0	2	4	2	1
G3	L_2	3	4	2	0	0
G3	L_3	9	0	0	0	0

We start with the COG technique, which evaluates a group's *quality performance*:

i) Use of the COG Technique

From the first row of Table 2 one finds that y_5=1 and y_1=y_2=y_3=y_4=0 for L_1 in G_1. Therefore, the first of equations (7) gives that $x_c = 9/2 = 4.5$. Similarly, from the fourth row of Table 2, one finds that y_1=y_2=0, $y_3=1/21$, $y_4=2/21$ and $y_5=18/21$. Therefore

$$x_c = \frac{1}{2}\left(\frac{5}{21} + \frac{14}{21} + \frac{162}{21}\right) \approx 4.31$$

for L_1 in G_2. In the same way one finds that the values of x_c are 2.5 for L_1 in G_3, 3.7, 3.07, 1.39 for L_2 in G_1, G_2, G_3 respectively and 2.75, 1.4, 0.5 for L_3 in G_1, G_2, G_3 respectively. Therefore, according to the first case of the assessment criterion of Section 2, G_1 demonstrated the best performance in the first three vH levels, followed by G_2 and G_3.

Also, on comparing the above values with 2.25, one concludes that G_1 demonstrated a more than satisfactory performance in these three levels, while G_2 demonstrated a more than satisfactory performance in the first two levels and G_3 demonstrated a more than satisfactory performance only in the first level.

Finally, since the mean value

$$\frac{4.5 + 3.7 + 2.75}{3} = 3.65,$$

G_1 demonstrated a more than satisfactory overall performance in the first three vH levels. In the same way, since the corresponding mean values for G_2 and G_3 are 2.93 and 1.46 respectively, G_2 demonstrated also a more than satisfactory overall performance, while the performance of G_3 was not satisfactory.

The calculation of the mean value of the scores assigned to each one of its members is the classical method of the bi-valued logic for assessing a group's *mean performance* with respect to an action. However, in cases involving a significant degree of uncertainty and/or fuzziness–as it happens with the data of Table 2 characterizing the student performance by qualitative grades and not by numerical scores–one can use either the TFNs instead or the corresponding system's uncertainty, because both of these fuzzy assessment methods are connected to the group's mean performance. We apply first the method of TFNs:

ii) Use of the TFNs

From the data of the first row of Table 1 one obtains that M=A and x(M)=92.5 for G_1 in L_1. Similarly, from the second row it is obtained that M= (F+3C+6B+10A)/20=(74, 81.38, 88.75) and x(M)=81.38 for G_1 in L_2. Therefore the first group demonstrates a very good (B) overall performance in level L_2. We keep going in the same way with the remaining rows. All the assessment outcomes of the paradigm are depicted in Table 3.

Table 3: Assessment outcomes

Group	vH level	M	x(M)	Group's performance
G1	L1	(85, 92.5, 100)	92.5	A
G1	L2	(74, 81.38, 88.75)	81.38	B
G1	L3	(60.75, 68.45, 76.15)	68.45	C
G2	L1	(82.86, 90.05, 97.24)	90.05	A
G2	L2	(69.05, 74.17, 79.69)	74.37	C
G2	L3	(32.14, 45.81, 59.48)	45.81	F
G3	L1	(63.89, 69.83, 75.78)	69.84	C
G3	L2	(35.56, 47.28, 59)	47.28	F
G3	L3	(0, 24.5, 59)	24.5	F

Observing the values of x(M) in Table 3 it becomes evident that G_1 demonstrates the best performance in all levels, followed by G_2 and G_3. Further, from the last column of Table 3 it turns out that. G_1 demonstrates excellent performance in L_1, very good in L_2 and good in L_3, G_2 demonstrates excellent performance in L_1, good in L_2 and nonatisfactory in L_3, while G_3 demonstrates good performance in L_1 and nonsatisfactory performance in L_2 and L_3. Finally, it seems logical to evaluate the overall performance of each group in the first three vH levels by calculating the mean value of its performances in each level. Thus, G_1 demonstrated a very good ($(92.5+81.38+68.45)/3 \approx 80.78$), G_2 demonstrated a good ($(90.05+7437+45.81)/3 \approx 70.08$) and G_3 demonstrated a nonsatisfactory ($(69.84+47.28+24.5)/3 \approx 47.21$) overall performance.

We shall finish the application of our fuzzy assessment methods by calculating the group total possibilistic uncertainty $T(r)$:

iii) Calculation of the group total possibilistic uncertainty

Perdikaris [8] used fuzzy possibilities and the corresponding fuzzy system's uncertainty to compare the intelligence of student groups in the vH level theory. He considered all the profiles of the form (x, y, z) with x, y and z in U representing a student's performance in the vH levels L_1, L_2 and L_3 respectively and he defined the membership degrees of those profiles by the product

$$\frac{n_x}{n} \cdot \frac{n_y}{n} \cdot \frac{n_z}{n}$$

of the corresponding frequencies. However, this definition is problematic, since it assigns non–zero membership degrees to profiles like (A, B, A), (B, A, D), etc. in which the student's performance in a vH level is assumed to be worse than that in the next level, which is impossible to happen.

This problem was resolved by Voskoglou in [21], where he developed a similar model for the process of learning, by assigning non-zero membership degrees only to *well defined* student profiles (x, y, z). In such profiles x is a grade better or equal than y, which is better or equal than z. This method, although it performs a useful for the instructor/researcher quantitative analysis of all student profiles in terms of their possibilities, it is very laborious even in its revised form [21] requiring the calculation of the membership degrees of 5^3 in total student profiles and the corresponding possibilities by formulas (4) and (5) of Section 3 or by formula (1) when one uses the group probabilistic uncertainty.

Here we shall sketch a much simpler (but still laborious) variation of the above method. For this, the data of the first row of Table 1, concerning the performance of G_1 in level L_1, imply that the ordered possibility distribution is $r_1=r(A)=1>r_2=r_3=r_4=r_5=0$ and therefore from formulas (4) and (5) of Section 3 one finds immediately that the total possibilistic uncertainty for G_1 in L_1 is $T_1(r)= 0$. Consider now the data of the second row of Table 1 concerning the performance of G_1 in level L_2. Then, one finds the membership degrees $m(F)=1/20$, $m(D)=0$, $m(C)= 3/20$, $m(B)= 6/20$ and $m(A)= 10/20$. Since $m(A)$ is the maximal membership degree, the corresponding ordered possibility distribution is

$$r_1=r(A)=1>r_2= r(B)= \frac{6}{10}>r_3=r(C)= \frac{3}{10}>r_4=r(F)= \frac{1}{10}>r_5=r(D)=0.$$

Therefore, since $r_{n+1} = r_5$, i.e. $n=4$, formula (4) gives that

$$ST(r) = \frac{1}{\log 2}\left[(r_2-r_3)\log\frac{2}{r_1+r_2} + (r_3-r_4)\log\frac{3}{r_1+r_2+r_3} + (r_4-r_5)\log\frac{4}{r_1+r_2+r_3+r_4}\right]$$

$$= \frac{1}{\log 2}[\frac{3}{10}\log(1.25) + + \frac{2}{10}\log(1.58) + \frac{1}{10}\log 2) \approx 0.29.$$

Also, formula (5) gives that

$$N(r) = \frac{1}{\log 2}[\frac{3}{10}\log 2 + \frac{2}{10}\log 3 + \frac{1}{10}\log 4] \approx 0.82.$$

Therefore, the total possibilistic uncertainty for G_1 in L_2 is $T_2(r) \approx 1.11$. In the same way one finds from the third row of Table 1 that $T_3(r) = 0.89$ for G_1 in L_3[‡]. Therefore, it is logical to accept that the mean value

$$T(r) = \frac{T_1(r)+T_2(r)+T_3(r)}{3} \approx 0.66$$

measures the total possibilistic uncertainty for G_1 in the first three vH levels L_1, L_2 and L_3.

Repeating the same process for the groups G_2 and G_3 one finds for $T(r)$ the values 1.05 and 2.01 respectively. Under the hypothesis that the existing uncertainty before the experiment was the same for the three groups (*which is not true in general*) this shows that G_1 performed better than G_2, which performed better than G_3.

7 Conclusion and Hints for Future Research

In this work, using the data of an experiment performed by Gutierrez et al. [4], we applied three different fuzzy assessment methods for the evaluation of student group geometric reasoning in terms of the vH level theory. More explicitly, the COG technique (RFAM) was used to assess the group quality performance, while the TFNs and the total possibilistic uncertainty were used for assessing the group mean performance. The calculation of the total possibilistic uncertainty, although it was performed in a simpler way than that used in earlier works, it remained still laborious. Moreover, this method can be used for comparing the performance of different groups only under the assumption that the existing in these groups uncertainty is the same before the experiment, which is not always true. On the contrary, the application of the TFNs provides clear outcomes, which can be easily interpreted. This suggests the use of the TFNs rather instead of the uncertainty, when one wants to evaluate the group mean performance under fuzzy conditions. Concerning the group quality performance the RFAM must be preferred in cases where the groups share the same value of the traditional GPA index, because in such cases the application of the GPA index could not lead to logically based conclusions.

In closing, our plans for future research include the application of the above fuzzy methods for the evaluation of several other human activities in order to obtain stronger conclusions about their advantages and disadvantages.

References

[1] Burger, W.P., and J.M. Shaughnessy, Characterization of the van Hiele levels of development in geometry, *Journal for Research in Mathematics Education*, vol.17, pp.31-48, 1986.

[2] Freudenthal, H., *Mathematics as an Educational Task*, D. Reidel, Dordrecht, 1973.

[3] Fuys, D., Geddes, D., and R. Tischler, The van Hiele model of thinking in geometry among adolescents, *Journal for Research in Mathematics Education, Monograph*, vol.3, 1988.

[4] Gutierrez, A., Jaine, A., and J.K. Fortuny, An alternative paradigm to evaluate the acquisition of the van Hiele levels, *Journal for Research in Mathematics Education*, vol.22, pp.237–251, 1991.

[5] Kaufmman, A., and M. Gupta, *Introduction to Fuzzy Arithmetic*, van Nostrand Reinhold Company, New York, 1991.

[6] Klir, G.J., and T.A. Folger, *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London, 1988.

[7] Klir, G.J., Principles of Uncertainty: What are they? Why do we need them?, *Fuzzy Sets and Systems*, vol.74, no.1, pp.15–31, 1995.

[8] Perdikaris, S.C., Measuring the student group capacity for obtaining geometric information in the van Hiele development thought process: A fuzzy approach, Fuzzy Systems and Mathematics, vol.16, no.3, pp.81–86, 2002.

[9] Perdikaris, S.C., Using fuzzy sets to determine the continuity of the van Hiele levels, *Journal of Mathematical Sciences and Mathematics Education*, vol.6, no.3, pp.81–86, 2011.

[‡] The fact that $T_3(r) < T_3(r)$ does not mean that G_1 demonstrated a better performance in L_3 than in L_2, because the initially existing uncertainty was obviously different in these two levels.

[10] Ruziyeva, A., and S. Dempe, Yager ranking in fuzzy bi-level optimization, *Artificial Intelligence Research*, vol.2, no.1, pp.55–58, 2013.

[11] Shackle, G.L.S., *Decision, Order and Time in Human Affairs*, Cambridge University Press, Cambridge, UK, 1961.

[12] Shannon, C., A mathematical theory of communications, *Bell Systems Technical Journal*, vol.27, pp.379–423 and 623–656, 1948.

[13] Subbotin, I.Y., Badkoobehi, H., and N.N. Bilotckii, Application of fuzzy logic to learning assessment, *Didactics of Mathematics: Problems and Investigations*, vol.22, pp.38–41, 2004.

[14] Subbotin, I.Y., and N.N. Bilotckii, Triangular fuzzy logic model for learning assessment, *Didactics of Mathematics: Problems and Investigations*, vol.41, pp.84–88, 2014.

[15] Subbotin, I.Y., and M.G. Voskoglou, An application of the generalized rectangular model to critical thinking assessment, *American Journal of Educational Research*, vol.4, no.5, pp.397–403, 2016.

[16] Swinburne.edu.au, Grade point average assessment, retrieved on October 15, 2014 from: http://www.swinburne.edu.au/studentadministration/assessment/gpa.html.

[17] van Broekhoven, E., and B. De Baets, Fast and accurate centre of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions, *Fuzzy Sets and Systems*, vol.157, no.7, pp.904–918, 2006.

[18] van Hiele, P.M., and D. van Hiele-Geldov, *Report on Methods of Initiation into Geometry*, edited by Freudenthal, H., and J.B. Wolters, Groningen, Netherlands, pp.67–80, 1958.

[19] van Hiele, P.M., *Structure and Insight*, Academic Press, New York, 1986.

[20] Voskoglou, M.G., An application of fuzzy sets to the process of learning, *Heuristics and Didactics of Exact Sciences*, vol.10, pp.9–13, 1999.

[21] Voskoglou, M.G., Fuzziness or probability in the process of learning: a general question illustrated by examples from teaching mathematics, *The Journal of Fuzzy Mathematics, International Fuzzy Mathematics Institute* (*Los Angeles*), vol.17, no.3, pp.679–686, 2009.

[22] Voskoglou, M.G., *Stochastic and Fuzzy Models in Mathematics Education, Artificial Intelligence and Management*, Lambert Academic Publishing, Saarbrucken, Germany, 2011.

[23] Voskoglou, M.G., A study on fuzzy systems, *American Journal of Computational and Applied Mathematics*, vol.2, no.5, pp.232–240, 2012.

[24] Voskoglou, M.G., Case-based reasoning in computers and human cognition: a mathematical framework, *International Journal of Machine Intelligence and Sensory Signal Processing*, vol.1, pp.3–22, 2013.

[25] Voskoglou, M.G., and I.Y. Subbotin, Dealing with the fuzziness of human reasoning, *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*, vol.3, pp.91–106, 2013.

[26] Voskoglou, M.G., Defuzzification of fuzzy numbers for student assessment, *American Journal of Applied Mathematics and Statistics*, vol.3, no.5, pp.206–210, 2015.

[27] Voskoglou, M.G., Comparison of the COG defuzzification technique and its variations to the GPA index, *American Journal of Computational and Applied Mathematics*, vol.6, no.5, pp.187–193, 2016.

[28] Voskoglou, M.G., *Finite Markov Chain and Fuzzy Models in Management and Education*, GIAN Program, Course No. 16102K03/2015-16, National Institute of Technology, Durgapur, India, 2016.

[29] Wikipedia, Center of mass: definition, last retrieval on October 10, 2014 from http://en.wikipedia.org/wiki/Center_of_mass #Definition.

[30] Wilson, M., Measuring a van Hiele geometric sequence: a reanalysis, *Journal for Research in Mathematics Education*, vol.21, pp.230–237, 1990.

[31] Yager, R., A procedure for ordering fuzzy subsets of the unit interval, *Information Sciences*, vol.24, pp.143–161, 1981.