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# Interval Uncertainty Reduction via Division-by-2 Dichotomization based on Expert Estimations for Short-termed Observations

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**Abstract**

A problem of reducing interval uncertainty is considered. The interval contains admissible values of an object's parameter (property). The observed object's parameter cannot be measured directly or deductively computed, so it is evaluated by expert judgments and estimations. Terms of observations are short, and the object's statistical data are poor. Besides, the term of the parameter's application tends to be the shortest, so any statistical methods to reduce the interval uncertainty are unreliable. Thus an algorithm of reducing interval uncertainty is designed via adjusting the parameter by expert procedures. The interval reduction ensues from the adjustment. While the parameter is adjusted forward, the interval becomes narrower after every next expert procedure and the interval uncertainty seems to be progressively reduced. The narrowing is performed via division-by-2 dichotomization cutting off the halves from the left and right. If the current parameter's value falls outside of the interval, forward adjustment is canceled. Then backward adjustment is executed, where one of the endpoints is moved backwards. For conforming the rate of experts' proficiencies and quality of the adjustment, hard and softer backward adjustment modes can be switched. If the current parameter's value belonging to the interval is too close to either left or right endpoint, then this endpoint is not changed. The closeness is treated differently from both sides by the given relative tolerances. Adjustment is not executed when the current parameter's value enclosed within the interval is simultaneously too close to both left and right endpoints. On this base, an early stop condition is given.


**Keywords:** interval uncertainty reduction, expert procedure, expert estimations, an object's parameter adjustment, dichotomization

## 1 Introduction

Any parameters or attributes of the predicted and controlled processes (events, mechanisms, systems, etc.) are either identified or defined by experience. The identification relates to statistical observations and their successive handling for correcting the mathematical model, which is identified [6, 31]. Otherwise, if the observed object's properties cannot be measured directly or deductively computed, they are evaluated by expert judgments and estimations [2, 15]. Practically, those evaluations are enclosed within intervals including probable values fit to be applied on equal footing when no probability distributions are available [11, 13, 14]. Thus interval uncertainty commonly issues.

Mathematical modeling deals with interval uncertainties also. One of its main problems is to reduce interval uncertainty rather than execute calculations based on interval analysis. When identifying a model, uncertainty is reduced step-by-step, going to be ultimately removed if possible and desired [6, 23]. For expert procedures, number of those steps is generally lesser. Besides, expert data are not so reliable due to their subjectivity [24]. And, moreover, expert procedures themselves need some additional parameters (sometimes called hyperparameters) to be properly and effectively conducted [2, 15, 26, 36]. So, reduction of interval uncertainty by expert estimations and procedures is much harder than that by statistical observations (measurements) and stronger mathematical frameworks (modeling).

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## 2 Related Works and Motivation

Although intervals are common type of estimation in statistics and socio-environmental sciences, they bring us undesirable uncertainties. In statistics, in contrast to point estimation, which is a single number, interval estimation is the use of sample data to calculate an interval of possible (or probable) values of an unknown population parameter [28, 30]. The most prevalent forms of interval estimation are confidence intervals (a frequentist method) and credible intervals (a Bayesian method). In Bayesian statistics, a credible interval is an interval in the domain of a posterior probability distribution or predictive distribution used for interval estimation [4]. Credible intervals are analogous to confidence intervals in frequentist statistics, although they differ on a philosophical basis treating the interval bounds differently. Bayesian intervals treat their bounds as fixed and the estimated parameter as a random variable, whereas frequentist confidence intervals treat their bounds as random variables and the parameter as a fixed value [19, 30, 35].

Other common approaches to interval estimation, which are encompassed by statistical theory, are tolerance, prediction, and likelihood intervals [8, 10, 22, 30]. Fiducial inference, another approach to statistical inference, also considers interval estimation [9, 32]. A lot of non-statistical methods including fuzzy logic lead to interval estimates as well [12, 17, 25].

The interval estimate is an outcome of statistical analysis and related fields of study. The wider interval is, the severer uncertainty grows, though some probabilistic properties may be applied to that interval [7, 14, 16, 18]. Reduction of interval uncertainty is motivated by both statisticians and engineers. This is because any practically reliable and robust decisions on short-term processes (events, objects) can be made only by outcomes, which are point estimates [27, 33]. Nothing but long-term processes allow us practicing with Bayesian decisions using expected values as a version of point estimates [5, 30]. However, assumptions on validity of long-term strategies usually fail due to unpredictable volatility of natural circumstances and engineering conditions. Therefore, interval estimates or interval data are practically inconsistent (Figure 1), where an acceptable decision should be single and precise, rather than an expected value or a set of satisfactory values.

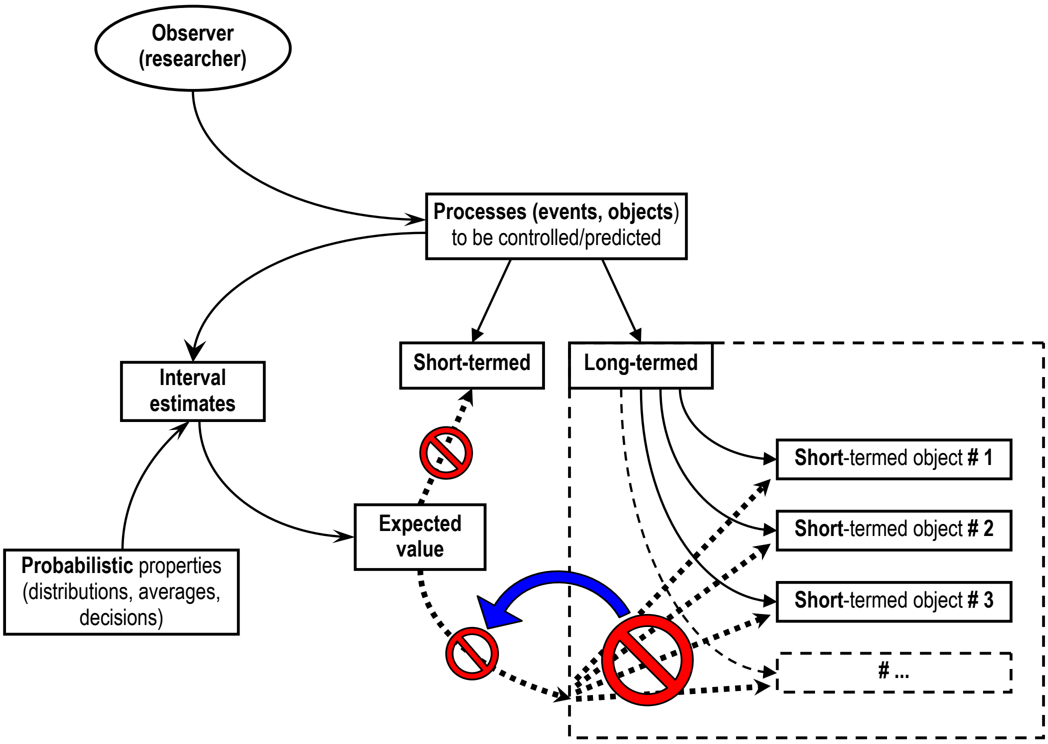


Figure 1: A sketch explaining practical inconsistency of interval estimates whose reliable probabilistic properties, if any, are tied to only short-termed objects

A real challenge is to reduce interval uncertainty under conditions of short-termed observations and poor knowledge [1, 3, 34]. Unlike mathematical modeling based on direct measurements or deductive computations, expert procedures are widely subjected to such conditions. Mainly, the following six causes build them [20, 21, 26]:

1. Judgments and estimations of experts are susceptible to personal impressions, making subjectivism influence dominant.
2. In specific fields of study, expert estimations may be biased, especially if experts' competences and proficiencies are poor.
3. Numbers of expert procedures and estimations commonly are much less than those of manipulations with ready data taken from computer-aided observations for mathematical model identification.
4. Number of experts is usually strongly limited, unless they put their finite set judgments through social computer networks.
5. For the most ingenuous expert judgments similar to likes and dislikes through social computer networks, finding a valid consensus concerning the considered interval is an issue.
6. A lot of criteria to find the consensus have similar ranks, and thus taking a superior single criterion is impossible.

An adjuvant condition to expert procedures is that often experts have an interval whose bounds are exaggerated, but they do not know how much. In such cases, the interval uncertainty can be actually reduced. Whatever the case, interval uncertainty reduction is additionally motivated by that the shorter interval uncertainty is always better to make decisions. Compared to longer interval uncertainty, it is nearer to removal for obtaining a single point decision.

# 3 Goals and Tasks to be Fulfilled

There are two chief goals of this work. They are to design an algorithm of reducing interval uncertainty via adjusting a parameter by expert procedures, and provide regulating distinctness of the adjustment. The regularization is needful to conform the rate of experts' proficiencies and quality of the adjustment. Naively, high quality of adjustment based on estimations of experts with poor proficiencies is not required.

To achieve those goals, the eight tasks are going to be fulfilled:

1. To state principles of every step of the adjustment.
2. To unify these steps into a definite disambiguated sequence.
3. To provide a condition when, at a definite step, the adjustment is not necessary and thus it is not executed.
4. To develop a condition of an early stop.
5. To provide regularization of the adjustment distinctness.
6. To describe specificities of application of the designed algorithm.
7. To discuss merits and drawbacks of reducing interval uncertainty according to the designed algorithm.
8. To conclude and explain a further work outline.

The work is organized according to the sequence of these tasks. In the two last sections corresponding to the task #7 and task #8, a place of the designed algorithm within the field of interval uncertainty reduction is going to be described and sketched. This should help in fairly comprehending the contribution to the field.

# 4 Forward Adjustment

Let an initial interval $[a;\, b] \subset \mathbb{R}$ by $b > a$ be fixed. The interval is supposed to be at least a little bit wider than it is in reality. A parameter $Y \in [a;\, b]$ is to be determined more accurately. Formally, this is to find a strict inclusion

$$[\eta;\, \mu] \subset [a;\, b] \tag{1}$$

using data of expert procedures of adjusting the parameter $Y$. In every procedure, experts deliver their single-point judgments or estimations on the parameter's value. A new subinterval $[\eta;\, \mu]$ is desired to be as narrow as possible.

Let $H$ be a number of expert procedures to be conducted, $H \in \mathbb{N} \setminus \{1\}$. This number is optional. After the $h$-th expert procedure, $h \in \mathbb{N}$ and $h \leqslant H$, a value $y$ of the parameter $Y$ is deduced or calculated based on the experts' estimations (say, it can be a consensus). Number of experts is not specified. As of the first procedure, preassumptions

$$\mu = \frac{y+b}{2} \tag{2}$$

and

$$\eta = \frac{a+y}{2} \tag{3}$$

are irrejectable. Note that partitions (2) and (3) give a half of the initial interval. Since the second procedure, such dichotomy should be kept if $y \in (\eta;\ \mu)$.

Let $j$ be a counter for the previously determined left endpoints of the subinterval $[\eta;\ \mu]$ and $k$ be a counter for the previously determined right endpoints of this subinterval, $j \in \mathbb{N}$ and $k \in \mathbb{N}$. Those endpoints become obsolete as new endpoints are determined. The first obsolete endpoints are $\mu_{\text{obs}}^{\langle 1 \rangle} = b$ and $\eta_{\text{obs}}^{\langle 1 \rangle} = a$ prior to preassumptions (2) and (3). If $y \in (\eta;\ \mu)$ then both the counters $j$ and $k$ are increased by 1, and

$$\mu_{\text{new}} = \frac{y+\mu}{2}, \quad \mu_{\text{obs}}^{\langle k \rangle} = \mu, \quad \mu = \mu_{\text{new}}, \tag{4}$$

$$\eta_{\text{new}} = \frac{\eta+y}{2}, \quad \eta_{\text{obs}}^{\langle j \rangle} = \eta, \quad \eta = \eta_{\text{new}}. \tag{5}$$

Steps (2) and (3), (4) and (5) constitute forward adjustment of the parameter $Y$. While the parameter is adjusted forward, subinterval $[\eta;\ \mu]$ becomes narrower after every next expert procedure and the interval uncertainty seems to be progressively reduced (Figure 2).

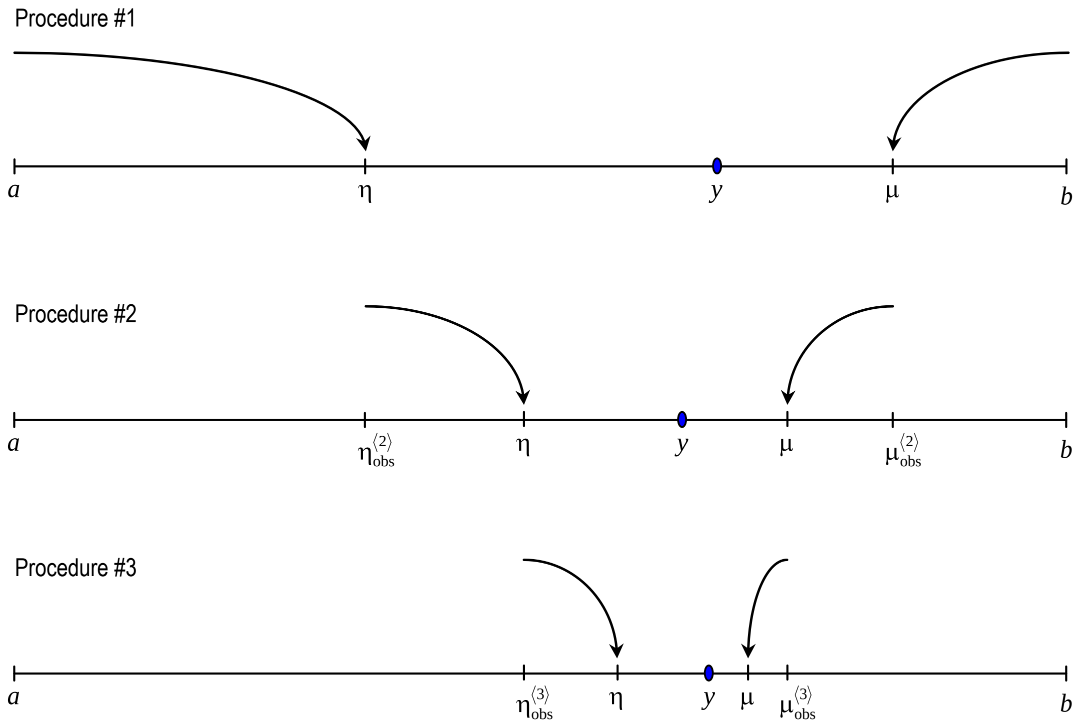


Figure 2: An example of the forward adjustment executed in the start three procedures

If $y \in (\eta;\ \mu)$ and the current parameter's value $y$ is too close to either left or right endpoint, then this endpoint must not be changed. Of course, the same happens if $y = \eta$ or $y = \mu$. The closeness can be treated differently from both sides. Given a relative tolerance $\vartheta$ for the left side, the inequality

$$\frac{y-\eta}{b-a} \leqslant \vartheta \tag{6}$$

for $y \geqslant \eta$ implies that the counter $j$ is not increased (remains the same), and the left endpoint $\eta$ is not updated (to the right). Obviously, setting the tolerance within the unit interval $(0; 1)$ is purely formal and $\vartheta \geqslant 0.5$ is meaningless. At the most, tolerance $\vartheta = 0.25$ can be actuated just after the second procedure. Therefore, tolerance $\vartheta \in (0; 0.25)$ but a chance of infinitely small tolerance is purely formal again (it is straitened to put any constraints on the relative tolerance from bottom). In real applications, however, the relative tolerance cannot exceed 0.1 (or 0.2) or about that.

Similarly to inequality (6), if

$$\frac{\mu - y}{b - a} \leqslant \xi \tag{7}$$

for $y \leqslant \mu$ by a relative tolerance $\xi$ for the right side, the counter $k$ remains the same, and the right endpoint $\mu$ is not moved to the left. Although $\xi \in (0; 0.25)$ and $\xi \leqslant 0.1$ for overwhelming majority of practicable reductions, orders of the left and right tolerances may be dissimilar treating closeness from both sides differently.

**Theorem 1.** After the $h$-th expert procedure, contributed to $h$ successive forward adjustments where neither inequality (6) nor inequality (7) was true, $h \in \mathbb{N}$, the length of the subinterval $[\eta; \mu]$ is $2^h$ times shorter than the length of the initial interval $[a; b]$.

**Proof.** After the first procedure, as it was above-mentioned, partitions (2) and (3) give a half of the initial interval. While forward adjustment, updates (4) and (5) give a new subinterval $[\eta_{\text{new}}; \mu_{\text{new}}]$ whose length is

$$\mu_{\text{new}} - \eta_{\text{new}} = \frac{y + \mu}{2} - \frac{\eta + y}{2} = \frac{\mu - \eta}{2} \tag{8}$$

meaning that every new length is twice shorter than the previous one. For $h$ successive forward adjustments, it gives a new subinterval, which is $2^h$ times shorter than the length of the initial interval $[a; b]$.

□

Clearly, the assertion of Theorem 1 breaks if either inequality (6) or inequality (7) turns true. Thus the interval uncertainty reduction is retarded. When $y \notin [\eta; \mu]$ after some procedure, forward adjustment breaks as well. Whether it should retard the reduction or should not, the answer is going to be given below, where the endpoints' obsolescence is exploited.

It is an obvious fact that, for the forward adjustment, sequences of obsolete endpoints are monotonic. A sequence of obsolete right endpoints is monotonic decreasing (i. e., it moves to the left), and a sequence of obsolete left endpoints is monotonic increasing (moves to the right). In strict words, inequalities

$$\mu_{\text{obs}}^{\langle k+1 \rangle} < \mu_{\text{obs}}^{\langle k \rangle} \tag{9}$$

and

$$\eta_{\text{obs}}^{\langle j \rangle} < \eta_{\text{obs}}^{\langle j+1 \rangle} \tag{10}$$

must hold for any counters $k$ and $j$. However, rate of monotonicity is not monotonic itself. This means that inequalities

$$\mu_{\text{obs}}^{\langle k \rangle} - \mu_{\text{obs}}^{\langle k+1 \rangle} \geqslant \mu_{\text{obs}}^{\langle k+1 \rangle} - \mu_{\text{obs}}^{\langle k+2 \rangle} \tag{11}$$

and

$$\eta_{\text{obs}}^{\langle j+1 \rangle} - \eta_{\text{obs}}^{\langle j \rangle} \geqslant \eta_{\text{obs}}^{\langle j+2 \rangle} - \eta_{\text{obs}}^{\langle j+1 \rangle} \tag{12}$$

do not necessarily hold simultaneously for any $k$ and $j$. The example in Figure 2, where (11) and (12) both are true for $k = j = 1$, is an occurrence typical for the three starting expert procedures. An occurrence when both (11) and (12) are false is impossible, because for the pure forward adjustment, due to Theorem 1,

$$\mu_{\text{obs}}^{\langle k \rangle} - \mu_{\text{obs}}^{\langle k+1 \rangle} + \eta_{\text{obs}}^{\langle j+1 \rangle} - \eta_{\text{obs}}^{\langle j \rangle} = 2\left(\mu_{\text{obs}}^{\langle k+1 \rangle} - \mu_{\text{obs}}^{\langle k+2 \rangle} + \eta_{\text{obs}}^{\langle j+2 \rangle} - \eta_{\text{obs}}^{\langle j+1 \rangle}\right)$$

that cannot violate inequalities (11) and (12) simultaneously. It also should be noted that strictness of inequalities (9) and (10) can be violated when $y \notin [\eta;\ \mu]$.

## 5 Backward Adjustment

If $y \notin [\eta;\ \mu]$ then, at the current stage, forward adjustment is canceled. If $y > \mu$ then the current right endpoint $\mu$ is discarded and not counted to be an obsolete one, being updated to the right as (2). Along with that, the counter $j$ is increased by 1 and the left endpoint $\eta$ is updated to the right as (5). This is backward adjustment (Figure 3), where one of the endpoints is moved backwards.

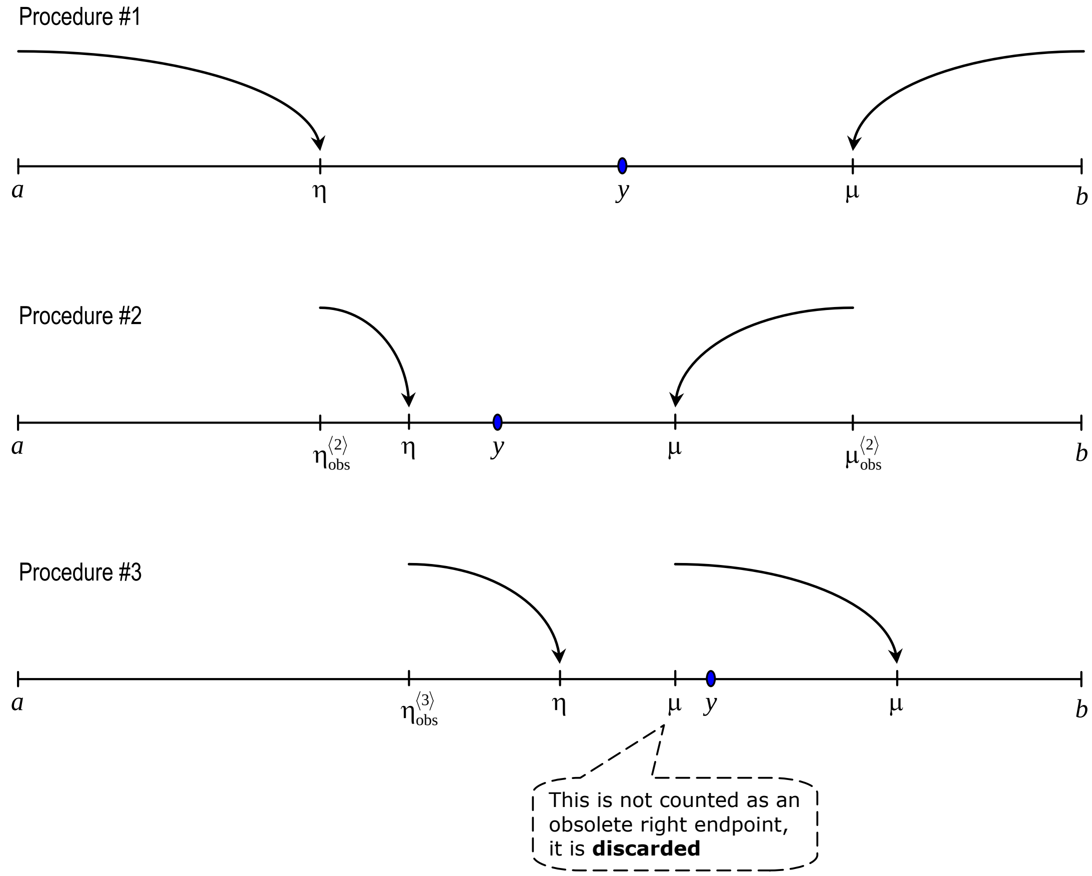


Figure 3: An example of the backward adjustment executed straight after the third procedure

Similarly, if $y < \eta$ then the left endpoint $\eta$ is discarded and not counted to be an obsolete one, being updated to the left as (3). Along with that, the counter $k$ is increased by 1 and the right endpoint $\mu$ is updated to the left as (4). Since here the left endpoint is moved to the left, the left backward adjustment and right backward adjustment are discerned.

The example in Figure 3, which is scaled precisely, shows that the new subinterval $[\eta;\ \mu]$ became wider after the right backward adjustment. But does it descend always? Does backward adjustment imply a new subinterval $[\eta;\ \mu]$ becomes wider? The answer to this question is negative owing to a counterexample.

**Theorem 2.** When backward adjustment occurs in the second expert procedure, the new subinterval $[\eta;\ \mu]$ is always shorter than the initial interval $[a;\ b]$.

**Proof.** Suppose that $y > \mu$ in the second procedure. Then the length of the new subinterval $[\eta;\ \mu]$ is

$$\frac{y+b}{2}-\frac{\eta_{\mathrm{obs}}^{\langle 2\rangle}+y}{2}=\frac{b-\eta_{\mathrm{obs}}^{\langle 2\rangle}}{2}. \tag{13}$$

The length of the first subinterval is $(b-a)/2$, so the difference

$$\frac{b-a}{2}-\frac{b-\eta_{\mathrm{obs}}^{\langle 2\rangle}}{2}=\frac{\eta_{\mathrm{obs}}^{\langle 2\rangle}-a}{2}$$

is always positive as $\eta_{\mathrm{obs}}^{\langle 2\rangle}>a$, confirming the theorem assertion for the right backward adjustment. For the left backward adjustment, $y<\eta$ in the second procedure, and the length of the new subinterval $[\eta;\ \mu]$ is

$$\frac{y+\mu_{\mathrm{obs}}^{\langle 2\rangle}}{2}-\frac{a+y}{2}=\frac{\mu_{\mathrm{obs}}^{\langle 2\rangle}-a}{2}. \tag{14}$$

The difference

$$\frac{b-a}{2}-\frac{\mu_{\mathrm{obs}}^{\langle 2\rangle}-a}{2}=\frac{b-\mu_{\mathrm{obs}}^{\langle 2\rangle}}{2}$$

is always positive as $b>\mu_{\mathrm{obs}}^{\langle 2\rangle}$. This completes confirmation of the theorem assertion.

□

Surely, the length of the new subinterval $[\eta;\ \mu]$ after backward adjustment occurring in the $h$-th expert procedure is calculated similarly to (13) and (14), $h\in\mathbb{N}\setminus\{1\}$. It is

$$\frac{y+b}{2}-\frac{\eta_{\mathrm{obs}}^{\langle h\rangle}+y}{2}=\frac{b-\eta_{\mathrm{obs}}^{\langle h\rangle}}{2} \tag{15}$$

and

$$\frac{y+\mu_{\mathrm{obs}}^{\langle h\rangle}}{2}-\frac{a+y}{2}=\frac{\mu_{\mathrm{obs}}^{\langle h\rangle}-a}{2} \tag{16}$$

for the right and left backward adjustment, respectively. This is kind of hard backward adjustment inasmuch as new lengths (15) and (16) may come much (unexpectedly) longer compared to preceding ones.

# 6 Softer Backward Adjustment

Backward adjustment updates (2) and (3) may be good at a few starting procedures. However, when the interval $[\eta;\ \mu]$ is sufficiently narrow, those updates appear grosser. A prompt is in Figure 3, where instead of (2), the new value of $\mu$ can be found as $(y+\mu_{\mathrm{obs}}^{\langle 2\rangle})/2$.

Generally, if

$$y\leqslant\mu_{\mathrm{obs}}^{\langle k\rangle} \tag{17}$$

then a new right endpoint is

$$\mu=\frac{y+\mu_{\mathrm{obs}}^{\langle k\rangle}}{2}. \tag{18}$$

While (17) is false, the counter $k$ is decreased by 1. And a new left endpoint is

$$\eta=\frac{\eta_{\mathrm{obs}}^{\langle j\rangle}+y}{2} \tag{19}$$

by

$$y \geqslant \eta_{\text{obs}}^{\langle j \rangle}. \tag{20}$$

While inequality (20) is false, the counter $j$ is decreased by 1. An appropriate right endpoint must not be changed if

$$\frac{\mu_{\text{obs}}^{\langle k \rangle} - y}{b-a} \leqslant \xi \tag{21}$$

by inequality (17), i.e. the current parameter's value $y$ is too close to this endpoint. Passing down with (21), the appropriate endpoint $\mu = \mu_{\text{obs}}^{\langle k \rangle}$ is set. If

$$\frac{y - \eta_{\text{obs}}^{\langle j \rangle}}{b-a} \leqslant \vartheta \tag{22}$$

by inequality (20) then an appropriate left endpoint is not changed as well, and $\eta = \eta_{\text{obs}}^{\langle j \rangle}$. No obsolescence counter is increased.

The backward adjustment by (17), (18), and (20), (19), and (21), (22), is far softer than that by hard updates (2) and (3). The softer updates (18) and (19) ensure higher quality of adjustment fitting experts with improved (improvable) proficiencies. Commonly, getting started with hard backward adjustment, it is shifted to softer backward adjustment after a few starting procedures.

# 7 An Early Stop Condition

Adjustment is not executed when the current parameter's value $y$ enclosed within the subinterval $[\eta;\ \mu]$ is simultaneously too close to both left and right endpoints. Algebraically, this occurs when $y \in [\eta;\ \mu]$ and both inequalities (6) and (7) are true. Apparently, if this both-sided closeness recurs in succession, the corresponding statistical stability is likely to be induced. Then the further uncertainty reduction is unlikely, and so the adjustment can be stopped.

Let $H_*$ be a maximal number of successive expert procedures, during which the both-sided closeness recurred, whereupon the adjustment is stopped. Of course, $H_* < H$ by $H_* \in \mathbb{N} \setminus \{1\}$ if $H$ is given, but then $H_*$ is optional. If $H$ is not given, then $H_*$ must be given obligatorily as that $H_* \in \mathbb{N} \setminus \{1\}$.

For algorithmic representation of a condition of an early stop, we need flags indicating directions of branching and subsequent actions. Denote the flags by $r_\eta$ and $r_\mu$ for the left and right endpoints, respectively. A convention is that the flag is equal to 2 when $y \in [\eta;\ \mu]$ and the current parameter's value $y$ is too close to the endpoint. If $y \in [\eta;\ \mu]$ but $y$ is not close to the endpoint, and while the hard backward adjustment, the flag is set to 0. While the softer backward adjustment, the flag is set to 1 if $y$ is too close to the endpoint, otherwise it is set to 0. It is easy to see that a case $r_\eta = r_\mu = 1$ is impossible because only one of the endpoints is moved backwards while the backward adjustment.

The flag for the given $H$ is $d = 1$. The flag for the given $H_*$ is $d_* = 1$. So when $r_\eta \cdot r_\mu = 4$ and $d_* = 1$, a counter $h_*$ for successive expert procedures, during which the both-sided closeness recurred, is increased by 1. The adjustment is stopped when the counter $h_*$ reaches its maximum, that is, $h_* = H_*$. When $r_\eta \cdot r_\mu = 2$ and $d_* = 1$, the counting for those successive expert procedures starts afresh by setting $h_* = 1$ back.

# 8 Interval Uncertainty Reduction Algorithm

The above-described adjustment steps constitute a partially-paralleled sequence represented as an algorithm of reducing interval uncertainty via adjusting the parameter by expert procedures (Figure 4). Depending on the rate of experts' proficiencies, hard and softer backward adjustment modes can be switched. The backward adjustment modes of the endpoints do not necessarily coincide, especially since a single endpoint is backwards updated at a stage. These modes are regularized so that the adjustment distinctive quality conforms to the rate of experts' proficiencies.

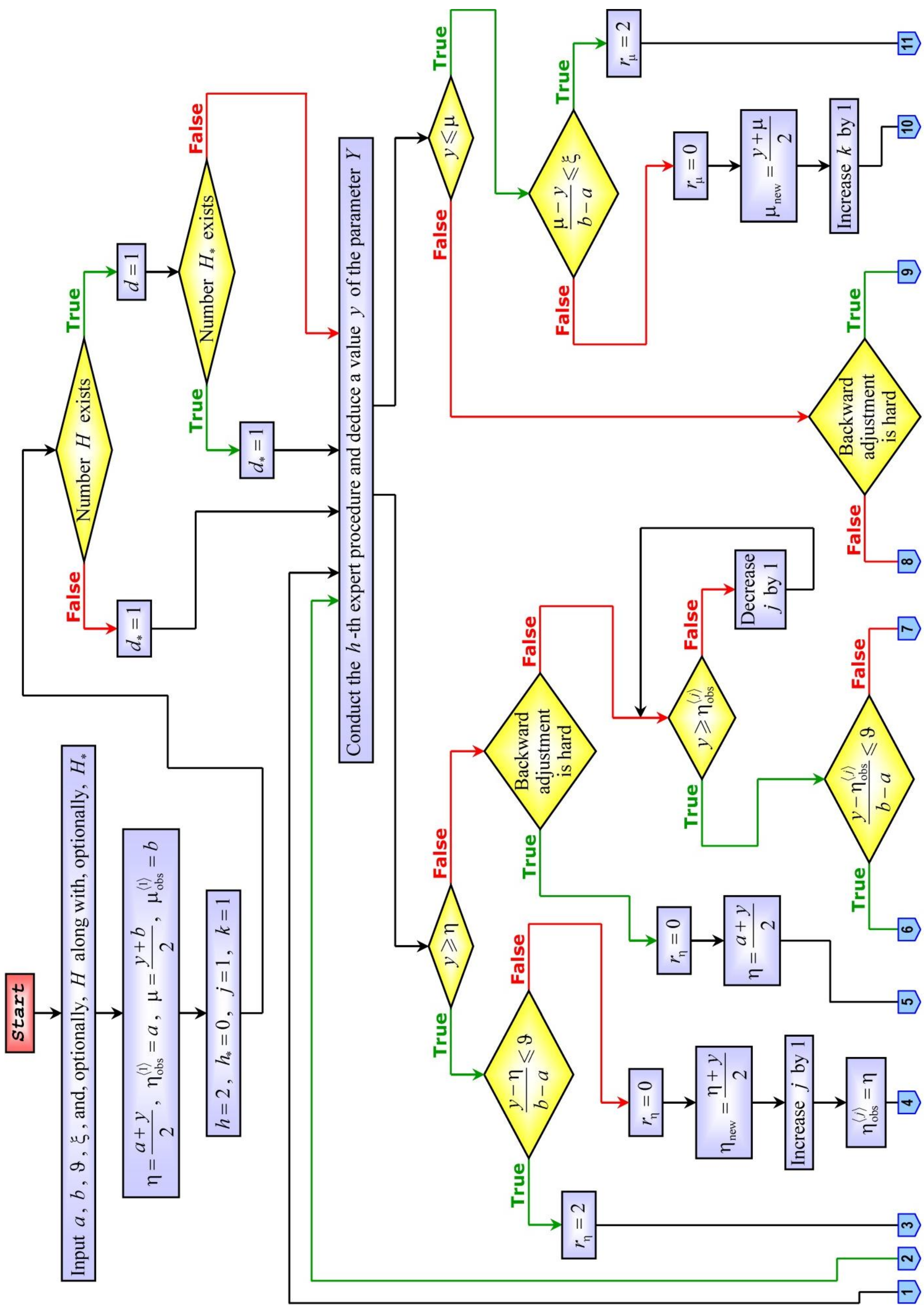


Figure 4: An algorithm of reducing interval uncertainty via adjusting the parameter by expert procedures (beginning)

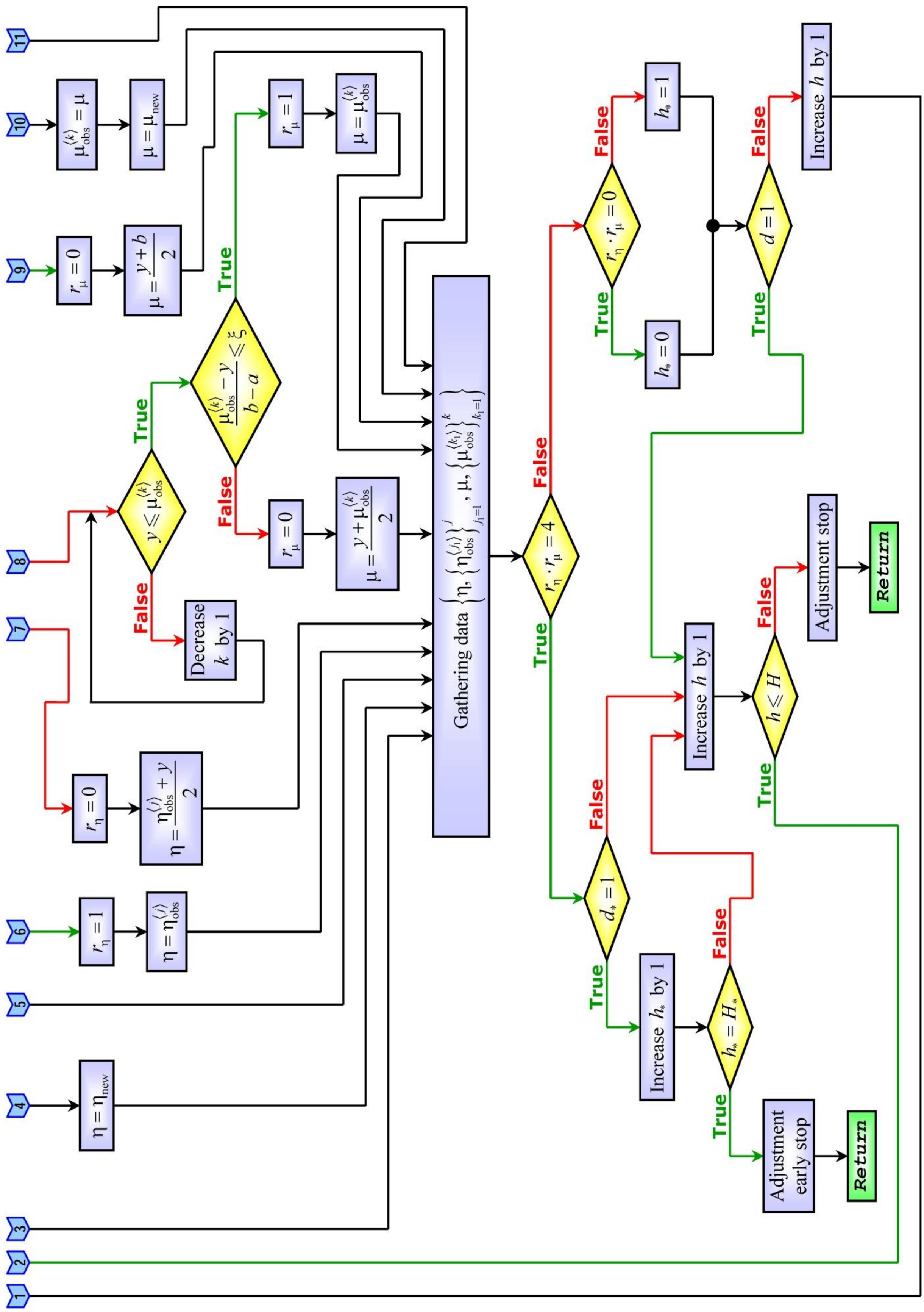


Figure 4: An algorithm of reducing interval uncertainty via adjusting the parameter by expert procedures (completion)

Parallelization begins when the next expert procedure conduction is completed and a value $y$ of the parameter $Y$ is deduced (calculated). Parallelization ends in gathering data

$$\left\{\eta, \left\{\eta_{\text{obs}}^{\langle j_1 \rangle}\right\}_{j_1=1}^{j}, \mu, \left\{\mu_{\text{obs}}^{\langle k_1 \rangle}\right\}_{k_1=1}^{k}\right\}$$

before deciding on whether to continue or stop. The early stop is possible after a succession of $H_*$ forward adjustments, when the endpoints are not changed.

**Theorem 3.** If

$$\mu - \eta > \min\left\{\vartheta \cdot (b-a), \xi \cdot (b-a)\right\}, \tag{23}$$

then a changeover of the subinterval $[\eta; \mu]$ is possible. Otherwise, if

$$\mu - \eta \leqslant \min\left\{\vartheta \cdot (b-a), \xi \cdot (b-a)\right\}, \tag{24}$$

then a changeover happens if only $y < \eta$ or $y > \mu$ (i. e., outside of the subinterval).

**Proof.** From inequality (23), we can get either

$$\mu - \eta > \vartheta \cdot (b-a) \tag{25}$$

or

$$\mu - \eta > \xi \cdot (b-a). \tag{26}$$

Inequality (25) implies that

$$(\mu - \delta_\mu) - \eta > \vartheta \cdot (b-a)$$

by sufficiently small $\delta_\mu > 0$, that gives the inverse inequality (6)

$$\frac{y-\eta}{b-a} > \vartheta$$

for $y \geqslant \eta$ and $y \leqslant \mu$. Similarly, inequality (26) implies that

$$\mu - (\eta + \delta_\eta) > \xi \cdot (b-a)$$

by sufficiently small $\delta_\eta > 0$, that gives the inverse inequality (7)

$$\frac{\mu - y}{b-a} > \xi$$

for $y \leqslant \mu$ and $y \geqslant \eta$. Both cases are followed with the changeover by updates (5) and (4), respectively. On the other hand, inequality (24) gives simultaneous inequalities (6) and (7) implying no changes for $\eta$ and $\mu$ if only $y \in [\eta; \mu]$. If the current parameter's value falls outside of the subinterval, a changeover is inevitable.

□

Clearly, forward adjustment is meant to be, but backward adjustment is undesirable. If experts learn to make their judgments or estimations more accurate then, as procedures go by, backward adjustment is expected to disappear completely. This is very urgent for unlimited number of procedures to be conducted, when just integer $H_*$ is given. In this case, once next backward adjustment is made, a succession of forward adjustments or non-changed endpoints must happen before the early stop. Then, what is the length of the subinterval $[\eta; \mu]$ expected to be? The following assertion answers this question.

**Theorem 4.** For unlimited number of procedures to be conducted, if the ultimate backward adjustment is made at a stage, the length of the final subinterval $[\eta;\ \mu]$ can be less than

$$\max\{\vartheta \cdot (b-a),\ \xi \cdot (b-a)\} \tag{27}$$

followed a succession of forward adjustments or non-changed endpoints and the corresponding early stop.

**Proof.** Suppose that

$$\mu - \eta > (\vartheta + \xi) \cdot (b-a) \tag{28}$$

after the ultimate backward adjustment is made at a stage. Without losing generality, let $\vartheta > \xi$. We always can take $\vartheta$ and $\xi$ such that

$$\vartheta - \xi = m\xi \quad \text{by} \quad m \in \mathbb{N} \setminus \{1\}$$

and

$$\mu - \eta - (\vartheta + \xi) \cdot (b-a) = \beta < \xi \cdot (b-a),$$

whence

$$\vartheta = (m+1)\xi$$

and

$$\mu - \eta = (b-a)(m+1)\xi + \beta + \xi \cdot (b-a) = (b-a)(m+2)\xi + \beta. \tag{29}$$

Assume that, after a procedure,

$$y = \eta + \delta \quad \text{by} \quad \delta < \xi \cdot (b-a). \tag{30}$$

Then $y \in [\eta;\ \mu]$, inequality (6) is true, and inequality (7) is false. Therefore, the left endpoint remains the same, and the new right endpoint is

$$\mu_{\text{new}} = \frac{y + \mu_{\text{obs}}^{\langle k_* \rangle}}{2} \quad \text{by} \quad \mu_{\text{obs}}^{\langle k_* \rangle} = \mu \quad \text{for some} \quad k_* \in \mathbb{N}.$$

With statement (29),

$$\mu_{\text{new}} = \frac{y + \mu_{\text{obs}}^{\langle k_* \rangle}}{2} = \frac{y + \eta + (b-a)(m+2)\xi + \beta}{2}$$
$$= \frac{2\eta + \delta + (b-a)(m+2)\xi + \beta}{2}.$$

The new subinterval length is

$$\mu_{\text{new}} - \eta = \frac{y + \mu_{\text{obs}}^{\langle k_* \rangle}}{2} - \eta$$
$$= \frac{2\eta + \delta + (b-a)(m+2)\xi + \beta}{2} - \eta$$
$$= \frac{\delta}{2} + \frac{(b-a)(m+2)\xi + \beta}{2}$$
$$= \frac{\delta + \beta}{2} + \frac{(b-a)(m+1)\xi}{2} + \frac{\xi \cdot (b-a)}{2}.$$

It must be compared to

$$\max\left\{\vartheta\cdot(b-a),\ \xi\cdot(b-a)\right\}=\vartheta\cdot(b-a)=(b-a)(m+1)\xi\,. \tag{31}$$

Inasmuch as

$$\delta<\xi\cdot(b-a)\ \text{ and }\ \beta<\xi\cdot(b-a),$$

we conclude that

$$\frac{\delta+\beta}{2}<\xi\cdot(b-a),$$

whence

$$\mu_{\text{new}}-\eta<\frac{(b-a)(m+1)\xi}{2}+\frac{3\xi\cdot(b-a)}{2}=\frac{\xi\cdot(b-a)(m+4)}{2}\,. \tag{32}$$

The difference between (31) and the right term of inequality (32) is

$$(b-a)(m+1)\xi-\frac{\xi\cdot(b-a)(m+4)}{2}=\xi\cdot(b-a)\left(\frac{m}{2}-1\right),$$

which is positive by $m>2$ and turns into zero by $m=2$. Thus the length of the current subinterval

$$[\eta;\ \mu]=[\eta;\ \mu_{\text{new}}]$$

is less than (27):

$$\mu-\eta<\max\left\{\vartheta\cdot(b-a),\ \xi\cdot(b-a)\right\}. \tag{33}$$

If a succession of forward adjustments or non-changed endpoints follows further, this subinterval is either narrowed, due to inequality (23) until inequality (24) turns true, or not changed, until the corresponding early stop becomes. Another assumption of that $\vartheta<\xi$ induces a symmetrically proof for inequality (33).

□

Theorem 4 shows possibility of inequality (33) owing to the assumption (30) and the condition of that the current parameter's value $y$ is "trapped" in the subinterval $[\eta;\ \mu]$. Hence one more question is what the maximal length of the final subinterval $[\eta;\ \mu]$ is when the conditions of Theorem 4 hold.

**Theorem 5.** For unlimited number of procedures to be conducted, if the ultimate backward adjustment is made at a stage, the maximal length of the final subinterval $[\eta;\ \mu]$ is limited to

$$(\vartheta+\xi)\cdot(b-a) \tag{34}$$

but never reaches the value (34).

**Proof.** Suppose that inequality (28) holds after the ultimate backward adjustment is made at a stage. Then, after every next procedure, value $\mu-\eta$ becomes smaller due to forward adjustments for both endpoints by

$$y\in\left(\eta+\vartheta\cdot(b-a);\ \mu-\xi\cdot(b-a)\right), \tag{35}$$

or by a non-changed endpoint and the other endpoint's update by either

$$y\in\left[\eta;\ \eta+\vartheta\cdot(b-a)\right] \tag{36}$$

or

$$y\in\left[\mu-\xi\cdot(b-a);\ \mu\right]. \tag{37}$$

Consequently, in a finite number of stages, we get

$$\mu-\eta-(\vartheta+\xi)\cdot(b-a)=\beta\leqslant\min\left\{\vartheta\cdot(b-a),\ \xi\cdot(b-a)\right\}. \tag{38}$$

If, in the next procedure, membership (35) is true, then the new subinterval length is

$$\begin{aligned}\mu_{\text{new}}-\eta_{\text{new}}&=\frac{y+\eta+\vartheta\cdot(b-a)+\beta+\xi\cdot(b-a)}{2}-\frac{\eta+y}{2}\\&=\frac{\vartheta\cdot(b-a)+\beta+\xi\cdot(b-a)}{2}<(\vartheta+\xi)\cdot(b-a),\end{aligned}$$

where (38) is used. Otherwise, if either membership (36) or (37) is true, then the new subinterval length is either

$$\begin{aligned}\mu_{\text{new}}-\eta&=\frac{y+\eta+\vartheta\cdot(b-a)+\beta+\xi\cdot(b-a)}{2}-\eta\\&=\frac{y-\eta+\vartheta\cdot(b-a)+\beta+\xi\cdot(b-a)}{2}\leqslant(\vartheta+\xi)\cdot(b-a)\end{aligned}$$

by using (38) and (36), or

$$\begin{aligned}\mu-\eta_{\text{new}}&=\eta+\vartheta\cdot(b-a)+\beta+\xi\cdot(b-a)-\frac{\eta+y}{2}\\&=\frac{\eta-y}{2}+\vartheta\cdot(b-a)+\beta+\xi\cdot(b-a)\leqslant(\vartheta+\xi)\cdot(b-a)\end{aligned}$$

by using (38) and $y-\eta\geqslant2\beta$ ensuing from (37), respectively. Nonetheless, if the event with the equality

$$\mu-\eta=(\vartheta+\xi)\cdot(b-a) \tag{39}$$

happens, there is zero probability of that $y=\eta+\vartheta\cdot(b-a)$ in any next procedure before the early stop, because the current parameter's value $y$ here is a value of a factual continuous variate co-named by $Y$.

□

Assertions of Theorem 4 and Theorem 5 might appear those whose dealing with ideal conditions (without backward adjustment) is unrealizable. This is really so for inexperienced experts and poorly selected endpoints of the initial interval $[a;\ b]$ along with groundlessly diminished $\vartheta$ and $\xi$. Nevertheless, $a$ and $b$ are presumed to be taken such that the initial interval would be a little bit wider than it would have been according to non-preconceived estimation. This lets experts learn longer and get more proficient before estimating and deciding on pretty narrow subinterval $[\eta;\ \mu]$. Thus, along with appropriate tolerances, conditions of Theorem 4 and Theorem 5 seem realizable. These theorems and Theorem 3 help to reckon an aftermath of the interval uncertainty reduction algorithm application.

# 9 Application

The algorithm is applicable in study fields engaging expert estimations and procedures prior to any statistical inferences or point estimations. Figure 4 suggests a preparatory refinement of the initial interval using only data of expert judgments. One should remember that the interval uncertainty reduction is realized by the adjustment of the parameter belonging to a subinterval, which is to be finally found at early stop or when all the scheduled procedures are conducted. Although one of integers $H$ and $H_*$ is optional, them both are recommended to be set. Both tolerances should be equal, i. e. $\vartheta=\xi$, unless there are significant reasons to set them unequal. Figure 5 features a real-scaled interval uncertainty reduction in 18 stages by three backward adjustments, where equal tolerances are one eighth of the initial interval, and $H_*=5$ by non-specified number $H$. The parameter is a duration of the warm run-in process of a four-stroke engine. In such a way, the duration is time interval $[\eta;\ \mu]$-optimized by minimizing wear.

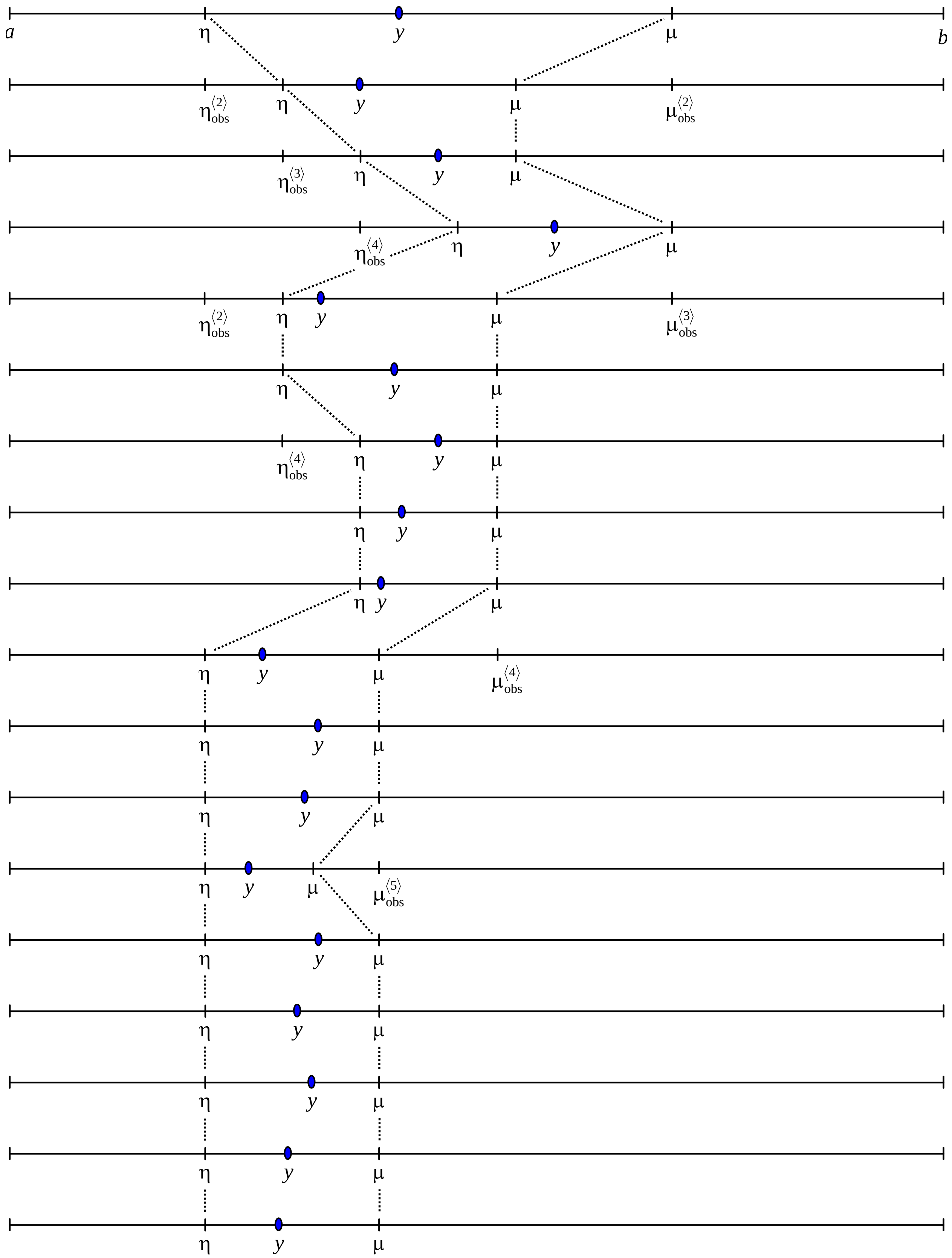


Figure 5: An example of the run-in process duration uncertainty reduction applied for settling a low-cost run-in [27] (the wear experience is poorer at the starting stages, but it is bettering further and its trace is the decreasing scatter of the current parameter's values, which are like to become stable; so the early stop has fired after the 18-th procedure)

In addition to the reasonable run-in duration settling, the uncertainty reduction algorithm in Figure 4 can be applied for non-destructive checkup, production (assembly) line supervision, construction works (handling the building tolerances), and for solving other problems in technical and technological systems dealing with intervals. Many rules of thumb are to be rationalized more. This concerns problems with time windows, dispatching, scheduling, etc. Besides, the algorithm is ready to adjust parameters of neural networks whose values are ordinarily set loose for new statistical approximation problems. The adjustment fits to be straightforwardly applied in modeling social systems, where ranges of uncertainty are very high and the observed object's properties cannot be studied without expert judgments.

The algorithm, however, does not guarantee that the width of the interval will be reduced by half or whatsoever. If a number of expert procedures to be conducted is specified, then a situation with a series of alternating forward and backward adjustments is possible. Then the subinterval in (1) may be pretty much the same as the initial interval. Such situations may be caused with one or several of the following reasons:

1. A specified number of expert procedures to be conducted is insufficient, i. e., it is too small.
2. The initial interval does not reflect the real admissibility of the object's parameter values, meaning that the initial interval is either biased or taken narrower (that is, not all admissible values of the object's parameter are included).
3. Small group of experts is involved (their consensus value is inconsistent).
4. The interval uncertainty cannot be principally reduced due to its stochastic nature. For instance, the uncertainty reduction is impossible in weather prediction based only on expert judgments, without referring to statistical analysis of the corresponding natural factors and meteorological conditions.

If an item of those four reasons is true, the interval uncertainty reduction may be canceled. It is absolutely canceled if items #1 and #2 are true. A small group of experts can render the uncertainty only when the number of expert procedures is sufficiently great or not specified.

# 10 Discussion

The adjustment herein is associated with uncertainty reduction and vice versa. Forward adjustment is naturally favorable, but backward adjustment is practically inevitable. Too many backward adjustments is a bad event hinting at we got poor initial endpoints or/and inexperienced experts. The worse event is when forward and backward adjustments shift one another for a long series. At such an event, the early stop cannot fire (in the run-in process duration uncertainty reduction example by Figure 5, the early stop is thwarted twice — at the 10-th stage and directly at the 14-th stage). The interval uncertainty reduction algorithm, after all, does not predict a series of alternating forward and backward adjustments. If such a series happens, integers $H$ and $H_*$ are unlikely of their influence. Probably, endpoints of the initial interval were selected inappropriately or (and) the tolerances are smaller than they should have been.

The example of the run-in process duration uncertainty reduction in Figure 5 is a pattern where the closeness is not treated differently from both sides. Indeed, wear and its intensity come at saturation for longer run-in processes. But even if it is so, the tolerances are seen to be set equal at the opening studies. Any complexifications may lead to the interval reduction misinterpretation and delays.

The adjustment is performed by dichotomization. An important question is raised thereof. Why do we divide exactly by 2? Not, say, by 3, cutting off the thirds from the left and right? The answer is the dichotomization should be the simplest, which is the division by 2. All the above-mentioned fields are studied under conditions of short-termed observations and poor knowledge, so any other division cannot be perfectly substantiated.

The proved theorems prompt us about plausible trends of the algorithm. Theorem 5 states that once $y$ is "trapped" within $(\vartheta+\xi)\cdot(b-a)$-length interval, the subinterval $[\eta;\ \mu]$ will not become wider and

$$\mu-\eta<(\vartheta+\xi)\cdot(b-a)$$

for accurate enough expert estimations following it. However, if $y$ was restricted to discrete values, then equality (39) would be possible. In general, for discrete interval uncertainty, the uncertainty reduction algorithm in Figure 4 is hardly applicable.

An open issue exists in the adjustment stop by having conducted all $H$ procedures and not reaching the statistical stability related to the number $H_*$. This issue seems grander when the adjustment stops after a backward adjustment update. Another drawback is that the designed algorithm is weakly connected to the law of large numbers. For sufficiently great $H_*$ serving to reach the statistical stability, however, the connection is clearer.

The designed algorithm contributes a short-termed interval narrowing to the field of interval uncertainty reduction. As number of observations increases (duration of observations extends) by the longer term of application, the algorithm tends to be a statistical method for cutting off outermost parts of the initial interval, which are hit the least (Figure 6). Operations over intervals (interval calculus) have no relation to the algorithm. An influence upon fuzzy optimization and decision making is revealed when short-termed events are studied by minor number of observations [12, 17, 25, 29]. A critical number of observations exists (some $H$ or $H_*$), below which the algorithm cannot produce credible results. Above this number and a little bit up (to the right), operations over intervals still make sense.

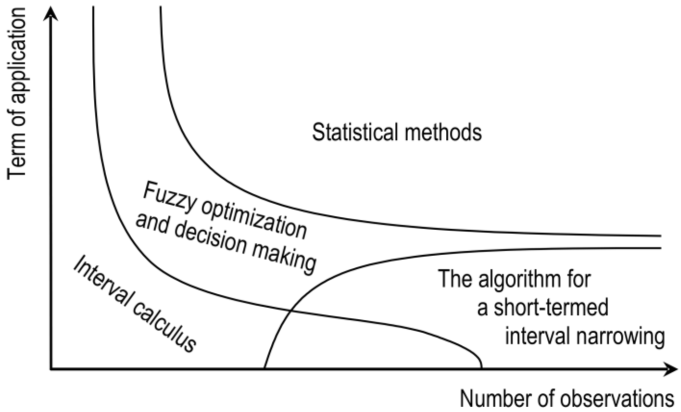

Figure 6: The designed algorithm within the field of interval uncertainty reduction

# 11 Conclusion

The designed algorithm allows reducing interval uncertainty based on expert estimations and weak statistical inferences. The weakness implies immediately narrowing an interval towards an average-like (consensus) of expert observations (judgments). The narrowing is equivalent to successive adjustment performed via division-by-2 dichotomization. In fact, the reduction of an interval of the parameter's values is equivalent to the adjustment of this parameter. This is a supplement "under" statistical methods to the field of interval uncertainty reduction.

The rate of experts' proficiencies is conformed to the adjustment quality by switching hard and softer backward adjustment modes. Despite estimations of experts with poor proficiencies, the adjustment quality is improved by proper switching. At a few starting procedures, hard backward adjustment updates may be good, especially if the initial interval is given intentionally wider. As the number of the passed procedures grows and the interval becomes narrower, it is reasonable to stand by at the softer backward adjustment.

Applicability of the interval uncertainty reduction algorithm calls for the fine-tuned integers $H$ and $H_*$ by appropriate tolerances. The algorithm is applicable to processes whose stochastic nature is minimal or moderate. It fits best when the object's statistical data are poor and terms of observations are short. Despite the integer $H_*$ is more crucial (in particular, for the early stop condition), number of procedures to be conducted is recommended to be specified, or else it is running to be unlimited. The infiniteness of expert procedures may happen if $H_*$ is too great or tolerances are small. Nevertheless, parallelization of the algorithm ensures twice faster data processing.

For further research development, division-by-$q$ dichotomization must be considered for $q \neq 2$. The value $q$ implies cutting off the $q^{-1}$-th parts from the left and right, so $q > 1$. It is believed to be an adaptation to various situations while expert procedures are conducted. At a few starting procedures, taking $q < 2$ is suitable. As the interval becomes narrower, taking $q > 2$ is prudent for cutting off fewer parts. Cutting fewer parts is also reasonable for the shorter terms of observations.

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