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Analysis and Projective Synchronization of New 4D Hyperchaotic System

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Abstract

In this manuscript, a new 4D autonomous hyperchaotic system has been introduced which is obtained by the three dimensional autonomous chaotic system. We analyse the hyperchaotic properties of the new system such as dissipation, equilibrium, Lyapunov exponent, stability, time series, phase portraits, Poincare map and bifurcation diagram. Furthermore, the projective synchronization of the new hyperchaotic system is analysed by using the active nonlinear control method. Brief theoretical analysis and numerical results are presented to prove the dynamical behaviour of the new 4D hyperchaotic system.

Keywords: hyperchaos, Poincare map, bifurcation diagram, Lyapunov exponent, projective synchronization

1 Introduction

Chaotic dynamics is a fascinating area of nonlinear sciences which has been extensively studied during the past few decades. Chaotic behaviour is observed in different fields for instance chemical systems, electrical engineering, biological systems, secure communication and so on [3]. Since Pecora and Carrol (1990) has proposed chaos synchronization [20]. In current years more and more attention has been diverted towards the control and synchronization of chaotic systems [22, 10]. Various kinds of synchronization phenomena have been studied, such as complete synchronization [15], anticipated synchronization [27], hybrid synchronization [8], projective synchronization [30] etc. Among all projective synchronization is one of the particular type of synchronization which has been suggested by [16]. In projective synchronization the master and slave systems could be synchronized up to a scaling factor, which can be used to widen binary digital to M-nary digital communications for obtaining rapid communication [25].

Hyperchaos dynamics has been comprehensively studied over the last two - three decades due to its great potential applications in many engineering oriented practical fields, such as secure communication [24], nonlinear circuits [1], laser's [26], control [6], synchronization [11, 9] and many more. In secure communication, the message to be transmitted is masked by a chaotic signal. We know that, the chaotic systems have one positive Lyapunov exponent. Perez and Cerderia justified that the messages masked by chaotic systems are not always secure [21]. However, Pecora established that this situation can be overwhelmed by adapting the higher dimensional - hyperchaotic systems, that have growing randomness and higher unpredictability [19]. A hyperchaotic system in general is a chaotic system having atleast two positive Lyapunov exponents, indicating that its dynamics is spreading in various distinct directions simultaneously. It establishes that the hyperchaotic systems possesses higher complex dynamical behaviour, which is used to enhance the safety of chaotic communication systems. Therefore, the analytical design and circuitry recognition of numerous hyperchaotic signals have currently become the interesting area of research [12].

Historically, Rossler in 1979 was the first who reported hyperchaos and noted 4D hyperchaotic Rössler system [23]. But in electronic circuits Matsumoto and his colleagues first discovered hyperchaos [17]. There are various systems which were discovered over the last two - three decades from the higher dimensional systems. There are Rössler's hyperchaotic system [23], Lorenz - Haken hyperchaotic system [18], Chau's hyperchaotic system [7], hyperchaotic finance system [31]. Since the past few years, hyperchaos was found experimentally and numerically by adding another state variable to the chaotic systems [13, 14, 5], in the generalized Lorenz system [2], Chen system [4], and a modified Lü system [28].

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In this manuscript, a new 4D hyperchaotic system is designed which is based on the 3D chaotic system [32] by introducing one more state variable in order to get the new 4D system. Firstly some basic analysis of the new 4D hyperchaotic system has been done by means of dissipation, equilibrium, stability, time series, phase portrait, Lyapunov exponents, Poincare map and bifurcation diagram. Then a projective synchronization approach is applied on the new 4D hyperchaotic system via active nonlinear control method. Simulations results are used to demonstrate the efficiency and feasibility of the applied synchronization scheme.

The paper is arranged as. In Section 2, the new 4D hyperchaotic system and its construction is described. In Section 3, some dynamical analysis of the new 4D hyperchaotic system has been numerically investigated. In Section 4, Lyapunov exponents and Kaplan -Yorke dimension has been calculated. In Section 5, the Poincare map and bifurcation diagram of the new 4D system has been established. In Section 6, the projective synchronization via nonlinear control method has been examined. Numerical results are used to verify this technique. Finally in Section 7, conclusions are being drawn.

2 The New 4D Hyperchaotic System and Its Construction

First,consider the newly constructed 3D chaotic system, constructed by Zhu et. al [32], described as

$$\begin{aligned} \dot{x} &= -x - ay + yz \\ \dot{y} &= by - xz \\ \dot{z} &= -cz + xy \end{aligned} \tag{1}$$

where a, b and c are all positive real parameters. For $a = 1.5,\ b = 2.5$ and $c = 4.9$, the system shows the chaotic behaviour.

In order to generate hyperchaos from the dissipative autonomous system, following two basic conditions should be satisfied by the state equations which are as follows:

• The dimension of the state equation should be atleast four and the order of the state equations should be atleast two.

• The system has atleast two positive Lyapunov exponents satisfying the condition that the sum of all Lyapunov exponents is less than zero.

The new hyperchaotic system which is based on the system (1) and the above two properties can be generated by adding an additional variable w, described as:

$$\begin{aligned} \dot{x} &= -x - ay + yz \\ \dot{y} &= by - xz - w \\ \dot{z} &= -cz + xy \\ \dot{w} &= dw + y \end{aligned} \tag{2}$$

where $(x,\ y,\ z,\ w) \in \mathbb{R}^4$ is a state vector. a, b, c, and d are the positive real parameters of the system (2). The corresponding phase portraits and time series of the new hyperchaotic system (2), for $a = 1.5,\ b = 2.5,\ c = 4.9$ and $d = 0.10$ are illustrated in Figs. 1(a-d) and Figs. 2(a-d).

3 Dynamical Analysis of the Newly Constructed Hyperchaotic System

In this section, some basic properties of the new hyperchaotic system (2) are investigated. For that, Lyapunov exponents, Poincare section and bifurcation diagram are used to illustrate the dynamics of the hyperchaotic system. The new hyperchaotic system has the following properties:

3.1 Symmetry

We define the set of new coordinates as $(x,y,z,w) \to (-x,\ -y,\ z,\ -w)$, then under this coordinate transformation, the new hyperchaotic system is invarient.Therefore it follows that our new hyperchaotic system (2) has the rotational axis about z-axis.

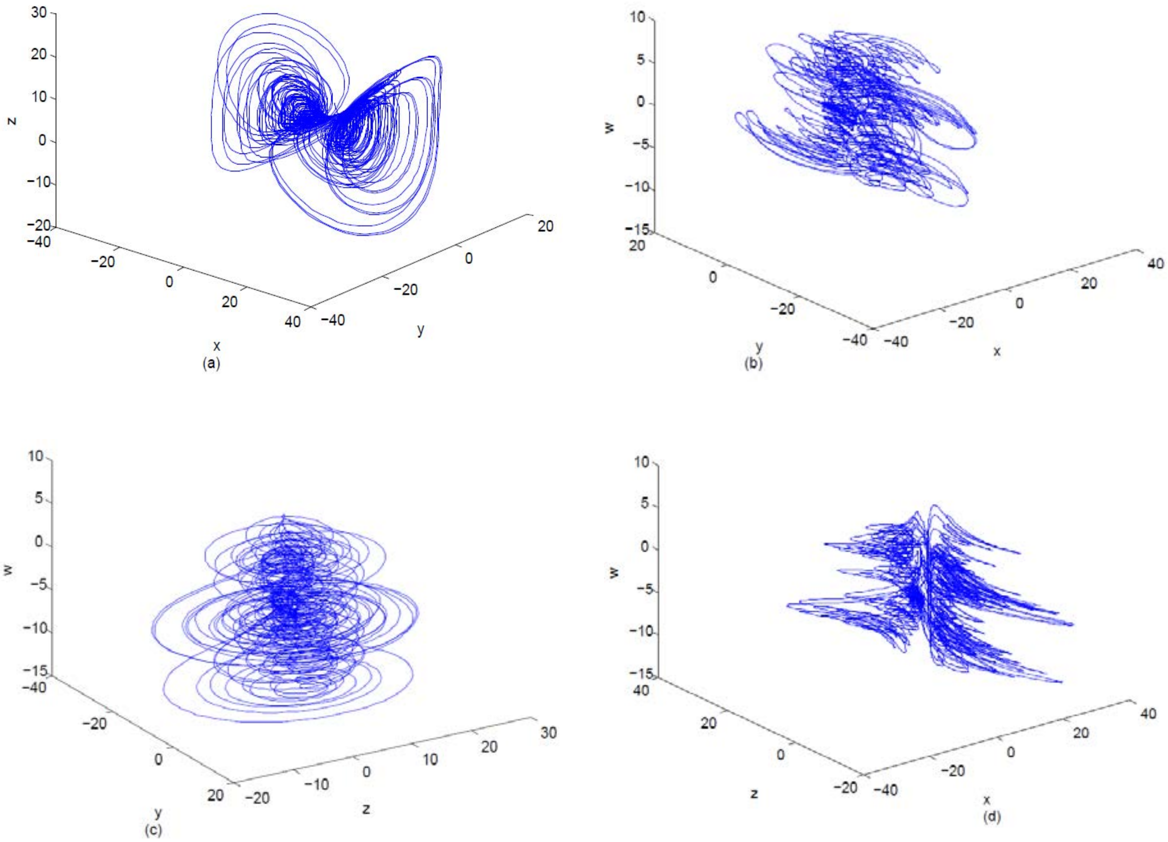

Figure 1: (a) Strange attractor in x-y-z space (b) Strange attractor in x-y-w space (c) Strange attractor in y-z-w space (d) Strange attractor in x-z-w space

3.2 Dissipation and Existence of Hyperchaotic Attractors

In vector notation, the new hyperchaotic system (2) can be expressed as

$$x' = f(x) = \begin{pmatrix} f_1(x,y,z,w) \\ f_2(x,y,z,w) \\ f_3(x,y,z,w) \\ f_4(x,y,z,w) \end{pmatrix}$$

where

$$\begin{aligned} f_1(x,y,z,w) &= -x - ay + yz \\ f_2(x,y,z,w) &= by - xz - w \\ f_3(x,y,z,w) &= -cz + xy \\ f_4(x,y,z,w) &= dw + y. \end{aligned}$$

Let χ be any region in $\mathbb{R}^4$ with a smooth boundary and also let $\chi(t) = \Psi(t)$, where $\Psi(t)$ is the flow of f. Furthermore let $V(t)$ denotes the volume of $\chi(t)$. By Liouville's theorem, we have

$$V'(t) = \int_{\chi(t)} (\nabla \bullet f) dxdydzdw. \tag{3}$$

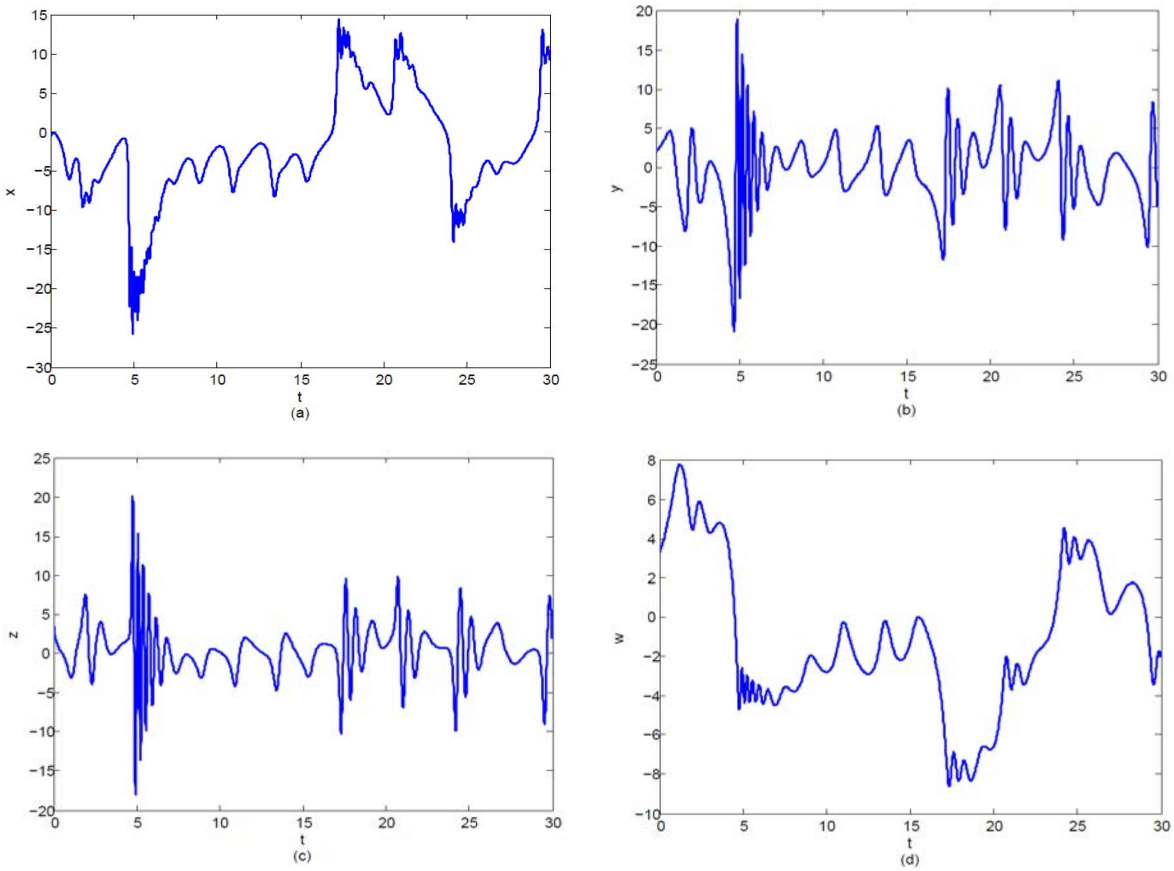

Figure 2: (a) Time series of state variable x (b) Time series of state variable y (c) Time series of state variable z (d) Time series of state variable w

The divergence of the vector field 'f' of the system (2) can be found as

$$\begin{aligned} \nabla \bullet f &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w} \\ &= -1 + b - c + d \\ &= -\alpha \end{aligned}$$

where $\alpha = 1 - b + c + d$. For a system to be dissipative it is required that $\nabla \bullet f < 0$. For the choice of parameter values as given in (2), we find that $\alpha > 0$.

Therefore from (3), we get

$$\begin{aligned} V'(t) &= \int_{\chi(t)} (-\alpha) dxdydzdw \quad (4) \\ &= -\alpha v(t). \quad (5) \end{aligned}$$

Integrating (4), we get

$$V(t) = exp(-\alpha t)V(0). \quad (6)$$

Since $\alpha > 0$, this follows from (6) that $V(t) \to 0$ exponentially as $t \to \infty$. It means that each volume containing the trajectory of this dynamical system (2) shrinks to zero. Subsequently, all the trajectories of the new system ultimately arrive to an attractor. This shows that the new hyperchaotic system (2) is dissipative.

3.3 Invariance

The invariant motion characterized by the scalar dynamics along z- axis

$$z' = -cz, \quad (c > 0)$$

which is globally exponentially stable.

3.4 Equilibrium Points and Their Stability

The equilibrium points of the new hyperchaotic system (2) are obtained by solving the followig equations

$$\begin{aligned} -x - ay + yz &= 0 \\ by - xz - w &= 0 \\ -cz + xy &= 0 \\ dw + y &= 0. \end{aligned} \tag{7}$$

In the above system of equations we take the parameter values as in the equation (2) and on solving, we get the five equilibrium points.

$$E_1 = (0,0,0,0), \quad E_2 = (7.82624, 2.73243, 4.36421, -27.3242), \quad E_3 = (-7.82624, 1.79328, -2.86421, -17.9328),$$
$$E_4 = (7.82624, -1.74328, -2.86421, 17.9328), \quad E_5 = (-7.82624, -2.73243, 4.36421, 27.3243).$$

3.4.1 Proposition:

The equilibrium point E_1 of the system with chosen parameter $a = 1.5,\ b = 2.5,\ c = 4.9$ and c = 0.10 is a saddle and unstable point.
Proof: For the equilibrium point E_1, the jacobian matrix is as follows

$$J = \begin{pmatrix} -1 & -a & 0 & 0 \\ 0 & b & 0 & -1 \\ 0 & 0 & -c & 0 \\ 0 & 1 & 0 & d \end{pmatrix}.$$

The eigenvalues of J are

$$\lambda_1 = 4.9,\ \ \lambda_2 = 1.96332,\ \ \lambda_3 = -1,\ \ \lambda_4 = 0.636675.$$

Here λ_1 and λ_3 are negative real numbers, λ_2 and λ_4 are positive real numbers. This implies that the equilibrium point E_0 is saddle and unstable.

3.4.2 Proposition:

The equilibrium points E_2, E_3, E_4 and E_5 of the system (2) with chosen parameter $a = 1.5,\ b = 2.5,\ c = 4.9$, and $d = 0.10$ are saddle - focus and unstable.
Proof: For the equilibrium point E_2, the jacobian matrix J is as follows

$$\begin{pmatrix} -1 & -a + 4.36421 & 2.73243 & 0 \\ -4.36421 & b & -7.82624 & -1 \\ 2.73243 & 7.82624 & -c & 0 \\ 0 & 1 & 0 & d \end{pmatrix}. \tag{8}$$

The corresponding eigenvalues of jacobian matrix J are as follows

$$\lambda_1 = 0.598476 + 7.9126i,\ \ \lambda_2 = 0.598476 - 7.9126i,\ \ \lambda_3 = -4.6036,\ \ \lambda_4 = 0.10665.$$

Here $\lambda_1,\ \lambda_2$ are complex but have a positive real part, λ_4 is also positive and λ_3 is also negative. It follows that the eigenvalue E_2 is a saddle focus and unstable.
Remark: The equilibrium points $E_3,\ E_4$ and E_5 are also saddle - focus and unstable and can be shown in a similar way as proved in Proposition 3.4.2.

4 Lyapunov Exponents and Kaplan-Yorke Dimension

Based on the chaos theory, Lyapunov exponent is a key component of Chaotic dynamics, and also tells the rate of convergence and divergence of nearby trajectories in the phase space of the system. As it is known that for a four dimensional autonomous system to be hyperchaotic, there should be more than one positive Lyapunov exponent.

For the new system (2) we set the parameters values and initial conditions as, $a = 1.5,\ b = 2.5,\ c = 4.9,\ d = 0.10.$ and $(x_0,\ y_0,\ z_0,\ w_0) = (-0.5,\ 2,\ 3.5,\ 3.3)$, the corresponding spectrum Lyapunov exponents which are calculated by using Wolf Algorithm [29], shown in Fig. 3. The numerical values of the Lyapunov exponents are $L_1 = 0.41812,\ L_2 = 0.107,\ L_3 = 0,\ L_4 = -3.7988$. By theoretical analysis and numerical simulations, the Kaplan - Yorke dimension is given by

$$D_{YK} = j + \frac{1}{\mid \lambda_{L_{j+1}} \mid} \sum_{i=1}^{j} \lambda_{L_i} \tag{9}$$

where j is the largest number satisfying $\sum_{i=1}^{j} \lambda_{L_i} \geq 0$ and $\sum_{i=1}^{j+1} \lambda_{L_i} < 0$. Therefore the Kaplan- Yorke dimension for the hyperchaotic system (2) is $D_{YK} = 3.13135332$, which shows that the Kaplan - Yorke dimension of system (2) is a fractional dimension.

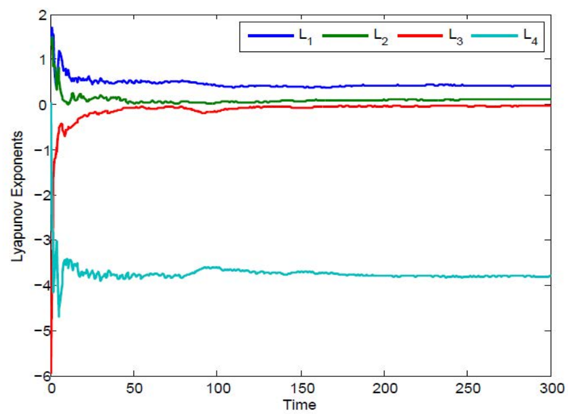

Figure 3: Lyapunov Exponent diagram

5 Poincare Section and Bifurcations Analysis

The Poincare section or surface of section is a very effective technique to describe the bifurcation and folding properties of chaos. The Poincare section of system (2) projected on x-y plane are shown in Fig.4.

In order to analyse the dynamics of the new system (2) with respect to parameter b, we let the parameter b to vary in the interval $[0,\ 2.6]$ and all the other parameters fixed. we fix $a = 1.5,\ b = 2.5,\ c = 4.9$ and $d = 0.10$, with initial conditions given as $(-0.5,\ 2,\ 3.5,\ 3.3)$. According to numerical and theoretical analysis, the function of the bifurcation diagram is to demonstrate how the hyperchaotic system (2) alters with increasing values of the parameter b, see Fig. 5. It displays generous and complex dynamical behaviour. From Fig. 5, it is clear that when b varies between 0.8 and 2.6, the hyperchaotic behaviour of the system (2) arises. When $b = 2.5$ the strange attractors occurs as shown in Fig. 1(a-d).

6 Projective Synchronization of New 4D Hyperchaotic System

Projective Synchronization is attractive because of its equivalence between the synchronized dynamical states, which has a very wide application in secure communication. In this section, we investigate the projective synchronization of the new 4D hyperchaotic system via active nonlinear control approach. The master and slave systems are respectively

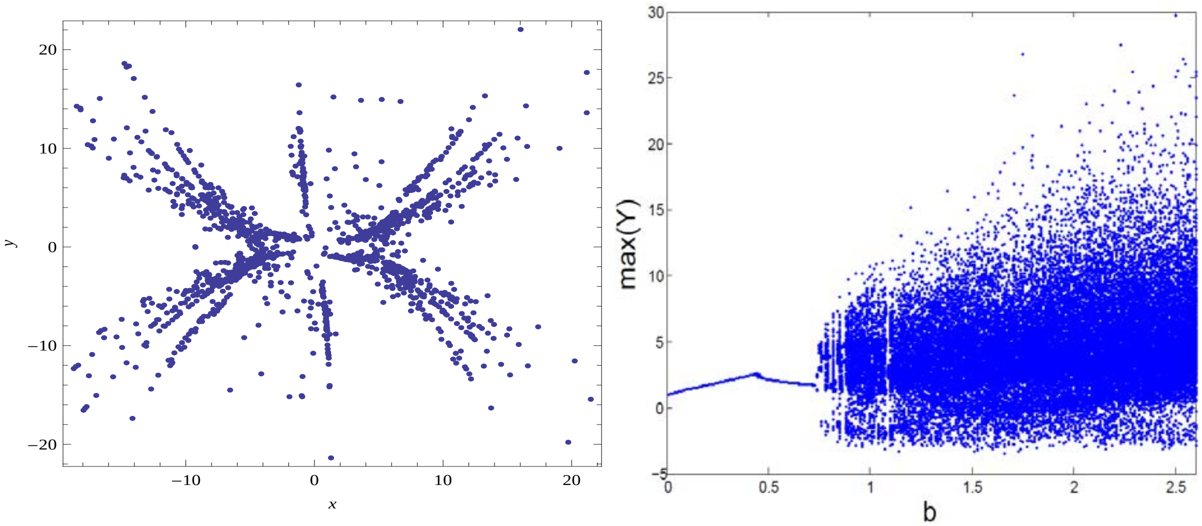

Figure 4: Poincare map and Bifurcation diagram of system (2)

defined as

$$\begin{aligned} \dot{x}_1 &= -x_1 - ay_1 + y_1z_1 \\ \dot{y}_1 &= by_1 - x_1z_1 - w_1 \\ \dot{z}_1 &= -cz_1 + x_1y_1 \\ \dot{w}_1 &= dw_1 + y_1 \end{aligned} \tag{10}$$

and

$$\begin{aligned} \dot{x}_2 &= -x_2 - ay_2 + y_2z_2 + u_1 \\ \dot{y}_2 &= by_2 - x_2z_2 - w_2 + u_2 \\ \dot{z}_2 &= -cz_2 + x_2y_2 + u_3 \\ \dot{w}_2 &= dw_2 + y_2 + u_4 \end{aligned} \tag{11}$$

where $u_1,\ u_2,\ u_3,\ u_4$ are nonlinear controllers to be constructed such that the two new hyperchaotic systems synchronize in the way of projective synchronization.

Define the error system for the projective synchronization as follows:

$$e = y - \theta x$$

where θ is a constant called as scaling factor.

From systems (10) and (11), the error dynamical system is as follows

$$\begin{aligned} \dot{e}_1 &= -e_1 - ae_2 + (y_1z_1) - \theta y_2z_2) - \theta u_1 \\ \dot{e}_2 &= be_2 - (x_1z_2 - \theta x_2z_2) - e_4 - \theta u_2 \\ \dot{e}_3 &= -ce_3 + (x_1y_1 - \theta x_2y_2) - \theta u_3 \\ \dot{e}_4 &= de_4 + e_2 - \theta u_4. \end{aligned} \tag{12}$$

By using the technique of active control method, we define the four controller functions $u_i\ (i = 1,\ 2,\ 3,\ 4)$ as:

$$\begin{aligned} u_1 &= \frac{y_1z_1}{\theta} - y_2z_2 - \frac{v_1}{\theta} \\ u_2 &= \frac{-x_1z_1}{\theta} + x_2z_2 - \frac{v_2}{\theta} \\ u_3 &= \frac{x_1y_1}{\theta} - x_2y_2 - \frac{v_3}{\theta} \\ u_4 &= \frac{-v_4}{\theta}. \end{aligned} \tag{13}$$

Subsequently the error dynamical system (11) becomes

$$\begin{aligned}\dot{e}_1 &= -e_1 - ae_2 + v_1\\ \dot{e}_2 &= be_2 - e_4 + v_2\\ \dot{e}_3 &= -ce_3 + v_3\\ \dot{e}_4 &= de_4 + e_2 + v_4.\end{aligned} \tag{14}$$

Thus the system (14) which is to be controlled is a linear system with control inputs $v_1,\ v_2,\ v_3$ and v_4 which are the functions of $e_1,\ e_2,\ e_3$ and e_4. Now in order to stabilize the system (14) we must choose the controllers $v_1,\ v_2,\ v_3$ and v_4 such that $e_1,\ e_2,\ e_3$ and e_4 converges to zero as time tends to infinity., which shows that projective synchronization between the systems (10) and (11) is achieved with a scaling factor θ. There are various choices for selecting the controllers $v_1,\ v_2,\ v_3$ and v_4. Let us suppose that

$$\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix} = A\begin{pmatrix} e_1\\ e_2\\ e_3\\ e_4\end{pmatrix} \tag{15}$$

where A is a 4×4 matrix to be determined, we choose the matrix A as follows

$$A = \begin{pmatrix} 0 & a & 0 & 0\\ 0 & -2b & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & -2d\end{pmatrix}.$$

Now consider the Quadratic Lyapunov function

$$V(e) = \frac{1}{2}e^Te = \frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2) \tag{16}$$

which is positive definite in $\mathbb{R}^4$. Differentiating (16) along the trajectories of the system (14), we get

$$\dot{V}(e) = -e_1^2 - be_2^2 - ce_3^2 - de_4^2 \tag{17}$$

which is negative definite in $\mathbb{R}^4$. Thus by Lyapunov stability theory we conclude that the system (14) is stable and the projective synchronization between the systems (10) and (11) is achieved.

6.1 Numerical Results

In this subsection, we investigate the theoretical results which are being defined in the section 6. The parameter values of the new 4D hyperchaotic system are taken as $a = 1.5,\ b = 2.5,\ c = 4.9$ and $d = 0.10$. The initial values of the master system (10) and slave system (10) are taken as $(x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)) = (-0.5,\ 2,\ 3.5,\ 3.3)$ and $(x_2(0),\ y_2(0),\ z_2(0),\ w_2(0)) = (0.3,\ 1.3,\ 1.5,\ 1.8)$ respectively. Now we consider two cases:

Case I: When the scaling factor θ is taken to be 2. Simulations results of the new 4D hyperchaotic system using projective synchronization via active nonlinear controller are exhibited in fig 5 and fig 6 . In Figs. 5(a-d) , the concrete lines indicates the states of the drive system (10) and the dotted lines indicates the states of the response system (11). From Figs. 5(a-d), we see that the states of the response system (11) is declined by a half in contrast with the master system (10) in the same phase synchronization design. The synchronization errors e_1,e_2,e_3 and e_4 tend to zero when $t > 12$ as indicated in Fig. 6, it shows that the state variables are synchronized in a proportional way. This concludes that the projective synchronization of the new 4D hyperchaotic system is accomplished.

Case II: When the scaling factor θ is taken to be 1. Simulations results of the new 4D hyperchaotic system using projective synchronization via active nonlinear controller are exhibited in Fig.7 and Fig. 8. In Figs. 7(a-d), the concrete lines indicates the states of the drive system (10) and the dotted lines indicates the states of the response system (11). From Figs. 7(a-d), we see that the states of the response system (11) are completely synchronized with the master system (10). The synchronization errors $e_1,\ e_2,\ e_3$ and e_4 tend to zero, as indicated in Fig. 8, it shows that the state variables are synchronized in a proportional way.

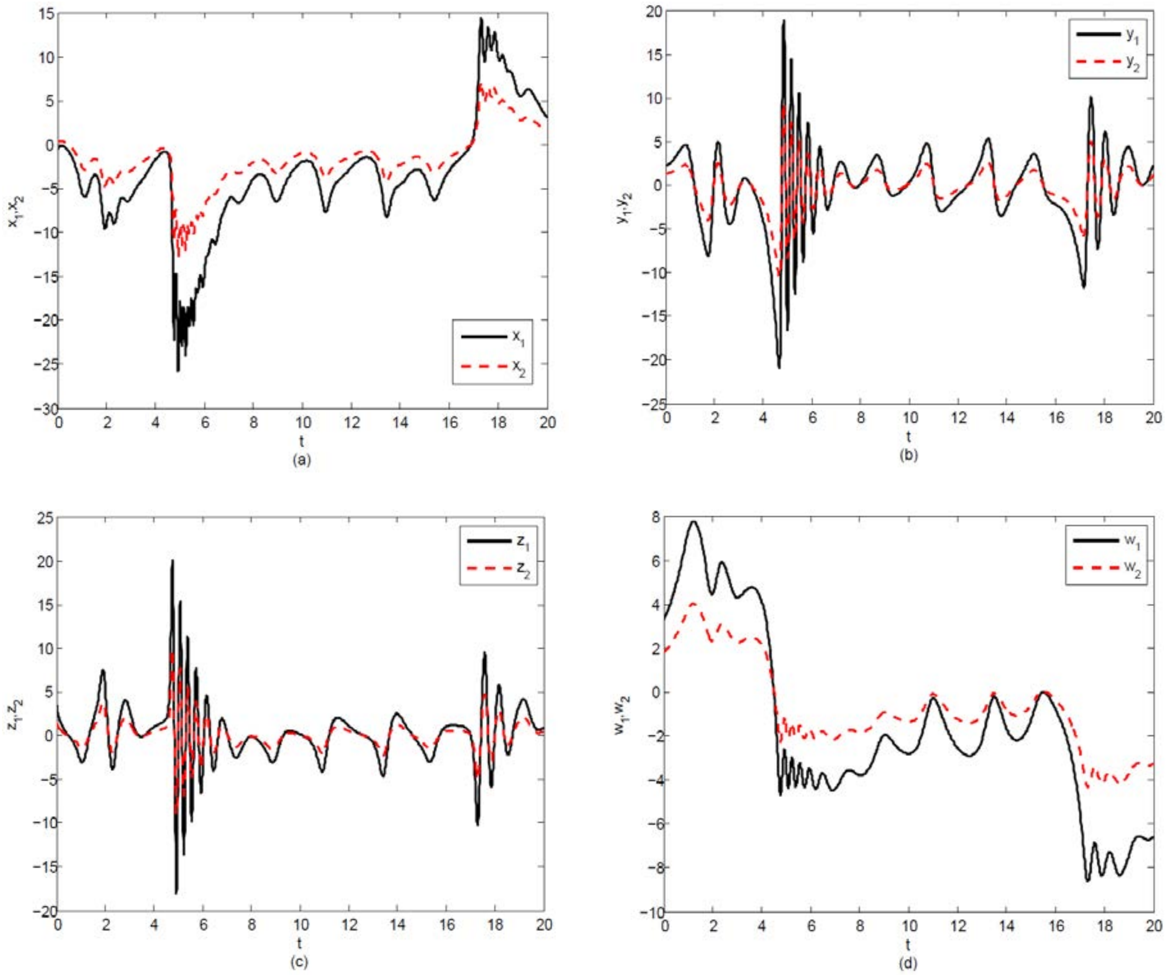

Figure 5: Numerical results of generalized projective synchronization of systems (10) and (11) with $\theta = 2$: (a) Time series of $x_1(t)$ and $x_2(t)$ (b) Time series of $y_1(t)$ and $y_2(t)$ (c) Time series of $z_1(t)$ and $z_2(t)$ (d) Time series of $w_1(t)$ and $w_2(t)$

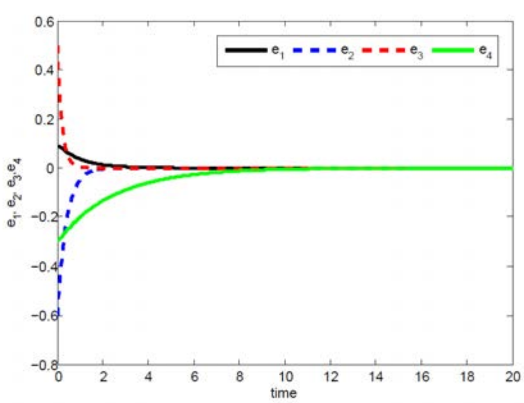

Figure 6: Time series of synchronization errors

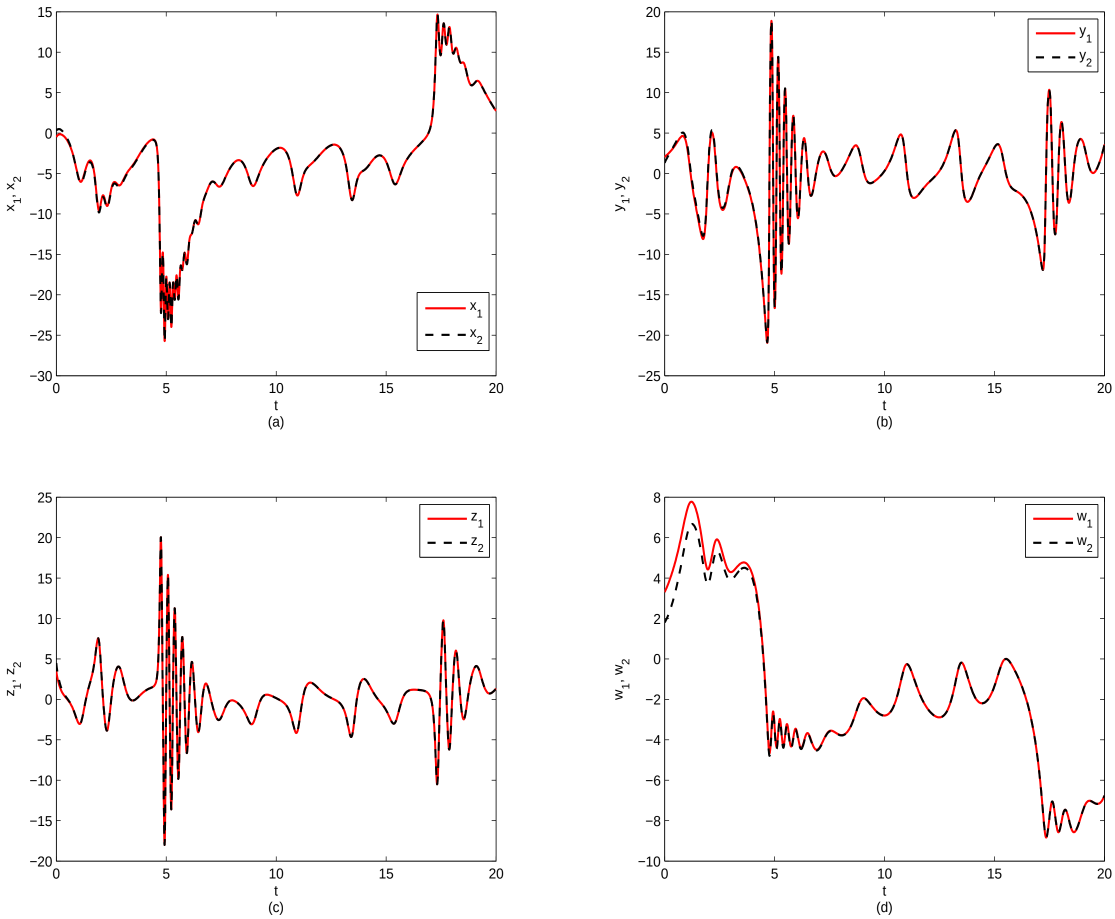

Figure 7: Numerical results of generalized projective synchronization of systems (10) and (11) with $\theta = 1$: (a) Time series of $x_1(t)$ and $x_2(t)$ (b) Time series of $y_1(t)$ and $y_2(t)$ (c) Time series of $z_1(t)$ and $z_2(t)$ (d) Time series of $w_1(t)$ and $w_2(t)$

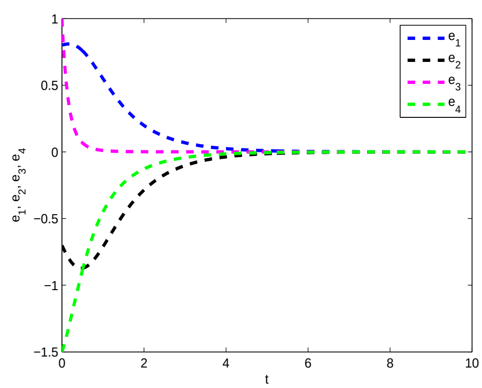

Figure 8: Time series of synchronization errors

7 Conclusions

In this manuscript, a new four - dimensional continuous autonomous hyperchaotic system has been presented, in which each equation consists of almost single cross - product term. Basic properties of the new system have been analysed in terms of Lyapunov exponents, equilibria, fractal dimension, Poincare mapping, and hyperchaotic behaviour. Furthermore, the projective synchronization approach via active nonlinear control method has been simultaneous performed. By establishing the suitable controllers, we have achieved the synchronization. Lastly, numerical results are given to confirm the efficiency of the proposed synchronization scheme. Theoretical and numerical results are in excellent argument.

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