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# Cost-Time Trade-Off Pairs for Cost Varying Transportation Problem

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**Abstract**

This paper draws attention on the problem formulation and solution procedure of a bi-objective transportation problem(TP). In this TP, one objective is to minimize the total transportation cost and other is to minimize the time for completion of the transportation. In this TP, the transportation cost is not constant; it varies by depending on the capacity of vehicles as well as amount of transport quantity. An algorithm is used to determine unit transportation costs which deals this TP in a bi-level bi-objective mathematical model of mixed-integer type. To solve this model, use an initial allocation procedure like Modified Vogels Approximation Method(MVAM) and then find optimal solution by well known UV-method. After that find out the total elapsed time. This model suggests several cost-time pairs which is more realistic today. Numerical examples is presented to support this model.


**Keywords:** bi-objective transportation problem, cost varying transportation problem(CVTP), bi-level programming, mixed-integer programming, MVAM

## 1 Introduction

Transportation problem is famous in operation research for its wide application in real life. This is a special kind of the network optimization problems in which goods are transported from a set of sources to a set of destinations subject to the supply and demand of the source and destination, respectively, such that the total cost of transportation is minimized. The basic transportation problem was originally developed by Hitchcock in 1941 [7]. The first solution procedure of TP was developed by G. B. Dantzig [5] and referred as North West Corner Method (NWCM) by Charnes and Cooper [4]. This is the method of finding an IBFS of TP which consider the north-west-corner cost cell at every stage of allocation. Then the Least Cost Method (LCM) [1, 8] consists in allocating as much as possible in the lowest cost cell of the Transportation Table (TT) in making allocation in every stage. Vogels Approximation Method (VAM) [24, 12, 3, 11] and Extremum Difference Method (EDM) [9] provides comparatively better Initial Basic Feasible Solution. The problem of minimizing transportation cost has been studied since long and is well known [24, 9, 3, 23, 11]. TP in general are concerned with distributing any single commodity from any group of supply centre, called sources, to any group of receiving centre, called destinations. A destination can receive its demand from one or more sources. Each source has a fixed supply of units, where the entire supply must be distributed to the destinations. Similarly, each destination has fixed demand of units, where the entire demand must be received from the sources.

In TP the following information are to be needed:
(P1) Available amount of the commodity at different origins.
(P2) Amounts demanded at different destinations.
(P3) The transportation cost of one unit of commodity from various origin to various destination.

In [20] Panda and Das considered that "the transportation cost of one unit of commodity from various origin to various destination" is not constant and it is varied by depending on capacity of vehicles. This type of transportation problem is named by us as cost varying transportation problem (CVTP). In [13, 14, 15, 17, 16, 18, 19] Panda and Das introduced some multi-objective CVTP models,but these objectives are

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homogeneous type(i.e. directly related to quantity). This paper represents a bi-objective CVTP where one objective is cost minimization and other is time minimization. The cost is directly related to quantity but time is not. The solution is made through proposed MVAM to allocate the initial basic feasible solution and optimality test.

In Classical TP is a single/multi objective problem where objective(s) is minimize total transportation cost or minimize to total time or both. But in proposed model not only variable cost is consider but maximum time taken for transporting the quantities is determined. This time is not total time of transportation. This time is obtained after determining the optimal allocation for cost i.e. after determining minimum transportation cost. Proposed model suggests trade pairs that means if any one need very urgent to fulfill his demands(requirements) he will have to pay more. The initial allocation and optimal techniques are similar. The mathematical model of a classical TP is LPP where as in proposed TP is a bi-level model.

Section 1 represents brief introduction. Section 2 presents basic model, some definitions, theorems, procedure of initial allocations and determination of optimal solution of classical TP. In subsection 2.2, proposed CVTP model with bi-objective is builded by determining unit transportation cost through an algorithm. Section 3 discusses the similar methodologies of initial allocation like North West Corner Rule(NWCR), Modified Matrix Minimum Method(MMMM),Modified Row Minimum Method(MRMM), Modified Column Minimum Method(MVAM), Modified Vogel's Approximation Method(MVAM) etc. and Solution procedure of proposed model. Lastly in this section give little discussion about proposed model. In section 4 numerical examples are presented which support the discission of proposed model. Finally in section 5, some conclusions are given.

# 2 Mathematical Description and Model Formulation

## 2.1 Classical Transportation Problem

Transportation problem is a special type of networks problems that for shipping a commodity from source (e.g., factories) to destinations (e.g., warehouse). Transportation model deal with get the minimum-cost plan to transport a commodity from a number of sources (m) to number of destination (n). Let $a_i$ is the number of supply units required at source $i(i = 1, 2, \ldots, m)$, $b_j$ is the number of demand units required at destination $j$ $(j = 1, 2, \ldots, n)$ and $c_{ij}$ represent the unit transportation cost for transporting the units from sources i to destination j. Using linear programming method to solve transportation problem, we determine the value of objective function which minimize the cost for transporting and also determine the number of unit can be transported from source $i$ to destination $j$.

If $x_{ij}$ is number of units shipped from source $i$ to destination $j$; the equivalent linear programming model will be **Model 1** as follows:
**Model 1**

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m \qquad (1) \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n \qquad (2) \\ & \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \\ & x_{ij} \geq 0 \quad \forall i, \quad \forall j. \end{aligned}$$

A transportation problem can be represent in the following tabulated form.

Table: Tabular representation of a multi-objective transportation problem

| | $D_1$ | $D_2$ | .. | $D_n$ | *stock* |
|---|---|---|---|---|---|
| $O_1$ | $c_{11}$ | $c_{12},$ | .... | $c_{1n},$ | $a_1$ |
| $O_2$ | $c_{21}$ | $c_{22},$ | .... | $c_{2n},$ | $a_2$ |
| .... | .... | .... | .... | .... | .... |
| $O_m$ | $c_{n1}$ | $c_{n2},$ | .... | $c_{nn},$ | $a_m$ |
| *Demand* | $b_1$ | $b_2$ | .... | $b_n$ | |

where $a_i$ is the quantity of material available at source $O_i, i = 1, \dots, m$, $b_j$ is the quantity of material required at destination $D_j, j = 1, \dots, n$, $c_{ij}$ is the unit cost of transportation from st source $O_i$ to destination $D_j$.

The following terms are to be defined with reference to the transportation problems.

**Definition 1: Feasible Solution (F.S.):**
*A set of non-negative allocations $x_{ij} \geq 0$ which satisfies (1), (2) is known as feasible solution.*

**Definition 2: Basic Feasible Solution (B.F.S.):**
*A feasible solution to a m-origin and n-destination problem is said to be basic feasible solution if number of positive allocations are $(m + n - 1)$.*

If the number of allocations in a basic feasible solutions are less than (m+n-1), it is called degenerate basic feasible solution (DBFS) otherwise non-degenerate basic feasible solution (NDBFS).

**Definition 3: Optimal Solution:**
*A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.*

**Theorem 2.1:** *The number of basic variables in a Transportation Problem(T.P.) is at most $(m + n - 1)$.*

**Theorem 2.2:** *There exits a F.S. in each Transportation Problem(T.P.).*

**Theorem 2.3:** *In each T.P. there exits at least one B.F.S. which makes the objective function a minimum.*

**Theorem 2.4:** *The solution of a T.P. is never unbounded.*

**Definition 4: Loop:**
*In the Transportation table, a sequence of cells is said to form a loop, if*
*(i) each adjacent pair of cells either lies in the same column or in the same row;*
*(ii) not more than two consecutive cells in the sequence lie in the same row or in the same column;*
*(iii) the first and the last cells in the sequence lie either in the same row or in the same column;*
*(iv) the sequence must involve at least two rows or two columns of the table.*

**Theorem 2.5:** *A sub-set of the columns of the coefficient matrix of a T.P. are linearly dependent, iff, the corresponding cells or a sub-set of them can be sequenced to form a loop.*

There are many procedures to determine initial basic feasible solution like North West Corner Rule, Row minimum method, Column minimum method, VAM method etc. In the following subsection,VAM method is depicted.

### 2.1.1 Vogel's Approximation Method(VAM)

In this method the allocation is made on the basis of the opportunity (or penalty or extra) cost that would be incurred if allocation in certain cells with minimum unit transportation cost were missed. The steps in Vogel's approximation method(VAM) are as follows:

**Step i.** Calculate the penalties for each row(column) by taking the differences between the smallest and next smallest unit transportation cost in the same row (column) and write them in brackets against the corresponding row (column).

**Step ii.** Select the row or column with the largest penalty. If there is a tie in the values of penalties, then it can be broken by selecting the cell where the maximum allocation can be made.

**Step iii.** Allocate as much as possible in the lowest cost of the row( or column) which is defined by the **Step ii.**

**Step iv.** Adjust the supply and demand and cross-out the satisfied row or column.

**Step v.** Repeat **Step i. Step ii.** until the entire available supply at various sources and demand at various destinations are fully satisfied.

### 2.1.2 Optimality Test:

In order to test for optimality we should follow the procedure as given bellow:

**Step O1.** Start with B.F.S. consisting of $m + n - 1$ allocation in independent positions.

**Step O2.** Determine a set of $m+n$ numbers $u_i, i=1,\dots,m$ and $v_j, j=1,\dots,n$ such that in each cell $(i,j)$ $c_{ij}=u_i+v_j$.
**Step O3.** Calculate cell evaluations (unit cost difference) $d_{ij}$ for each empty cell $(i,j)$ by using formula $d_{ij}=c_{ij}-(u_i+v_j)$.
**Step O4.** Examine the matrix of cell evaluation $d_{ij}$ for negative entries and conclude that
(i) If all $d_{ij}>0$, then Solution is optimal and unique.
(ii) If all $d_{ij}\geq 0$ and at least one $d_{ij}=0$, then solution is optimal and alternative solution also exists.
(iii) If at least one $d_{ij}<0$, then solution is not optimal.
If it is so, further improvement is required by repeating the above process after Step 5 and onwards.
**Step O5.** (i) See the most negative cell in the matrix $[d_{ij}]$.
(ii) Allocate $\theta$ to this empty cell in the final allocation table. Subtract and add the amount of this allocation to other corners of the loop in order to restore feasibility.
(iii) This value of $\theta$, in general is obtained by equating to zero the minimum of the allocations containing $-\theta$ (not $+\theta$) only at the corners of the closed loop.
(iv) Substitute the value of $\theta$ and find a fresh allocation table.
**Step O6.** Again, apply the above test for optimality till we find $d_{ij}\geq 0$.

## 2.2 Cost Varying Transportation Problem

Cost varying transportation problem (CVTP) represented as follows.
**Model 2**

$$\min \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij}=a_i, \quad i=1,\dots,m \tag{3}$$

$$\sum_{i=1}^{m} x_{ij}=b_j, \quad j=1,\dots,n \tag{4}$$

$$\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$$

$$x_{ij}\geq 0 \;\; \forall i, \;\; \forall j$$

$$\text{where } c_{ij} \text{ is not constant.} \tag{5}$$

Panda and Das [16, 20] considered that there are two types off vehicles $V_1, V_2$ from each source to each destination. Let $C_1$ and $C_2(>C_1)$ are the capacities(in unit) of the vehicles $V_1$ and $V_2$ respectively. $R_{ij}=(R^1_{ij}, R^2_{ij})$ represent transportation cost for each cell $(i,j)$; where $R^1_{ij}$ is the transportation cost from source $O_i, i=1,\dots,m$ to the destination $D_j, j=1,\dots,n$ by the vehicle $V_1$. And $R^2_{ij}$ is the transportation cost from source $O_i, i=1,\dots,m$ to the destination $D_j, j=1,\dots,n$ by the vehicle $V_2$. So, cost varying transportation problem can be represent in the following tabulated form.

Table: Tabular representation of cost varying transportation problem

| | $D_1$ | $D_2$ | .. | $D_n$ | *stock* |
|---|---|---|---|---|---|
| $O_1$ | $R^1_{11}, R^2_{11}$ | $R^1_{12}, R^2_{12}$ | .... | $R^1_{1n}, R^2_{1n}$ | $a_1$ |
| $O_2$ | $R^1_{21}, R^2_{21}$ | $R^1_{22}, R^2_{22}$ | .... | $R^1_{2n}, R^2_{2n}$ | $a_2$ |
| .... | .... | .... | .... | .... | .... |
| $O_m$ | $R^1_{m1}, R^2_{m1}$ | $R^1_{m2}, R^2_{m2}$ | .... | $R^1_{mn}, R^2_{mn}$ | $a_m$ |
| *Demand* | $b_1$ | $b_2$ | .... | $b_n$ | |

### 2.2.1 Determining $c_{ij}$

**Algorithm A1:**
**Step A1.1.** Since unit cost is not determined (because it depends on quantity of transport), so North-west

corner rule (because it does not depend on unit transportation cost) is applicable to allocate initial B.F.S.
**Step A1.2.** After the allocate $x_{ij}$ by North-west corner rule, for basic cell we determine $c^r_{ij}$ (unit transportation cost from source $O_i$ to destination $D_j$) as

$$c_{ij} = \begin{cases} \frac{p1_{ij}R1_{ij}+p2_{ij}R2_{ij}}{x_{ij}}, & \text{if } x_{ij} \neq 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases} \tag{6}$$

$$\begin{aligned} \text{where} \quad & p1_{ij},\ p2_{ij}, i = 1,\ldots,m; j = 1,\ldots,n \text{ are integer solution of} \\ \min \quad & p1_{ij}R1_{ij} + p2_{ij}R2_{ij} \\ \text{s.t.} \quad & x_{ij} \leq p1_{ij}C_1 + p2_{ij}C_2. \end{aligned}$$

**Step A1.3.** For non-basic cell $(i,j)$ possible allocation is the minimum of allocations in $i^{th}$ row and $j^{th}$ column (for possible loop). If possible allocation be $x_{ij}$, then for non-basic cell $c_{ij}$ (unit transportation cost from source $O_i$ to destination $D_j$) as

$$c_{ij} = \begin{cases} \frac{p1_{ij}R1_{ij}+p2_{ij}R2_{ij}}{x_{ij}}, & \text{if } x_{ij} \neq 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases} \tag{7}$$

$$\begin{aligned} \text{where} \quad & p1_{ij},\ p2_{ij}, i = 1,\ldots,m; j = 1,\ldots,n \text{ are integer solution of} \\ \min \quad & p1_{ij}R1_{ij} + p2_{ij}R2_{ij} \\ \text{s.t.} \quad & x_{ij} \leq p1_{ij}C_1 + p2_{ij}C_2. \end{aligned}$$

In this manner we convert cost varying transportation problem to a usual transportation problem but $c_{ij}$ is not fixed, it may be changed (when this allocation will not serve optimal value) during optimality test.
**Step A1.4.** During optimality test some basic cell changes to non-basic cell and some non-basic cell changes to basic cell, depends on running basic cell we first fix $c_{ij}$ by **Step A1.2** and for non-basic we fix $c_{ij}$ by **Step A1.3**.
**Step A1.5.** Repeat **Step A1.2.** to **Step A1.4.** until we obtain optimal solution.

### 2.2.2 Bi-level Mathematical Programming Cost Varying Transportation Problem

The Bi-level mathematical programming for 2-vehicle cost varying transportation problem is formulated in **Model 3** as follows:
**Model 3**

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \tag{8}$$

where $c_{ij}$ is determined by following mathematical programming

$$c_{ij} = \begin{cases} \frac{p1_{ij}R1_{ij}+p2_{ij}R2_{ij}}{x_{ij}}, & \text{if } x_{ij} \neq 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases}$$

$$\begin{aligned} \min \quad & p1_{ij}R1_{ij} + p2_{ij}R2_{ij} \qquad (9) \\ \text{s. t.} \quad & x_{ij} \leq p1_{ij}C_1 + p2_{ij}C_2 \\ & \sum_{i=1}^{m} x_{ij} = a_i, \quad i = 1,\ldots,m \\ & \sum_{j=1}^{n} x_{ij} = b_j, \quad j = 1,\ldots,n \\ & \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \\ & x_{ij} \geq 0 \ \ \forall i, \ \ \forall j \end{aligned}$$

where $p1_{ij},\ p2_{ij}, i = 1,\ldots,m; j = 1,\ldots,n$ are integers.

### 2.2.3 Bi-criteria Cost Varying Transportation Problem

Most of the practical transportation problems have two objectives : minimizing of cost and minimizing of time. The cost minimizing problem and time minimizing problem cannot be viewed as two independent problems. Most of the methods develop so far have given importance to minimize cost then time or to minimize time then cost. If one is interested in obtaining a solution which minimizes cost and time simultaneously is called bi-criteria transportation problem.

A bi-criteria cost varying transportation problem formulated as

**Model 4**

$$\min\ Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \tag{10}$$

$$\min\ T = \max[t_{ij}\ \ /x_{ij} > 0] \tag{11}$$

where $c_{ij}$ is determined by following mathematical programming

$$c_{ij} = \begin{cases} \frac{p1_{ij}R1_{ij}+p2_{ij}R2_{ij}}{x_{ij}}, & \text{if } x_{ij} \neq 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases}$$

$$\min\ p1_{ij}R1_{ij} + p2_{ij}R2_{ij} \tag{12}$$

s. t. $x_{ij} \leq p1_{ij}C_1 + p2_{ij}C_2$

$$\sum_{i=1}^{m} x_{ij} = a_i, \quad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} x_{ij} = b_j, \quad j = 1, \ldots, n$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

$$x_{ij} \geq 0\ \ \forall i,\ \ \forall j$$

where $p1_{ij},\ p2_{ij}, i = 1, \ldots, m; j = 1, \ldots, n$ are integers.

# 3 Solution Procedure of CVTP

## 3.1 Determination of IBFS

**Example 1:** Consider a cost varying transportation problem as

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7, 16$ | $12, 16$ | $8, 12$ | 45 |
| $O_2$ | $4, 6$ | $14, 18$ | $9, 15$ | 35 |
| $O_3$ | $10, 15$ | $17, 22$ | $5, 7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

The capacities of vehicles of $V_1$ and $V_2$ are respectively, $C_1 = 8$ and $C_2 = 14$.

To determine the IBFS we apply any one of the following procedure

**North-West corner Method (NWCM)**

**Step 1.** Compute min $(a_1, b_1)$. If $a_1 < b_1$, min $(a_1, b_1) = a_1$ and if $a_1 > b_1$, min $(a_1, b_1) = b_1$. Select $x_{11} =$ min $(a_1, b_1)$ allocate the value of $x_{11}$ in the cell $(1, 1)$.

**Step 2.** If $a_1 < b_1$, compute min $(a_2, b_1 - a_1)$. Select $x_{21} =$ min $(a_2, b_1 - a_1)$ and allocate the value of $x_{21}$ in the cell $(2, 1)$.

If $a_1 > b_1$, compute min $(a_1 - b_1, b_2)$. Select $x_{12} =$ min $(a_1 - b_1, b_2)$ and allocate the value of $x_{12}$ in the cell $(1, 2)$.

Let us now make an assumption that $a_1 - b_1 < b_2$. With this assumption the next cell for which some allocation is to made, is the cell $(2, 2)$.

If $a_1 = b_1$ then allocate 0 only in one of two cells $(2,1)$ or $(1,2)$. The next allocation is to be made cell $(2,2)$.

In general, if an allocation is made in the cell $(i+1, j)$ in the current step, the next allocation will be made either in cell $(i,j)$ or $(i, j+1)$.

The feasible solution obtained by this away is always a BFS.

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 30$ <br> $7,16$ | $x_{12} = 15$ <br> $12,16$ | $8,12$ | 45 |
| $O_2$ | $4,6$ | $x_{22} = 5$ <br> $14,18$ | $x_{23} = 30$ <br> $9,15$ | 35 |
| $O_3$ | $10,15$ | $17,22$ | $x_{33} = 10$ <br> $5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Modified row-minima method(MRMM)**

In this method, we first consider the first row and find the minimum cost cell. Let $(1,k)$ cell be the cell in the first row with minimum cost entities $(R^1_{l,k}, R^2_{l,k})$. We allot in this cell the maximum allocation, i.e,$x_{1k} = min(a_1, b_k)$.If $a_1 < b_k$, then $x_{1k} = a_1$ and we cross out the first row and consider the remaining tableau and proceed in same way. Again if $a_1 > b_k$, then $x_{1k} = b_k$ and we cross out the $k^{th}$ column and consider the remaining row of the tableau and proceed next in the same way. If $a_1 = b_1$,then either 1st row or the $k^{th}$ column will be crossed out and the remaining tableau will be consider.

By MRMM, the IBFS **Example 1** is given as follows.

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 30$ <br> $7,16$ | $12,16$ | $x_{13} = 15$ <br> $8,12$ | 45 |
| $O_2$ | $4,6$ | $x_{22} = 10$ <br> $14,18$ | $x_{23} = 25$ <br> $9,15$ | 35 |
| $O_3$ | $10,15$ | $x_{32} = 10$ <br> $17,22$ | $5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Modified column-minima method(MCMM)** This method is exactly same as the Row-minima method. In this method, we are to start with first column instead of first row and the successive steps we consider only columns.

By MCMM, the IBFS of **Example 1** is given as follows.

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7,16$ | $x_{12} = 20$ <br> $12,16$ | $x_{13} = 25$ <br> $8,12$ | 45 |
| $O_2$ | $x_{21} = 30$ <br> $4,6$ | $14,18$ | $x_{23} = 5$ <br> $9,15$ | 35 |
| $O_3$ | $10,15$ | $17,22$ | $x_{33} = 10$ <br> $5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Modified matrix-minima method(MMMM)**

This method finds a better starting solution. In this method we first find out the cell with minimum cost entities in the cost matrix and allocate in that cell the maximum allowable amount. We then cross out the satisfied row or column and adjust the amounts of supply and demand accordingly. We repeat the process with uncrossed out matrix and we are left at the end with exactly one uncrossed out row or column. If the cell with minimum cost is not unique, then any one of these cells may be selected for allotment.

By MCMM, the IBFS of **Example 1** is given as follows.

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7,16$ | $x_{12}=20$<br>$12,16$ | $x_{13}=25$<br>$8,12$ | <br>45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | <br>$14,18$ | $x_{23}=5$<br>$9,15$ | <br>35 |
| $O_3$ | <br>$10,15$ | <br>$17,22$ | $x_{33}=10$<br>$5,7$ | <br>10 |
| *Demand* | 30 | 20 | 40 | |

**Modified Vogel's Approximation Method(MVAM)**

In this method the allocation is made on the basis of the opportunity (or penalty or extra) cost entities that would be incurred if allocation in certain cells with minimum unit transportation cost entities were missed. The steps in modified Vogel's approximation method(MVAM) are as follows:

**Step V.1.** Calculate the penalties for each row(column) by taking the differences between the smallest and next smallest transportation cost entities in the same row (column) and write them in brackets against the corresponding row (column).

**Step V.2.** Select the row or column with the largest penalty entities. If there is a tie in the values of penalties entities, then it can be broken by selecting the cell where the maximum allocation can be made.

**Step V.3.** Allocate as much as possible in the lowest cost entities of the row( or column) which is defined by the **Step V.2.**

**Step V.4.** Adjust the supply and demand and cross-out the satisfied row or column.

**Step V.5.** Repeat **Step V.1. Step V.2.** until the entire available supply at various sources and demand at various destinations are fully satisfied.

The solution of **Model 4** is described by following proposed algorithm. By modified VAM method initial B.F.S. of **Example 1** is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7,16$ | $x_{12}=20$<br>$12,16$ | $x_{13}=25$<br>$8,12$ | <br>45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | <br>$14,18$ | $x_{23}=5$<br>$9,15$ | <br>35 |
| $O_3$ | <br>$10,15$ | <br>$17,22$ | $x_{33}=10$<br>$5,7$ | <br>10 |
| *Demand* | 30 | 20 | 40 | |

## 3.2 Solution Algorithm

**Algorithm A2:**

**Step 1.** Find the initial basic feasible solution $\{x_{ij}\}$ by MVAM method ( One can calculate any other method). Let $B$ is the basis matrix.

**Step 2.** Determine all $c_{ij}$ by **Algorithm A1:**

**Step 3.** Set $u_i + v_j = c_{ij} \ \ \forall (i,j) \in B$. $u_i + v_j = z_{ij} \ \ \forall (i,j) \notin B$.

**Step 4.** Find $\Delta_{ij} = c_{ij} - z_{ij}$. If $\Delta_{ij} \geq 0$ goto **Step 5** else goto **Step 2.** as $(i,j)$ enters the basis.

**Step 5.** Let $Z^1$ be the optimal cost of **Model 2** yielded by the basic feasible solution $\{x^1_{ij}\}$.

**Step 6.** Find $T^1 = \max[t_{ij} \ \ /x^1_{ij} > 0]$. Then the corresponding pair $(Z^1, T^1)$ is the first cost-time trade off pair for the **Model 3**. To find the next best cost-time trade-off pair, goto **Step 7.**

**Step 7.** Define $c_{ij} = M$ if $t_{ij} \geq T^1$. Where $M$ is sufficiently large positive number.

Repeat the above process till we get the problem to be infeasible.

The complete set of cost-time trade off pairs of **Model 3** at the end of the $q^{th}$ iteration is given by $(Z^1, T^1)$, $(Z^2, T^2)$ , . . . , $(Z^q, T^q)$ where $Z^1 < Z^2 < \cdots < Z^q$ and $T^1 > T^2 > \cdots > T^q$.

**Discussion:**

In TP, the main objective is to minimize the total transportation cost so maximum needed time is determined after determining the optimal allocation for optimal cost. If time is first priory then optimal cost is not determined. Various trade pairs are obtained through proposed technique( in solution methodology). But this model does not serve minimum cost and minimum time.

Numerical examples are presented in the following section.

# 4 Numerical Example

**Example 1:** Recall the cost varying transportation problem as

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7, 16$ | $12, 16$ | $8, 12$ | 45 |
| $O_2$ | $4, 6$ | $14, 18$ | $9, 15$ | 35 |
| $O_3$ | $10, 15$ | $17, 22$ | $5, 7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

The capacities of vehicles of $V_1$ and $V_2$ are respectively, $C_1 = 8$ and $C_2 = 14$. The time is taken to deliver the quantities in the following table

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $8, 6$ | $14, 10$ | $4, 3$ | 45 |
| $O_2$ | $7, 5$ | $5, 4$ | $11, 9$ | 35 |
| $O_3$ | $12, 9$ | $9, 7$ | $16, 12$ | 10 |
| *Demand* | 30 | 20 | 40 | |

By **Algorithm A2** the results are as follows:
**Iteration 1:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7, 16$ | $x_{12} = 20$<br>$12, 16$ | $x_{13} = 25$<br>$8, 12$ | <br>45 |
| $O_2$ | $x_{21} = 30$<br>$4, 6$ | <br>$14, 18$ | $x_{23} = 5$<br>$9, 15$ | <br>35 |
| $O_3$ | <br>$10, 15$ | <br>$17, 22$ | $x_{33} = 10$<br>$5, 7$ | <br>10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1.** Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC : 1)$ : Unit cost in basic cell

| *cell*$(i, j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij} = \max(t1_{ij}, t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1, 2)$ | $x_{12} = 20$ | 1 | 1 | $1 * 12 + 1 * 16 = 28$ | $c_{12} = \frac{28}{20}$ | 14 | 10 | 14 |
| $(1, 3)$ | $x_{13} = 25$ | 0 | 2 | $0 * 8 + 2 * 12 = 24$ | $c_{13} = \frac{24}{25}$ | 0 | 3 | 3 |
| $(2, 1)$ | $x_{321} = 30$ | 2 | 1 | $2 * 4 + 1 * 6 = 14$ | $c_{21} = \frac{14}{30}$ | 7 | 5 | 7 |
| $(2, 3)$ | $x_{23} = 5$ | 1 | 0 | $1 * 9 + 0 * 15 = 9$ | $c_{23} = \frac{9}{5}$ | 11 | 0 | 11 |
| $(3, 3)$ | $x_{33} = 10$ | 0 | 1 | $0 * 5 + 1 * 7 = 7$ | $c_{33} = \frac{7}{10}$ | 0 | 12 | 12 |
| | | | | *Total cost* $Z = 82$ | | | | *Time* $T = 14$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC : 1)$ : Unit cost in non-basic cell

| *cell*$(i, j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1, 1)$ | $x_{11} = 25$ | 2 | 1 | $2 * 7 + 1 * 16 = 30$ | $c_{11} = \frac{30}{25}$ |
| $(2, 2)$ | $x_{22} = 5$ | 1 | 0 | $1 * 14 + 0 * 18 = 14$ | $c_{22} = \frac{14}{5}$ |
| $(3, 1)$ | $x_{31} = 10$ | 0 | 1 | $0 * 9 + 1 * 15 = 15$ | $c_{31} = \frac{15}{10}$ |
| $(3, 2)$ | $x_{32} = 10$ | 0 | 1 | $0 * 17 + 1 * 22 = 22$ | $c_{32} = \frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11} = \frac{30}{25},\ c_{12} = \frac{28}{20},\ c_{13} = \frac{24}{25},\ c_{21} = \frac{14}{30},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{9}{5},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij} \ \ \forall(i,j) \in B$. $u_i + v_j = z_{ij} \ \ \forall(i,j) \notin B$. Therefore,

$$u_1 = \frac{24}{25},\ u_2 = \frac{9}{5},\ u_3 = \frac{7}{10},\ v_1 = -\frac{4}{3},\ v_2 = \frac{11}{12},\ v_3 = 0.$$

**Step 4.** $\Delta_{ij} \geq 0 \ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^1 = 82$ and optimal time is $T^1 = 14$. Therefore first optimal cost-time trade pair is $(82, 14)$.
**Step 6.** Before going to next iteration setting $R1_{12} = 150 = M1$ and $R2_{12} = 200 = M1$.
**Iteration 2:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7, 16$ | $150, 200$ | $x_{13} = 40$<br>$8, 12$ | 45 |
| $O_2$ | $x_{21} = 30$<br>$4, 6$ | $x_{22} = 5$<br>$14, 18$ | $9, 15$ | 35 |
| $O_3$ | $10, 15$ | $x_{32} = 10$<br>$17, 22$ | $5, 7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ :Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij} = \max(t1_{ij}, t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $x_{11} = 5$ | 1 | 0 | $1*7+0*16=7$ | $c_{11} = \frac{7}{5}$ | 8 | 0 | 8 |
| $(1,3)$ | $x_{13} = 40$ | 5 | 0 | $5*8+0*12=40$ | $c_{13} = \frac{40}{40}$ | 4 | 0 | 4 |
| $(2,1)$ | $x_{21} = 25$ | 0 | 2 | $0*4+2*6=12$ | $c_{21} = \frac{12}{25}$ | 0 | 5 | 5 |
| $(2,2)$ | $x_{22} = 10$ | 0 | 1 | $0*14+1*18=18$ | $c_{22} = \frac{18}{10}$ | 0 | 4 | 4 |
| $(3,2)$ | $x_{32} = 10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32} = \frac{22}{10}$ | 0 | 7 | 7 |
| | | | | *Total cost* $Z = 99$ | | | | *Time* $T = 8$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12} = 5$ | 1 | 0 | $1*150+0*200=150$ | $c_{12} = \frac{150}{5}$ |
| $(2,3)$ | $x_{23} = 25$ | 0 | 2 | $0*9+2*15=30$ | $c_{23} = \frac{30}{25}$ |
| $(3,1)$ | $x_{31} = 10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31} = \frac{15}{10}$ |
| $(3,3)$ | $x_{32} = 10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33} = \frac{7}{10}$ |

Therefore unit cost in all cells are as follows:
Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11} = \frac{7}{5},\ c_{12} = \frac{150}{5},\ c_{13} = \frac{40}{40},\ c_{21} = \frac{14}{30},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{9}{5},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij} \ \ \forall(i,j) \in B$. $u_i + v_j = z_{ij} \ \ \forall(i,j) \notin B$. Therefore,

$$u_1 = 0,\ u_2 = -\frac{23}{25},\ u_3 = -\frac{13}{25},\ v_1 = \frac{7}{5},\ v_2 = \frac{68}{25},\ v_3 = 1.$$

**Step 4.** $\Delta_{ij} \geq 0 \ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^2 = 99$ and optimal time is $T^2 = 8$. Therefore second optimal cost-time trade pair

is $(99, 8)$.
**Step 6.** Before going to next iteration selling.

After this iteration, solution is infeasible. Then two cost-time trade off pairs $(Z, T)$ are (82,14), (99,8).
**N.B.** Solution by NWCR, MMMM, MRMM, MCMM are shown in **Appendix**.
**Example 2:** Suppose there are three of production of berk. They are at kolaghat, Burdwan and Bhubaneswar. Three units produces $40, 30, 50$ thousands respectively. A infrastructure company need berks at Midnapore, Arambag and Bhubaneswar $35, 55, 30$ thousands respectively. It is seen that only two types of vehicles, DCM and Track. DCM can carry 2 thousand berks and Track. DCM can carry 4 thousand berks in a single trip.

It is observed that the single trip cost( in thousand Rupees) for DCM from source station to destination cities are given in following table.

Consider a CVTP as

| | *Midnapore*($D_1$) | *Arambag*($D_2$) | *Bhubaneswar*($D_3$) |
|---|---|---|---|
| $Kolaghat(O_1)$ | 3 | 4 | 5 |
| $Burdwan(O_2)$ | 5 | 2 | 7 |
| $Bhubanewar(O_3)$ | 6 | 8 | 4 |

where as the single trip cost( in thousand Rupees) for track from source station to destination cities are

| | *Midnapore*($D_1$) | *Arambag*($D_2$) | *Bhubaneswar*($D_3$) |
|---|---|---|---|
| $Kolaghat(O_1)$ | 5 | 6 | 8 |
| $Burdwan(O_2)$ | 7 | 3 | 10 |
| $Bhubanewar(O_3)$ | 8 | 11 | 6 |

The time(in hour) taken by DCM are

| | *Midnapore*($D_1$) | *Arambag*($D_2$) | *Bhubaneswar*($D_3$) |
|---|---|---|---|
| $Kolaghat(O_1)$ | 4 | 5 | 4 |
| $Burdwan(O_2)$ | 5 | 3 | 7 |
| $Bhubanewar(O_3)$ | 4.7 | 4.5 | 4 |

The time(in hour) taken by truk are

| | *Midnapore*($D_1$) | *Arambag*($D_2$) | *Bhubaneswar*($D_3$) |
|---|---|---|---|
| $Kolaghat(O_1)$ | 2 | 3 | 3 |
| $Burdwan(O_2)$ | 3 | 2.5 | 5 |
| $Bhubanewar(O_3)$ | 3.8 | 2.5 | 2 |

So the problem is formulated in CVTP as

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $3, 5$ | $4, 6$ | $5, 8$ | 40 |
| $O_2$ | $5, 7$ | $2, 3$ | $7, 10$ | 30 |
| $O_3$ | $6, 8$ | $8, 11$ | $4, 6$ | 50 |
| *Demand* | 35 | 55 | 30 | |

The capacities of vehicles of $V_1$ and $V_2$ are respectively, $C_1 = 2$ and $C_2 = 4$.

The time is taken to deliver the quantities in the following table

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $4, 2$ | $5, 3$ | $4, 3$ | 40 |
| $O_2$ | $5, 3$ | $2, 2.5$ | $7, 5$ | 30 |
| $O_3$ | $4.7, 3.8$ | $4.5, 2.5$ | $7, 5$ | 50 |
| *Demand* | 35 | 55 | 30 | |

By **Algorithm A2** the results are as follows:
**Iteration 1:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 15$<br>$3, 5$ | $x_{12} = 25$<br>$4, 6$ | $5, 8$ | 40 |
| $O_2$ | $5, 7$ | $x_{22} = 30$<br>$2, 3$ | $7, 10$ | 30 |
| $O_3$ | $x_{31} = 20$<br>$6, 8$ | $8, 11$ | $x_{33} = 30$<br>$4, 6$ | 50 |
| *Demand* | 35 | 55 | 30 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i, j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij} = \max(t1_{ij}, t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1, 1)$ | $x_{11} = 15$ | 0 | 4 | $0*3+4*5 = 20$ | $c_{11} = \frac{20}{15}$ | 0 | 2 | 2 |
| $(1, 2)$ | $x_{12} = 25$ | 1 | 6 | $1*4+6*6 = 40$ | $c_{12} = \frac{40}{25}$ | 5 | 3 | 5 |
| $(2, 2)$ | $x_{22} = 30$ | 1 | 7 | $1*2+7*3 = 23$ | $c_{22} = \frac{23}{30}$ | 3 | 2.5 | 3 |
| $(3, 1)$ | $x_{31} = 20$ | 0 | 5 | $0*6+5*8 = 40$ | $c_{31} = \frac{40}{20}$ | 4.7 | 3.8 | 4.7 |
| $(3, 3)$ | $x_{33} = 30$ | 1 | 7 | $1*4+6*7 = 46$ | $c_{33} = \frac{46}{20}$ | 4 | 2 | 4 |
| | | | | *Total cost* $Z = 169$ | | | | *Time* $T = 5$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| *cell*$(i, j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1, 3)$ | $x_{13} = 15$ | 0 | 4 | $0*5+4*8 = 32$ | $c_{13} = \frac{32}{15}$ |
| $(2, 1)$ | $x_{21} = 15$ | 0 | 4 | $0*5+4*7 = 28$ | $c_{21} = \frac{28}{15}$ |
| $(2, 3)$ | $x_{23} = 30$ | 1 | 7 | $1*7+7*10 = 77$ | $c_{23} = \frac{77}{30}$ |
| $(3, 2)$ | $x_{32} = 20$ | 0 | 5 | $0*8+5*11 = 55$ | $c_{32} = \frac{55}{20}$ |

Therefore unit cost in all cells are as follows:

$$c_{11} = \frac{20}{15},\ c_{12} = \frac{40}{25},\ c_{13} = \frac{32}{15},\ c_{21} = \frac{28}{15},\ c_{22} = \frac{23}{30},\ c_{23} = \frac{77}{30},\ c_{31} = \frac{40}{20},\ c_{32} = \frac{55}{20},\ c_{33} = \frac{46}{30}.$$

**Step 3.** Set $u_i + v_j = c_{ij}\ \ \forall (i, j) \in B$. $u_i + v_j = z_{ij}\ \ \forall (i, j) \notin B$. Therefore,

$$u_1 = 0,\ u_2 = \frac{-6}{5},\ u_3 = \frac{3}{5},\ v_1 = \frac{20}{15},\ v_2 = \frac{40}{25},\ v_3 = \frac{14}{15}.$$

**Step 4.** $\Delta_{ij} \geq 0\ \ \forall i, j$.
**Step 5.** Optimal cost is $Z^1 = 169$ and optimal time is $T^1 = 5$. Therefore first optimal cost-time trade pair is $(169, 5)$.
**Step 6.** Before going to next iteration setting $R1_{12} = 40 = M1$ and $R2_{12} = 60 = M1$.
**Iteration 2:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 35$<br>$3, 5$ | $4, 6$ | $x_{13} = 5$<br>$5, 8$ | 40 |
| $O_2$ | $5, 7$ | $x_{22} = 30$<br>$2, 3$ | $7, 10$ | 30 |
| $O_3$ | $6, 8$ | $x_{32} = 25$<br>$8, 11$ | $x_{33} = 25$<br>$4, 6$ | 50 |
| *Demand* | 35 | 55 | 30 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ :Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij}=\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=35$ | 0 | 9 | $0*3+9*5=45$ | $c_{11}=\frac{45}{35}$ | 0 | 2 | 2 |
| $(1,3)$ | $x_{13}=5$ | 1 | 1 | $1*5+1*8=13$ | $c_{13}=\frac{13}{5}$ | 4 | 3 | 4 |
| $(2,2)$ | $x_{22}=30$ | 1 | 7 | $1*2+7*3=23$ | $c_{22}=\frac{23}{30}$ | 3 | 2.5 | 3 |
| $(3,2)$ | $x_{32}=25$ | 1 | 6 | $1*8+6*11=74$ | $c_{32}=\frac{74}{25}$ | 4.5 | 2.5 | 4.5 |
| $(3,3)$ | $x_{33}=25$ | 1 | 6 | $1*4+6*6=40$ | $c_{33}=\frac{40}{25}$ | 4 | 2 | 4 |
| | | | | *Total cost* $Z=195$ | | | | *Time* $T=4.5$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$: Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=5$ | 1 | 1 | $1*40+1*60=100$ | $c_{12}=\frac{100}{5}$ |
| $(2,1)$ | $x_{21}=30$ | 1 | 7 | $1*5+7*7=54$ | $c_{21}=\frac{54}{30}$ |
| $(2,3)$ | $x_{23}=25$ | 1 | 6 | $1*7+6*10=67$ | $c_{23}=\frac{67}{25}$ |
| $(3,1)$ | $x_{31}=25$ | 1 | 6 | $1*6+6*8=54$ | $c_{31}=\frac{54}{25}$ |

Therefore unit cost in all cells are as follows: Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11}=\frac{45}{35},\ c_{12}=\frac{100}{5},\ c_{13}=\frac{13}{5},\ c_{21}=\frac{54}{30},\ c_{22}=\frac{23}{30},\ c_{23}=\frac{67}{25},\ c_{31}=\frac{54}{25},\ c_{32}=\frac{74}{25},\ c_{33}=\frac{40}{25}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\ \ \forall(i,j)\in B$. $u_i+v_j=z_{ij}\ \ \forall(i,j)\notin B$. Therefore,

$$u_1=0,\ u_2=-\frac{479}{150},\ u_3=-1,\ v_1=\frac{45}{35},\ v_2=\frac{99}{25},\ v_3=\frac{13}{5}.$$

**Step 4.** $\Delta_{ij}\geq 0\ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^2=195$ and optimal time is $T^2=4.5$. Therefore second optimal cost-time trade pair is $(195,4.5)$.
**Step 6.** Before going to next iteration selling.

After this iteration, solution is infeasible. Then two cost-time trade off pairs $(Z,T)$ are (169,5), (195,4.5).

# 5 Conclusion

This paper, develop time dependent two-vehicle cost varying transportation problem. Time dependent cost varying transportation problem is transferred to usual transportation problem by by proposed algorithm with initial allocation by modified VAM method. Then apply optimality test where unit transportation cost vary from one table to another table. Finally, achieve optimal cost and then calculate total elapse time and get optimal time & pay-off pair. Several time & pay-off pairs are also calculated for various trade which lead more realistic. From numerical example it is seen that if transport the quantities in shortest time then transportation cost is higher. In other words if transportation cost is minimum then more time is needed. This problem is more real life problem than usual transportation problem. This model can be formulated by considering several vehicles. Multi-objective time dependent economic model also be formulated through this proposed model.

# Appendix

### NWCR

For example, (NWCR) **Example 1:** Consider a cost varying transportation problem as

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7,16$ | $12,16$ | $8,12$ | 45 |
| $O_2$ | $4,6$ | $14,18$ | $9,15$ | 35 |
| $O_3$ | $10,15$ | $17,22$ | $5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

The capacities of vehicles of $V_1$ and $V_2$ are respectively, $C_1 = 8$ and $C_2 = 14$.
**Step 1.** By modified NWCR method initial B.F.S. is **Iteration** 1

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 30$<br>$7,16$ | $x_{12} = 15$<br>$12,16$ | $8,12$ | 45 |
| $O_2$ | $4,6$ | $x_{22} = 5$<br>$14,18$ | $x_{23} = 30$<br>$9,15$ | 35 |
| $O_3$ | $10,15$ | $17,22$ | $x_{33} = 10$<br>$5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ :Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11} = 30$ | 0 | 4 | $4*7+0*16=28$ | $c_{11} = \frac{28}{30}$ |
| $(1,2)$ | $x_{12} = 15$ | 2 | 0 | $2*12+0*16=24$ | $c_{12} = \frac{24}{15}$ |
| $(2,2)$ | $x_{22} = 5$ | 1 | 0 | $1*14+0*18=14$ | $c_{22} = \frac{14}{5}$ |
| $(2,3)$ | $x_{23} = 30$ | 2 | 1 | $2*9+1*15=33$ | $c_{23} = \frac{33}{30}$ |
| $(3,2)$ | $x_{33} = 10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33} = \frac{7}{10}$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,3)$ | $x_{13} = 15$ | 2 | 0 | $2*8+0*12=16$ | $c_{13} = \frac{16}{15}$ |
| $(2,1)$ | $x_{21} = 5$ | 1 | 0 | $1*4+0*6=4$ | $c_{21} = \frac{4}{5}$ |
| $(3,1)$ | $x_{31} = 10$ | 0 | 1 | $0*10+1*15=15$ | $c_{31} = \frac{15}{10}$ |
| $(3,2)$ | $x_{32} = 10$ | 1 | 0 | $1*17+0*22=17$ | $c_{32} = \frac{17}{5}$ |

Therefore unit cost in all cells are as follows:
Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11} = \frac{28}{30},\ c_{12} = \frac{24}{15},\ c_{13} = \frac{16}{15},\ c_{21} = \frac{4}{5},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{33}{30},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{17}{5},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij} \ \ \forall (i,j) \in B$. $u_i + v_j = z_{ij} \ \ \forall (i,j) \notin B$. Therefore,

$$u_1 = \frac{28}{30},\ u_2 = \frac{54}{30},\ u_3 = \frac{51}{30},\ v_1 = 0,\ v_2 = \frac{20}{30},\ v_3 = \frac{-21}{30}.$$

**Step 4.** $\Delta_{13} < 0, \Delta_{21} < 0, \Delta_{31} < 0, \Delta_{32} < 0$.

**Iteration** 2

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7,16$ | $x_{12} = 20$<br>$12,16$ | $x_{13} = 25$<br>$8,12$ | 45 |
| $O_2$ | $x_{21} = 30$<br>$4,6$ | $14,18$ | $x_{23} = 5$<br>$9,15$ | 35 |
| $O_3$ | $10,15$ | $17,22$ | $x_{33} = 10$<br>$5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | | | $\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=20$ | 1 | 1 | $1*12+1*16=28$ | $c_{12}=\frac{28}{20}$ | 14 | 10 | 14 |
| $(1,3)$ | $x_{13}=25$ | 0 | 2 | $0*8+2*12=24$ | $c_{13}=\frac{24}{25}$ | 0 | 3 | 3 |
| $(2,1)$ | $x_{321}=30$ | 2 | 1 | $2*4+1*6=14$ | $c_{21}=\frac{14}{30}$ | 7 | 5 | 7 |
| $(2,3)$ | $x_{23}=5$ | 1 | 0 | $1*9+0*15=9$ | $c_{23}=\frac{9}{5}$ | 11 | 0 | 11 |
| $(3,3)$ | $x_{33}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33}=\frac{7}{10}$ | 0 | 12 | 12 |
| | | | | *Total cost* $Z=82$ | | | | *Time* $T=14$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=25$ | 2 | 1 | $2*7+1*16=30$ | $c_{11}=\frac{30}{25}$ |
| $(2,2)$ | $x_{22}=5$ | 1 | 0 | $1*14+0*18=14$ | $c_{22}=\frac{14}{5}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11}=\frac{30}{25},\ c_{12}=\frac{28}{20},\ c_{13}=\frac{24}{25},\ c_{21}=\frac{14}{30},\ c_{22}=\frac{14}{5},\ c_{23}=\frac{9}{5},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\ \ \forall(i,j)\in B$. $u_i+v_j=z_{ij}\ \ \forall(i,j)\notin B$. Therefore,

$$u_1=\frac{24}{25},\ u_2=\frac{9}{5},\ u_3=\frac{7}{10},\ v_1=-\frac{4}{3},\ v_2=\frac{11}{12},\ v_3=0.$$

**Step 4.** $\Delta_{ij}\geq 0\ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^1=82$ and optimal time is $T^1=14$. Therefore first optimal cost-time trade pair is $(82,14)$.
**Step 6.** Before going to next iteration setting $R1_{12}=150=M1$ and $R2_{12}=200=M1$.

**Iteration 2:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7,16$ | <br>$150,200$ | $x_{13}=40$<br>$8,12$ | 45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | $x_{22}=5$<br>$14,18$ | <br>$9,15$ | 35 |
| $O_3$ | <br>$10,15$ | $x_{32}=10$<br>$17,22$ | <br>$5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ :Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij}=\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=5$ | 1 | 0 | $1*7+0*16=7$ | $c_{11}=\frac{7}{5}$ | 8 | 0 | 8 |
| $(1,3)$ | $x_{13}=40$ | 5 | 0 | $5*8+0*12=40$ | $c_{13}=\frac{40}{40}$ | 4 | 0 | 4 |
| $(2,1)$ | $x_{21}=25$ | 0 | 2 | $0*4+2*6=12$ | $c_{21}=\frac{12}{25}$ | 0 | 5 | 5 |
| $(2,2)$ | $x_{22}=10$ | 0 | 1 | $0*14+1*18=18$ | $c_{22}=\frac{18}{10}$ | 0 | 4 | 4 |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ | 0 | 7 | 7 |
| | | | | *Total cost* $Z=99$ | | | | *Time* $T=8$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=5$ | 1 | 0 | $1*150+0*200=150$ | $c_{12}=\frac{150}{5}$ |
| $(2,3)$ | $x_{23}=25$ | 0 | 2 | $0*9+2*15=30$ | $c_{23}=\frac{30}{25}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,3)$ | $x_{32}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33}=\frac{7}{10}$ |

Therefore unit cost in all cells are as follows:

Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11}=\frac{7}{5},\ c_{12}=\frac{150}{5},\ c_{13}=\frac{40}{40},\ c_{21}=\frac{14}{30},\ c_{22}=\frac{14}{5},\ c_{23}=\frac{9}{5},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\ \ \forall(i,j)\in B$. $u_i+v_j=z_{ij}\ \ \forall(i,j)\notin B$. Therefore,

$$u_1=0,\ u_2=-\frac{23}{25},\ u_3=-\frac{13}{25},\ v_1=\frac{7}{5},\ v_2=\frac{68}{25},\ v_3=1.$$

**Step 4.** $\Delta_{ij}\geq 0\ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^2=99$ and optimal time is $T^2=8$. Therefore second optimal cost-time trade pair is $(99,8)$.
**Step 6.** Before going to next iteration selling.

After this iteration, solution is infeasible. Then two cost-time trade off pairs $(Z,T)$ are (82,14), (99,8).

**MMMM**

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7,16$ | $x_{12}=20$<br>$12,16$ | $x_{13}=25$<br>$8,12$ | 45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | <br>$14,18$ | $x_{23}=5$<br>$9,15$ | 35 |
| $O_3$ | <br>$10,15$ | <br>$17,22$ | $x_{33}=10$<br>$5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij}=\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=20$ | 1 | 1 | $1*12+1*16=28$ | $c_{12}=\frac{28}{20}$ | 14 | 10 | 14 |
| $(1,3)$ | $x_{13}=25$ | 0 | 2 | $0*8+2*12=24$ | $c_{13}=\frac{24}{25}$ | 0 | 3 | 3 |
| $(2,1)$ | $x_{21}=30$ | 2 | 1 | $2*4+1*6=14$ | $c_{21}=\frac{14}{30}$ | 7 | 5 | 7 |
| $(2,3)$ | $x_{23}=5$ | 1 | 0 | $1*9+0*15=9$ | $c_{23}=\frac{9}{5}$ | 11 | 0 | 11 |
| $(3,3)$ | $x_{33}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33}=\frac{7}{10}$ | 0 | 12 | 12 |
| | | | | *Total cost* $Z=82$ | | | | *Time* $T=14$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=25$ | 2 | 1 | $2*7+1*16=30$ | $c_{11}=\frac{30}{25}$ |
| $(2,2)$ | $x_{22}=5$ | 1 | 0 | $1*14+0*18=14$ | $c_{22}=\frac{14}{5}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11} = \frac{30}{25},\ c_{12} = \frac{28}{20},\ c_{13} = \frac{24}{25},\ c_{21} = \frac{14}{30},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{9}{5},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij}\ \ \forall (i,j) \in B$. $u_i + v_j = z_{ij}\ \ \forall (i,j) \notin B$. Therefore,

$$u_1 = \frac{24}{25},\ u_2 = \frac{9}{5},\ u_3 = \frac{7}{10},\ v_1 = -\frac{4}{3},\ v_2 = \frac{11}{12},\ v_3 = 0.$$

**Step 4.** $\Delta_{ij} \geq 0\ \ \forall i, j$.
**Step 5.** Optimal cost is $Z^1 = 82$ and optimal time is $T^1 = 14$. Therefore first optimal cost-time trade pair is $(82, 14)$.
**Step 6.** Before going to next iteration setting $R1_{12} = 150 = M1$ and $R2_{12} = 200 = M1$.

**Iteration 2:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7, 16$ | $150, 200$ | $x_{13} = 40$<br>$8, 12$ | 45 |
| $O_2$ | $x_{21} = 30$<br>$4, 6$ | $x_{22} = 5$<br>$14, 18$ | $9, 15$ | 35 |
| $O_3$ | $10, 15$ | $x_{32} = 10$<br>$17, 22$ | $5, 7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1.** Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ :Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij} = \max(t1_{ij}, t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $x_{11} = 5$ | 1 | 0 | $1 * 7 + 0 * 16 = 7$ | $c_{11} = \frac{7}{5}$ | 8 | 0 | 8 |
| $(1,3)$ | $x_{13} = 40$ | 5 | 0 | $5 * 8 + 0 * 12 = 40$ | $c_{13} = \frac{40}{40}$ | 4 | 0 | 4 |
| $(2,1)$ | $x_{21} = 25$ | 0 | 2 | $0 * 4 + 2 * 6 = 12$ | $c_{21} = \frac{12}{25}$ | 0 | 5 | 5 |
| $(2,2)$ | $x_{22} = 10$ | 0 | 1 | $0 * 14 + 1 * 18 = 18$ | $c_{22} = \frac{18}{10}$ | 0 | 4 | 4 |
| $(3,2)$ | $x_{32} = 10$ | 0 | 1 | $0 * 17 + 1 * 22 = 22$ | $c_{32} = \frac{22}{10}$ | 0 | 7 | 7 |
| | | | | *Total cost* $Z = 99$ | | | | *Time* $T = 8$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12} = 5$ | 1 | 0 | $1 * 150 + 0 * 200 = 150$ | $c_{12} = \frac{150}{5}$ |
| $(2,3)$ | $x_{23} = 25$ | 0 | 2 | $0 * 9 + 2 * 15 = 30$ | $c_{23} = \frac{30}{25}$ |
| $(3,1)$ | $x_{31} = 10$ | 0 | 1 | $0 * 9 + 1 * 15 = 15$ | $c_{31} = \frac{15}{10}$ |
| $(3,3)$ | $x_{32} = 10$ | 0 | 1 | $0 * 5 + 1 * 7 = 7$ | $c_{33} = \frac{7}{10}$ |

Therefore unit cost in all cells are as follows:
Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11} = \frac{7}{5},\ c_{12} = \frac{150}{5},\ c_{13} = \frac{40}{40},\ c_{21} = \frac{14}{30},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{9}{5},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij}\ \ \forall (i,j) \in B$. $u_i + v_j = z_{ij}\ \ \forall (i,j) \notin B$. Therefore,

$$u_1 = 0,\ u_2 = -\frac{23}{25},\ u_3 = -\frac{13}{25},\ v_1 = \frac{7}{5},\ v_2 = \frac{68}{25},\ v_3 = 1$$

**Step 4.** $\Delta_{ij} \geq 0 \ \ \forall i, j$.
**Step 5.** Optimal cost is $Z^2 = 99$ and optimal time is $T^2 = 8$. Therefore second optimal cost-time trade pair is $(99, 8)$.
**Step 6.** Before going to next iteration selling.

After this iteration, solution is infeasible. Then two cost-time trade off pairs $(Z, T)$ are (82,14), (99,8).

**MRMM**

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 30$<br>$7, 16$ | $12, 16$ | $x_{13} = 15$<br>$8, 12$ | 45 |
| $O_2$ | $4, 6$ | $x_{22} = 10$<br>$14, 18$ | $x_{23} = 25$<br>$9, 15$ | 35 |
| $O_3$ | $10, 15$ | $x_{32} = 10$<br>$17, 22$ | $5, 7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC : 1)$ : Unit cost in basic cell

| *cell*$(i, j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1, 1)$ | $x_{12} = 30$ | 2 | 1 | $2 * 12 + 1 * 16 = 30$ | $c_{12} = \frac{30}{30}$ |
| $(1, 3)$ | $x_{13} = 15$ | 2 | 0 | $2 * 8 + 0 * 12 = 16$ | $c_{13} = \frac{16}{15}$ |
| $(2, 2)$ | $x_{22} = 10$ | 0 | 1 | $0 * 14 + 1 * 18 = 18$ | $c_{22} = \frac{18}{10}$ |
| $(2, 3)$ | $x_{23} = 25$ | 0 | 2 | $0 * 9 + 2 * 15 = 30$ | $c_{23} = \frac{30}{25}$ |
| $(3, 2)$ | $x_{32} = 10$ | 0 | 1 | $0 * 17 + 1 * 22 = 22$ | $c_{32} = \frac{22}{10}$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC : 1)$ : Unit cost in non-basic cell

| *cell*$(i, j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1, 2)$ | $x_{12} = 10$ | 0 | 1 | $0 * 12 + 1 * 16 = 16$ | $c_{12} = \frac{16}{10}$ |
| $(2, 1)$ | $x_{21} = 25$ | 0 | 2 | $0 * 4 + 2 * 6 = 12$ | $c_{21} = \frac{12}{25}$ |
| $(3, 1)$ | $x_{31} = 10$ | 0 | 1 | $0 * 10 + 1 * 15 = 15$ | $c_{31} = \frac{15}{10}$ |
| $(3, 3)$ | $x_{33} = 10$ | 0 | 1 | $0 * 5 + 1 * 7 = 7$ | $c_{33} = \frac{7}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11} = \frac{30}{30},\ c_{12} = \frac{16}{10},\ c_{13} = \frac{16}{15},\ c_{21} = \frac{12}{25},\ c_{22} = \frac{18}{10},\ c_{23} = \frac{30}{25},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij} \ \ \forall (i, j) \in B$. $u_i + v_j = z_{ij} \ \ \forall (i, j) \notin B$. Therefore,

$$u_1 = 0,\ u_2 = \frac{2}{5},\ u_3 = \frac{8}{15},\ v_1 = -\frac{30}{30},\ v_2 = \frac{5}{3},\ v_3 = \frac{16}{15}.$$

**Step 4.** $\Delta_{33} < 0$.

**Iteration** 2

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11} = 30$<br>$7, 16$ | $12, 16$ | $x_{13} = 15$<br>$8, 12$ | 45 |
| $O_2$ | $4, 6$ | $x_{22} = 20$<br>$14, 18$ | $x_{23} = 15$<br>$9, 15$ | 35 |
| $O_3$ | $10, 15$ | $17, 22$ | $5, 7$ | $x_{33} = 10$<br>10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1** Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{12}=30$ | 2 | 1 | $2*12+1*16=30$ | $c_{12}=\frac{30}{30}$ |
| $(1,3)$ | $x_{13}=15$ | 2 | 0 | $2*8+0*12=16$ | $c_{13}=\frac{16}{15}$ |
| $(2,2)$ | $x_{22}=20$ | 1 | 1 | $1*14+1*18=32$ | $c_{22}=\frac{32}{10}$ |
| $(2,3)$ | $x_{23}=15$ | 2 | 0 | $2*9+0*15=18$ | $c_{23}=\frac{18}{15}$ |
| $(3,3)$ | $x_{33}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{32}=\frac{7}{10}$ |

**Step 3.** For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=15$ | 2 | 0 | $2*12+0*16=24$ | $c_{12}=\frac{24}{15}$ |
| $(2,1)$ | $x_{21}=15$ | 2 | 0 | $2*4+0*6=8$ | $c_{21}=\frac{8}{25}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*10+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11}=\frac{30}{30},\ c_{12}=\frac{24}{15},\ c_{13}=\frac{28}{30},\ c_{21}=\frac{8}{15},\ c_{22}=\frac{32}{20},\ c_{23}=\frac{18}{15},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\ \ \forall(i,j)\in B.\ u_i+v_j=z_{ij}\ \ \forall(i,j)\notin B.$ Therefore,

$$u_1=\frac{16}{15},\ u_2=\frac{18}{15},\ u_3=\frac{7}{10},\ v_1=-\frac{1}{15},\ v_2=\frac{2}{5},\ v_3=0.$$

**Step 4.** $\Delta_{21}<0$.

**Iteration** 3

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $x_{11}=15$<br>$7,16$ | <br>$12,16$ | $x_{13}=30$<br>$8,12$ | <br>45 |
| $O_2$ | $x_{21}=15$<br>$4,6$ | $x_{22}=20$<br>$14,18$ | <br>$9,15$ | <br>35 |
| $O_3$ | <br>$10,15$ | <br>$17,22$ | <br>$5,7$ | $x_{33}=10$<br>10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=15$ | 2 | 0 | $2*7+0*16=14$ | $c_{11}=\frac{14}{15}$ |
| $(1,3)$ | $x_{13}=30$ | 2 | 1 | $2*8+1*12=28$ | $c_{13}=\frac{28}{30}$ |
| $(2,1)$ | $x_{21}=15$ | 2 | 0 | $2*4+0*6=8$ | $c_{21}=\frac{8}{15}$ |
| $(2,2)$ | $x_{22}=20$ | 1 | 1 | $1*14+1*18=32$ | $c_{22}=\frac{32}{10}$ |
| $(3,3)$ | $x_{33}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{32}=\frac{7}{10}$ |

**Step 3.** For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| $cell(i,j)$ | $possible\ x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=15$ | 2 | 0 | $2*12+0*16=24$ | $c_{12}=\frac{24}{15}$ |
| $(2,3)$ | $x_{23}=15$ | 2 | 0 | $2*9+0*15=18$ | $c_{23}=\frac{18}{15}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*10+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11}=\frac{14}{15},\ c_{12}=\frac{24}{15},\ c_{13}=\frac{28}{30},\ c_{21}=\frac{8}{15},\ c_{22}=\frac{32}{20},\ c_{23}=\frac{18}{15},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\quad\forall(i,j)\in B.\ u_i+v_j=z_{ij}\quad\forall(i,j)\notin B$. Therefore,

$$u_1=\frac{14}{15},\ u_2=\frac{8}{15},\ u_3=\frac{7}{10},\ v_1=0,\ v_2=\frac{16}{15},\ v_3=0.$$

**Step 4.** $\Delta_{12}<0$.

**Iteration** 4

| | $D_1$ | $D_2$ | $D_3$ | $stock$ |
|---|---|---|---|---|
| $O_1$ | $x_{11}=15$<br>$7,16$ | $12,16$ | $x_{13}=30$<br>$8,12$ | 45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | $x_{22}=5$<br>$14,18$ | $9,15$ | 35 |
| $O_3$ | $10,15$ | $17,22$ | $5,7$ | $x_{33}=10$<br>10 |
| $Demand$ | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| $cell(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=15$ | 2 | 0 | $2*12+0*16=24$ | $c_{12}=\frac{24}{15}$ |
| $(1,3)$ | $x_{13}=30$ | 2 | 1 | $2*8+1*12=28$ | $c_{13}=\frac{28}{30}$ |
| $(2,1)$ | $x_{21}=30$ | 2 | 1 | $2*4+1*6=14$ | $c_{21}=\frac{14}{30}$ |
| $(2,2)$ | $x_{22}=5$ | 1 | 0 | $1*14+0*18=14$ | $c_{22}=\frac{14}{5}$ |
| $(3,3)$ | $x_{33}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{32}=\frac{7}{10}$ |

**Step 3.** For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| $cell(i,j)$ | $possible\ x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=15$ | 2 | 0 | $2*7+0*16=14$ | $c_{11}=\frac{14}{15}$ |
| $(2,3)$ | $x_{23}=5$ | 1 | 0 | $1*9+0*15=9$ | $c_{23}=\frac{9}{5}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*10+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11}=\frac{14}{15},\ c_{12}=\frac{24}{15},\ c_{13}=\frac{28}{30},\ c_{21}=\frac{14}{30},\ c_{22}=\frac{14}{5},\ c_{23}=\frac{9}{5},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\quad\forall(i,j)\in B.\ u_i+v_j=z_{ij}\quad\forall(i,j)\notin B$. Therefore,

$$u_1=0,\ u_2=\frac{18}{15},\ u_3=\frac{-7}{30},\ v_1=\frac{-11}{15},\ v_2=\frac{24}{15},\ v_3=\frac{14}{15}.$$

**Step 4.** $\Delta_{23}<0$.

**Iteration** 5

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7, 16$ | $x_{12} = 20$<br>$12, 16$ | $x_{13} = 25$<br>$8, 12$ | <br>45 |
| $O_2$ | $x_{21} = 30$<br>$4, 6$ | <br>$14, 18$ | $x_{23} = 5$<br>$9, 15$ | <br>35 |
| $O_3$ | <br>$10, 15$ | <br>$17, 22$ | $x_{33} = 10$<br>$5, 7$ | <br>10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij} = \max(t1_{ij}, t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,2)$ | $x_{12} = 20$ | 1 | 1 | $1*12+1*16=28$ | $c_{12} = \frac{28}{20}$ | 14 | 10 | 14 |
| $(1,3)$ | $x_{13} = 25$ | 0 | 2 | $0*8+2*12=24$ | $c_{13} = \frac{24}{25}$ | 0 | 3 | 3 |
| $(2,1)$ | $x_{321} = 30$ | 2 | 1 | $2*4+1*6=14$ | $c_{21} = \frac{14}{30}$ | 7 | 5 | 7 |
| $(2,3)$ | $x_{23} = 5$ | 1 | 0 | $1*9+0*15=9$ | $c_{23} = \frac{9}{5}$ | 11 | 0 | 11 |
| $(3,3)$ | $x_{33} = 10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33} = \frac{7}{10}$ | 0 | 12 | 12 |
| | | | | *Total cost* $Z = 82$ | | | | *Time* $T = 14$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij} + p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11} = 25$ | 2 | 1 | $2*7+1*16=30$ | $c_{11} = \frac{30}{25}$ |
| $(2,2)$ | $x_{22} = 5$ | 1 | 0 | $1*14+0*18=14$ | $c_{22} = \frac{14}{5}$ |
| $(3,1)$ | $x_{31} = 10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31} = \frac{15}{10}$ |
| $(3,2)$ | $x_{32} = 10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32} = \frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11} = \frac{30}{25},\ c_{12} = \frac{28}{20},\ c_{13} = \frac{24}{25},\ c_{21} = \frac{14}{30},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{9}{5},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij}\ \ \forall (i,j) \in B$. $u_i + v_j = z_{ij}\ \ \forall (i,j) \notin B$. Therefore,

$$u_1 = \frac{24}{25},\ u_2 = \frac{9}{5},\ u_3 = \frac{7}{10},\ v_1 = -\frac{4}{3},\ v_2 = \frac{11}{12},\ v_3 = 0.$$

**Step 4.** $\Delta_{ij} \geq 0\ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^1 = 82$ and optimal time is $T^1 = 14$. Therefore first optimal cost-time trade pair is $(82, 14)$.
**Step 6.** Before going to next iteration setting $R1_{12} = 150 = M1$ and $R2_{12} = 200 = M1$.

**Iteration 2:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7, 16$ | <br>$150, 200$ | $x_{13} = 40$<br>$8, 12$ | <br>45 |
| $O_2$ | $x_{21} = 30$<br>$4, 6$ | $x_{22} = 5$<br>$14, 18$ | <br>$9, 15$ | <br>35 |
| $O_3$ | <br>$10, 15$ | $x_{32} = 10$<br>$17, 22$ | <br>$5, 7$ | <br>10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij}=\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=5$ | 1 | 0 | $1*7+0*16=7$ | $c_{11}=\frac{7}{5}$ | 8 | 0 | 8 |
| $(1,3)$ | $x_{13}=40$ | 5 | 0 | $5*8+0*12=40$ | $c_{13}=\frac{40}{40}$ | 4 | 0 | 4 |
| $(2,1)$ | $x_{21}=25$ | 0 | 2 | $0*4+2*6=12$ | $c_{21}=\frac{12}{25}$ | 0 | 5 | 5 |
| $(2,2)$ | $x_{22}=10$ | 0 | 1 | $0*14+1*18=18$ | $c_{22}=\frac{18}{10}$ | 0 | 4 | 4 |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ | 0 | 7 | 7 |
| | | | | *Total cost* $Z=99$ | | | | *Time* $T=8$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=5$ | 1 | 0 | $1*150+0*200=150$ | $c_{12}=\frac{150}{5}$ |
| $(2,3)$ | $x_{23}=25$ | 0 | 2 | $0*9+2*15=30$ | $c_{23}=\frac{30}{25}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,3)$ | $x_{32}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33}=\frac{7}{10}$ |

Therefore unit cost in all cells are as follows:

Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11}=\frac{7}{5},\ c_{12}=\frac{150}{5},\ c_{13}=\frac{40}{40},\ c_{21}=\frac{14}{30},\ c_{22}=\frac{14}{5},\ c_{23}=\frac{9}{5},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\ \ \forall(i,j)\in B$. $u_i+v_j=z_{ij}\ \ \forall(i,j)\notin B$. Therefore,

$$u_1=0,\ u_2=-\frac{23}{25},\ u_3=-\frac{13}{25},\ v_1=\frac{7}{5},\ v_2=\frac{68}{25},\ v_3=1.$$

**Step 4.** $\Delta_{ij}\geq 0\ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^2=99$ and optimal time is $T^2=8$. Therefore second optimal cost-time trade pair is $(99,8)$.
**Step 6.** Before going to next iteration selling.

After this iteration, solution is infeasible. Then two cost-time trade off pairs $(Z,T)$ are (82,14), (99,8).

**MCMM**

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | <br>$7,16$ | $x_{12}=20$<br>$12,16$ | $x_{13}=25$<br>$8,12$ | 45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | <br>$14,18$ | $x_{23}=5$<br>$9,15$ | 35 |
| $O_3$ | <br>$10,15$ | <br>$17,22$ | $x_{33}=10$<br>$5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:1)$ : Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij}=\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=20$ | 1 | 1 | $1*12+1*16=28$ | $c_{12}=\frac{28}{20}$ | 14 | 10 | 14 |
| $(1,3)$ | $x_{13}=25$ | 0 | 2 | $0*8+2*12=24$ | $c_{13}=\frac{24}{25}$ | 0 | 3 | 3 |
| $(2,1)$ | $x_{321}=30$ | 2 | 1 | $2*4+1*6=14$ | $c_{21}=\frac{14}{30}$ | 7 | 5 | 7 |
| $(2,3)$ | $x_{23}=5$ | 1 | 0 | $1*9+0*15=9$ | $c_{23}=\frac{9}{5}$ | 11 | 0 | 11 |
| $(3,3)$ | $x_{33}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33}=\frac{7}{10}$ | 0 | 12 | 12 |
| | | | | *Total cost* $Z=82$ | | | | *Time* $T=14$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:1)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=25$ | 2 | 1 | $2*7+1*16=30$ | $c_{11}=\frac{30}{25}$ |
| $(2,2)$ | $x_{22}=5$ | 1 | 0 | $1*14+0*18=14$ | $c_{22}=\frac{14}{5}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ |

Therefore unit cost in all cells are as follows:

$$c_{11}=\frac{30}{25},\ c_{12}=\frac{28}{20},\ c_{13}=\frac{24}{25},\ c_{21}=\frac{14}{30},\ c_{22}=\frac{14}{5},\ c_{23}=\frac{9}{5},\ c_{31}=\frac{15}{10},\ c_{32}=\frac{22}{10},\ c_{33}=\frac{7}{10}.$$

**Step 3.** Set $u_i+v_j=c_{ij}\ \ \forall(i,j)\in B$. $u_i+v_j=z_{ij}\ \ \forall(i,j)\notin B$. Therefore,

$$u_1=\frac{24}{25},\ u_2=\frac{9}{5},\ u_3=\frac{7}{10},\ v_1=-\frac{4}{3},\ v_2=\frac{11}{12},\ v_3=0.$$

**Step 4.** $\Delta_{ij}\geq 0\ \ \forall i,j$.
**Step 5.** Optimal cost is $Z^1=82$ and optimal time is $T^1=14$. Therefore first optimal cost-time trade pair is $(82,14)$.
**Step 6.** Before going to next iteration setting $R1_{12}=150=M1$ and $R2_{12}=200=M1$.

**Iteration 2:**
**Step 1.** By modified VAM method initial B.F.S. is

| | $D_1$ | $D_2$ | $D_3$ | *stock* |
|---|---|---|---|---|
| $O_1$ | $7,16$ | $150,200$ | $x_{13}=40$<br>$8,12$ | 45 |
| $O_2$ | $x_{21}=30$<br>$4,6$ | $x_{22}=5$<br>$14,18$ | $9,15$ | 35 |
| $O_3$ | $10,15$ | $x_{32}=10$<br>$17,22$ | $5,7$ | 10 |
| *Demand* | 30 | 20 | 40 | |

**Step 2.** For basic cell, determine $c_{ij}$ by (6) of **Algorithm A1**. Now $c_{ij}$ and corresponding required time are presented in the following table:

Table-$P1(BC:2)$ :Unit cost in basic cell

| *cell*$(i,j)$ | $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ | $t1_{ij}$ | $t2_{ij}$ | $T_{ij}=\max(t1_{ij},t2_{ij})$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | $x_{11}=5$ | 1 | 0 | $1*7+0*16=7$ | $c_{11}=\frac{7}{5}$ | 8 | 0 | 8 |
| $(1,3)$ | $x_{13}=40$ | 5 | 0 | $5*8+0*12=40$ | $c_{13}=\frac{40}{40}$ | 4 | 0 | 4 |
| $(2,1)$ | $x_{21}=25$ | 0 | 2 | $0*4+2*6=12$ | $c_{21}=\frac{12}{25}$ | 0 | 5 | 5 |
| $(2,2)$ | $x_{22}=10$ | 0 | 1 | $0*14+1*18=18$ | $c_{22}=\frac{18}{10}$ | 0 | 4 | 4 |
| $(3,2)$ | $x_{32}=10$ | 0 | 1 | $0*17+1*22=22$ | $c_{32}=\frac{22}{10}$ | 0 | 7 | 7 |
| | | | | *Total cost* $Z=99$ | | | | *Time* $T=8$ |

For non-basic cell, determine $c_{ij}$ by (7) of **Algorithm A1**. These are presented in the following table:

Table-$P1(NBC:2)$ : Unit cost in non-basic cell

| *cell*$(i,j)$ | *possible* $x_{ij}$ | $p1_{ij}$ | $p2_{ij}$ | $p1_{ij}R1_{ij}+p2_{ij}R2_{ij}$ | $c_{ij}$ |
|---|---|---|---|---|---|
| $(1,2)$ | $x_{12}=5$ | 1 | 0 | $1*150+0*200=150$ | $c_{12}=\frac{150}{5}$ |
| $(2,3)$ | $x_{23}=25$ | 0 | 2 | $0*9+2*15=30$ | $c_{23}=\frac{30}{25}$ |
| $(3,1)$ | $x_{31}=10$ | 0 | 1 | $0*9+1*15=15$ | $c_{31}=\frac{15}{10}$ |
| $(3,3)$ | $x_{32}=10$ | 0 | 1 | $0*5+1*7=7$ | $c_{33}=\frac{7}{10}$ |

Therefore unit cost in all cells are as follows:

Determine all $c_{ij}$ by **Algorithm A1**.

$$c_{11} = \frac{7}{5},\ c_{12} = \frac{150}{5},\ c_{13} = \frac{40}{40},\ c_{21} = \frac{14}{30},\ c_{22} = \frac{14}{5},\ c_{23} = \frac{9}{5},\ c_{31} = \frac{15}{10},\ c_{32} = \frac{22}{10},\ c_{33} = \frac{7}{10}.$$

**Step 3.** Set $u_i + v_j = c_{ij} \quad \forall (i,j) \in B.\ u_i + v_j = z_{ij} \quad \forall (i,j) \notin B.$ Therefore,

$$u_1 = 0,\ u_2 = -\frac{23}{25},\ u_3 = -\frac{13}{25},\ v_1 = \frac{7}{5},\ v_2 = \frac{68}{25},\ v_3 = 1.$$

**Step 4.** $\Delta_{ij} \geq 0 \quad \forall i, j.$

**Step 5.** Optimal cost is $Z^2 = 99$ and optimal time is $T^2 = 8$. Therefore second optimal cost-time trade pair is $(99, 8)$.

**Step 6.** Before going to next iteration selling.

After this iteration, solution is infeasible. Then two cost-time trade off pairs $(Z, T)$ are (82,14), (99,8).

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