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Robust Optimization for the Single Allocation p-hub Median Problem under Discount Factor Uncertainty

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Abstract

This paper studies the robust optimization for the single allocation p-hub median problem under discount factor uncertainty. We develop the discount factor as a novel uncertain parameter, which represents different connection modes between hubs. The discount factor can be described by a random variable with discrete distribution in nominal case. We apply a robust optimization approach to deal with the perturbation of probability in special circumstances and control it by an interval uncertainty set. The robust counterpart model has been shown tractable by using commercial solver. Finally, computational experiments are given to demonstrate the robustness by comparing the performance with the nominal and robust solutions. The results indicate robust model can immunize the system against parameter perturbations relative to nominal model.

Keywords: robust optimization, hub median problem, discount factor, single allocation

1 Introduction

Hub location problems are used in many applications, like logistics, airlines, telecommunications and other types of transportation networks. The problem concerns with finding the location of hubs and allocating demands between an origin-destination pair via the hub. The hub is defined as particular facilities that lead to reducing time, cost and other parameters. The studying on the hub location problem has various extensions−−single allocation, multiple allocation, capacitated, uncapacitated.

In hub location problems, the location of hub facilities is usually highly uncertain and depends on many factors such as costs, demands, distances and other parameters. This paper considers the uncertainty with discount factor which is few studies in the existing research. For inter-hub transportation, there will be multiple modes of transportation corresponding to different discount factor values. In general, the discrete distribution of discount factor is available but subject to limited information. Many unforeseen factors such as traffic congestion, weather changes and road failure that have a direct impact on the distribution. In this regard, this paper consider a hub network with perturbation of probability distribution. Stochastic optimization is an important methodology for dealing with uncertainty, and it has a strong dependence on distribution. However, in many cases, it may be difficult to know the precise distribution. Another popular methodology−−robust optimization can find the optimal solution in worst case scenario with an unknown or partial known probability distributions.

In this paper, we develop a robust optimization model for the single allocation p-hub median problem under discount factor uncertainty. The objective of the model is to minimize the total cost of movement. The available information is the nominal probability distribution of the uncertain discount factor, which resides in an interval of uncertainty. More specifically, several discount factors are applied to the transportation costs, and each of discount factors corresponds to a nominal probability value. Obviously the sum of the perturbation of probability equals to zero. The goal of the robust optimization is to guarantee the location decisions processes against the discount factors uncertainty.

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The main contributions of this paper are the following. Firstly, we study a single allocation p-hub median problem, the discount factor of which has discrete probability distribution. Secondly, we consider the perturbation of probability distribution. Thirdly, we employ robust optimization strategy to deal with the perturbation which belongs to an interval uncertainty set.

The remainder of the paper is organized as follows. In Section 2, we provide a literature review. Section 3 gives a nominal model for the scenario based single allocation p-hub median model. In Section 4, in consideration of discount factors uncertainty, robust optimization is applied in the model. Section 5 presents a computational experiment. And finally, Section 6 concludes the paper.

2 Literature Review

Three streams of literature relevant to this paper are summarized: the p-hub median problem, robust p-hub problem and solving methods.

The p-hub median problem is first formulated by Campbell [4, 5]. Campbell [5] proposes linear integer programming formulations for four versions of hub location problem such as p-hub median problem, the uncapacitated hub location problem, p-hub center problems, and hub covering problem. [21] proposes a p-hub median problem that arises in the design of a star-star network. Parvaresh et al. [15] formulate the multiple allocation p-hub median problem under intentional disruptions as a bi-level game model. Yang et al. [23] present a new risk aversion p-hub center problem, in which value-at-risk criterion is adopted in the formulation of objection function. Xavier et al. [20] propose a continuous multiple allocation p-hub median problem, which corresponds to a strongly non-differentiable min-sum-min formulation. Talbi and Todosijević [18] consider an uncertain uncapacitated multiple allocation p-hub median problem and study several ways to deal with the uncertainty.

Robust optimization is a specific methodology that may be outperformed with an unknown or partial known probability distributions. Some researchers apply the robust optimization to study the highly unpredictable nature of p-hub location problem. Ghezavati et al. [9] design a robust location-allocation model with uncertainty in distances. Rahmaniani et al. [16] propose an extension of the capacitated facility location problem with uncertain demands and costs. Amin-Naseri et al. [1] present a robust bi-objective uncapacitated single allocation p-hub median problem, which minimizes the transportation cost and maximum uncertainty in network. A heuristic based on scatter search and variable neighborhood descent is developed to solve this problem. Zetina et al. [26] use discrete robust optimization techniques in hub location problem, in which the demands and transportation costs are interval of uncertainty. In most of robust p-hub location problem, the uncertain factors are always related to the transportation costs, demands and flows. For instance, Makui et al. [12] establish a multi-objective robust capacitated p-hub location problem with uncertain demands and processing commodity time. Shahabi and Unnikrishnan [17] also consider the uncertain demand, which is assumed to lie in an ellipsoidal uncertainty set. Hult et al. [10] consider the stochastic nature of travel times and build an uncapacitated single allocation p-hub center model. Ghaffari-Nasab et al. [8] considers the capacitated single and multiple allocation hub location problems with stochastic demands. Meraklı and Yaman [13] study the robust uncapacitated multiple allocation p-hub median problem under polyhedral demand uncertainty. Talbi and Todosijević [18] focus on the uncertainty of flows in hubs.

To the best of our knowledge, there exist few published articles study the discount factor with uncertain distribution. In general, the discount factor is deterministic parameter in p-hub location problem [23, 25, 8]. In fact, the discount factor can also changed with the various uncertain factors. Yang et al. [22] establishes a stochastic air freight hub location and flight routes planning problem, where the discount factors are assumed as stochastic variables with known distributions. In the real world, the mangers always make the decisions when only limited discrete discount factor distribution information is available. In this paper, we study a robust single allocation p-hub median problem, in which the discount factor has discrete uncertain probability distribution.

Since p-hub location problem is a classical mixed integer programming (MIP) (which is a NP-hard problem), the solving method is also a key issue in p-hub location problem. Some classical algorithms like branch and bound [7], Benders decomposition [6], are proposed to solve such MIP problem. Hult et al. [10] develop exact solution approaches based on variable reduction and a separation algorithm. For some complicated p-hub location problem, heuristic intelligent algorithms are designed, such as the simulated annealing [14, 15], genetic algorithms [19, 24], particle swarm optimization [23] and electromagnetism-like (EM) metaheuristic [11].

3 p-hub Median Model with Deterministic Distribution

In this section, we present a single allocation p-hub median model with deterministic distribution. Let $G = (N, A)$ be a network, where N represents the set of nodes and A is the set of arcs. The hub network is a complete graph, i.e., all hubs are connected to one another via a direct link. Direct links between the nodes are not permissible, which means every node $i \in N$ is connected to a hub node. Also, let C_0 and d_{ij} represent the unit transportation cost and the distance along the path $i \to k \to l \to j$, respectively. Then the cost is comprised of three segments: collection cost $C_0 d_{ik}$, distribution cost $C_0 d_{lj}$ and transfer cost $C_0 d_{kl}$. Decision variable X_{iklj} determines the allocation, and decision variable Y_{ik} decides hub locations. For notational convenience, we set decision vectors $X = X_{iklj}$ and $Y = Y_{ik}$.

The sets, parameters and variables used in the model are listed as follows:

Sets

N: set of nodes;

S: set of scenarios;

Parameters

d_{ij}: the distance across link (i, j);

α^s: discount factor under scenario s;

C_0: transportation cost per unit flow per unit distance;

p: number of hubs to be opened;

P^s_{kl}: the probability that scenario s occurs across pubs (k, l);

Variables

X_{iklj}: a binary decision variable indicating whether the flow across link (i, k, l, j) or not;

Y_{ik}: a binary decision variable indicating whether node i is located to hub k or not.

In the p-hub median problem, the node is allocated to a single hub, which is the nearest one. Due to higher transmission efficiency on the inter-hub links, the inter-hub transportation mode is generally discounted. Hence the inter-hub discount factor α plays an important role in determining the hub locations decisions processes. In several scenarios, some probabilistic information P^s_{kl} is known for these discount factors and can be used to minimize the total expected cost $P^s_{kl}\alpha^s$ by using stochastic programming techniques.

Based on the notations, the proposed mixed-integer programming model formulation is as follows:

$$\min_{X,Y} \quad \sum_{s\in S}\sum_{i\in N}\sum_{k\in N}\sum_{l\in N}\sum_{j\in N} C_0(d_{ik} + P^s_{kl}\alpha^s d_{kl} + d_{lj})X_{iklj} \tag{1}$$

$$\text{s. t.} \quad \sum_{k\in N} Y_{kk} = p \tag{2}$$

$$Y_{ik} \leq Y_{kk}, \quad \forall i, k \in N \tag{3}$$

$$\sum_{k\in N}\sum_{l\in N} X_{iklj} = 1, \quad \forall i, j \in N \tag{4}$$

$$\sum_{l\in N} X_{iklj} = Y_{ik}, \quad \forall i, j, k \in N \tag{5}$$

$$\sum_{k\in N} X_{iklj} = Y_{jl}, \quad \forall i, j, l \in N \tag{6}$$

$$X_{iklj} \in \{0, 1\}, \quad \forall i, k, l, j \in N \tag{7}$$

$$Y_{ik} \in \{0, 1\}, \quad \forall i, k \in N. \tag{8}$$

In the above formulation, the objective function (1) minimizes the total transportation costs of collection, transfer and distribution. Note that each one of scenarios in transfer cost has the probability P^s and minimizes the total expected cost. Constraint (2) ensures that exactly p hubs are opened. Constraints (3) guarantees that each node i can be assigned to hub k, and (4) states that the node i is just allocated to one hub. Constraints (5) and (6) mean that flow from origin i to designation j cannot be allocated to a hub pair k and l via path $i \to k \to l \to j$ unless node i is allocated to hub k and j is allocated to hub l. Constraints (7) and (8) are binary integrality constraints.

4 Robust Optimization Formulation

As mentioned above, only nominal discrete discount factor distribution information P_{kl}^s is available in advance. However, traffic, weather and unforeseen factors can lead to a significant impact on the distribution. In this regard, the model should be considered as a hub network with perturbation of probability distribution. To deal with the perturbation, we apply the robust optimization approach developed by Ben-Tal et al. [2] for the single allocation p-hub median model. The only information available about probability distribution of the discount factor is that they are varying in a given perturbation set with bounded supports, i.e., box uncertainty set U_{box}.

Next we define the box uncertainty set below:

$$U_{box} = \{P_j | P_j = \hat{P}_j + \xi_j, e^T \xi_j = 0, |\xi_j| \leq \theta, \forall j \in J_i\}, \tag{9}$$

where $\hat{P}_j$ represents the historical estimated probability distribution; ξ denotes the uncertain parameter vector, and e signifies the vector of one; θ is the adjustable parameter controlling the size of the uncertainty set. Note that the condition $e^T\xi_j = 0$ ensures P_j will meet the requirements of the probability distribution.

The robust counterpart formulation for the p-hub median problem can be formulated as a min-max problem:

$$\begin{array}{lll} \min\limits_{X,Y} \max\limits_{P_{kl}^s \in U_{box}} & \displaystyle\sum_{s\in S}\sum_{i\in N}\sum_{k\in N}\sum_{l\in N}\sum_{j\in N} C_0(d_{ik} + P_{kl}^s \alpha^s d_{kl} + d_{lj}) X_{iklj} & \\ \text{s. t.} & \displaystyle\sum_{k\in N} Y_{kk} = p & \\ & Y_{ik} \leq Y_{kk}, \quad \forall i,k \in N & \\ & \displaystyle\sum_{k\in N}\sum_{l\in N} X_{iklj} = 1, \quad \forall i,j \in N & \\ & \displaystyle\sum_{l\in N} X_{iklj} = Y_{ik}, \quad \forall i,j,k \in N & \\ & \displaystyle\sum_{k\in N} X_{iklj} = Y_{jl}, \quad \forall i,j,l \in N & \\ & X_{iklj} \in \{0,1\}, \quad \forall i,k,l,j \in N & \\ & Y_{ik} \in \{0,1\}, \quad \forall i,k \in N. & \end{array} \tag{10}$$

The above robust counterpart formulation minimizes the total cost of transportation cost with respect to the worst case realization of probability. In order to reduce the above formulation to a computationally tractable formulation, we transform the inner maximization problem into its conic dual, and then incorporate the dual problem into the original objective function.

If the distribution of probability P_{kl}^s belongs to a box uncertainty set defined in Eq. (9), we have

$$\max_{P_{kl}^s \in U_{box}} \sum_{s\in S} \sum_{(i,k,l,j)\in N} C_0 P_{kl}^s \alpha^s d_{kl} X_{iklj} = \max_{\xi_{kl}^s \in U_{box}} \sum_{s\in S} \sum_{(i,k,l,j)\in N} C_0(\hat{P}_{kl}^s + \xi_{kl}^s)\alpha^s d_{kl} X_{iklj}.$$

Then the inner maximization problem can be rewritten as follows with dual variables γ_{kl}, τ_{kl}^s and ν_{kl}^s.

$$\begin{aligned} &\max_{\xi_{kl}^s \in U_{box}} \left\{ \sum_{s\in S} \sum_{(i,k,l,j)\in N} C_0 \xi_{kl}^s \alpha^s d_{kl} X_{iklj} \middle| e^T \xi_{kl}^s = 0, |\xi_{kl}^s| \leq \theta \right\} \\ &= \min_{\gamma,\tau,\nu} \left\{ \sum_{s\in S} \sum_{(k,l)\in N} \theta(\tau_{kl}^s + \nu_{kl}^s) \middle| e^T\gamma_{kl} + \tau_{kl}^s - \nu_{kl}^s = \sum_{(k,l)\in N} C_0 \alpha^s d_{kl} X_{iklj}, \tau, \nu \geq 0 \right\}. \end{aligned} \tag{11}$$

Substituting (11) into the objective function of (10), the min operator in (11) can be omitted, since if the constraint holds for some γ_{kl}, τ^s_{kl} and ν^s_{kl}, then it holds for the minimum. We get the following formulation:

$$
\begin{aligned}
\min_{X,Y} \quad & \sum_{s\in S}\sum_{i\in N}\sum_{k\in N}\sum_{l\in N}\sum_{j\in N} C_0(d_{ik}+P^s_{kl}\alpha^s d_{kl}+d_{lj})X_{iklj}+\sum_{s\in S}\sum_{(k,l)\in N}\theta(\tau^s_{kl}+\nu^s_{kl}) \\
\text{s. t.} \quad & e^T\gamma_{kl}+\tau^s_{kl}-\nu^s_{kl}=\sum_{(k,l)\in N} C_0\alpha^s d_{kl}X_{iklj}, \quad \forall s\in S, i,j\in N \\
& \sum_{k\in N} Y_{kk}=p \\
& Y_{ik}\le Y_{kk}, \quad \forall i,k\in N \\
& \sum_{k\in N}\sum_{l\in N} X_{iklj}=1, \quad \forall i,j\in N \\
& \sum_{l\in N} X_{iklj}=Y_{ik}, \quad \forall i,j,k\in N \\
& \sum_{k\in N} X_{iklj}=Y_{jl}, \quad \forall i,j,l\in N \\
& X_{iklj}\in\{0,1\}, \quad \forall i,k,l,j\in N \\
& Y_{ik}\in\{0,1\}, \quad \forall i,k\in N \\
& \tau^s_{kl},\nu^s_{kl}\ge 0, \quad \forall s\in S, k,l\in N.
\end{aligned}
\tag{12}
$$

The model is a mixed integer linear program with binary variables, which has the same complexity properties as the deterministic problem. This property makes the problem efficiently solvable using standard optimization packages (maybe a standard branch-and-bound code) for test problems of small and moderate sizes.

5 Computational Experiments

In this section, we describe computational experiments to evaluate the solution of the robust formulation for single allocation p-hub median problem with discount factor uncertainty. All experiments were carried out using CPLEX 12.6 on a PC of AMD 1.80 GHz CPU and 4 GB RAM, running under Windows 8 operations system.

The network includes 15 nodes with 3 hubs. These coordinates of nodes are generated randomly. We multiply X and Y positions and then calculate Euclidean distances between nodes. All of the flows consist of 3 scenarios with a corresponding nominal probability distribution $P^s=\{0.5,0.3,0.2\}$. The transportation cost C_0 in the objective function is assumed to be 10, and the discount factor α^s is chosen randomly from an interval $[0,1]$. Table 1 summarizes values of these parameters for the problem instance.

Table 1: The input data of the instance

Title	Value
N	15
p	3
P^s	(0.5, 0.3 ,0.2)
α^s	[0,1]
C_0	10

After a preliminary setting, we first illustrate the differences between the robust counterpart solution and the nominal solution. The robust counterpart solution means the solution in uncertain distribution. In robust model, adjustable parameter θ controls the level of uncertainty of the robust counterpart solution. In this paper, we take the adjustable parameter value in an interval between $[0,0.4]$. Note that $\theta=0.4$ represents the most conservative statement. In addition, the nominal solution represents the solution in deterministic distribution, i.e., $\theta=0$.

Next we will test the model in two aspects: (a) discount factors α are vary in $[0,1]$ for every fixed value of θ in $[0,0.4]$; (b) adjustable parameter θ are vary in $[0,0.4]$ for every fixed value of α in $[0,1]$.

5.1 Impact of Discount Factors

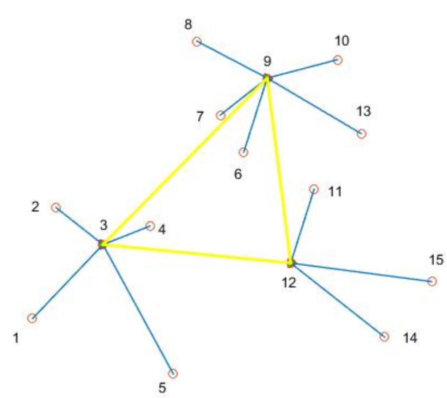
Firstly we test the performance of the robust model when discount factors change. Then we set the adjustable parameter value $\theta = 0.05$ and choose three kinds of scenario sets including $\alpha \in \{0.2, 0.3, 0.5\}$, $\alpha \in \{0.3, 0.4, 0.6\}$ and $\alpha \in \{0.5, 0.6, 0.8\}$. Table 2 shows the calculation results.

Table 2: Results of robust optimization model when $\theta = 0.05$

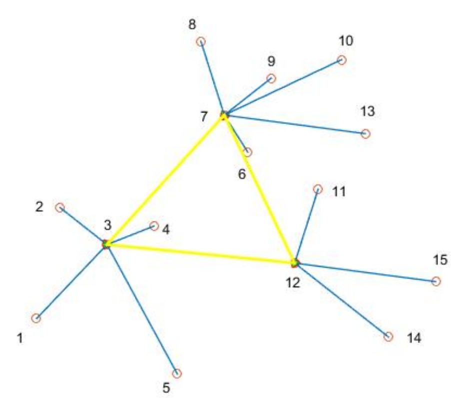
α	*Title*	*Nominal*	*Robust*
	hub locations	3, 9, 12	3, 7, 12
0.2, 0.3, 0.5	ideal cost	19996.84	20202.13
	violated cost	20220.85	20202.13
	hub locations	3, 7, 12	3, 6, 12
0.3, 0.4, 0.6	ideal cost	21263.44	21433.03
	violated cost	21450.76	21433.03
	hub locations	3, 6, 12	4, 6, 12
0.4, 0.6, 0.9	ideal cost	23067.06	23287.56
	violated cost	23330.62	23287.56

In Table 2, the column *Nominal* represents the value in deterministic distribution model. Meanwhile the column *Robust* means the value in robust model. The value for each group of α, the entries of the rows *hub locations* list the hub locations in nominal and robust models. The *ideal cost* row shows the value of the total nominal cost and *violated cost* row shows the actual cost which we may suffer when sticking to the nominal optimal solution when the actual data violate it. These two rows present the comparison of the two model values in the presence of whether a perturbation occurs.

Figure 1 shows the resulting hub networks from the nominal model and the robust counterpart. For the first group of discount factor $\alpha \in \{0.2, 0.3, 0.5\}$, the selected hubs in nominal model are $3, 9, 12$ and in robust model are $3, 7, 12$. The hub has changed from the nominal model 9 to the robust model 7. The row *ideal cost* in Table 2 shows the cost of robust is 1.03% larger than the nominal one, since the ideal cost ignores the uncertainty of the mentioned parameters and the robust cost is in a worst-case scenario. However, when the ideal data against it, the nominal model may suffer a bigger perturbation, i.e., 1.12% more than the nominal cost. The "price" of robustness is superior to its violated cost.

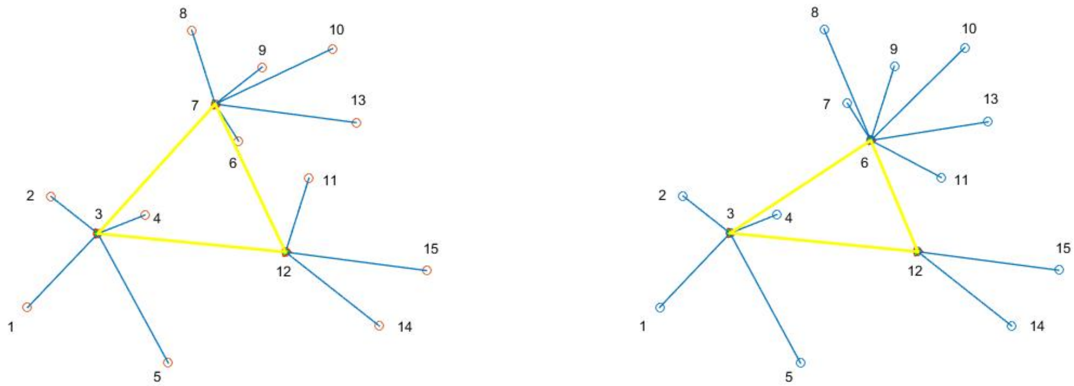


(a) Hub network of nominal model



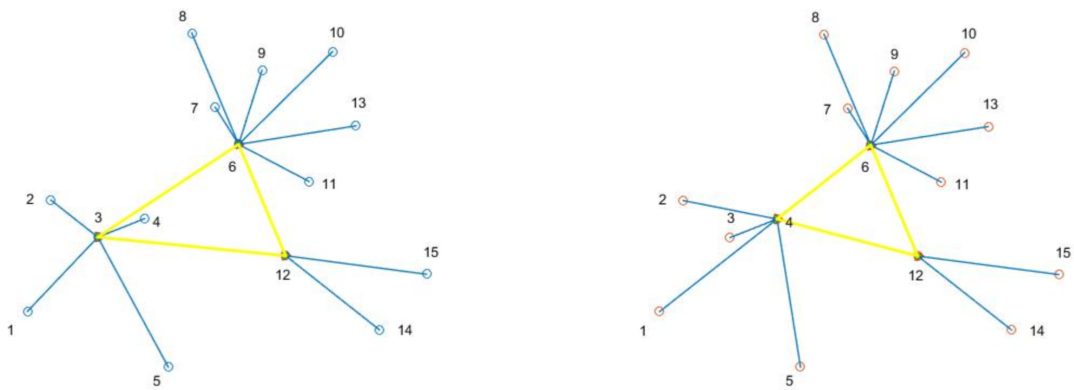
(b) Hub network of robust model

Figure 1: Results with hub networks when $\alpha \in \{0.1, 0.2, 0.4\}$



(a) Hub network of nominal model (b) Hub network of robust model

Figure 2: Results with hub networks when $\alpha \in \{0.2, 0.4, 0.8\}$



(a) Hub network of nominal model (b) Hub network of robust model

Figure 3: Results with hub networks when $\alpha \in \{0.4, 0.6, 0.9\}$

Increasing the discount factor by 0.1, the second group of discount factor $\alpha \in \{0.2, 0.3, 0.5\}$ chooses hubs in nominal model are $3, 7, 12$ and in robust model are $3, 6, 12$ as shown in Figure 2. At the same time the hub changes, the route of node 11 changes greatly. With α growing, the cost in nominal and robust models also enlarge.

By increasing the discount factor and enlarging the scope of the discount factor, the third group $\alpha \in \{0.2, 0.3, 0.5\}$ further changes the locations of hubs, i.e., from hubs $3, 6, 12$ to $4, 6, 12$ as in Figure 3. Meanwhile the violated cost is 1.14% more than the ideal cost. For that, the larger the scope of the discount factor, the higher the violated cost caused by uncertainty.

Note that the changes of discount factor can have a significant impact on the choice of hub locations. The structure of the robust optimal solution is quite different from the nominal optimal solution, i.e., the solutions of nominal model have perturbation by uncertainty whereas the robust solutions hold unchanged. Therefore, the solutions indeed become more robust.

5.2 Impact of Adjustable Parameter

In this subsection, we compare the solutions from the nominal and robust models with varied adjustable parameter values of θ. Then we set a larger scope discount factor $\alpha \in \{0.2, 0.4, 0.8\}$ and calculate θ varied from 0 to 0.4 in intervals of 0.1. The detailed results of the comparison are given in Table 4.

Table 3: Results of robust optimization model when $\alpha \in \{0.2, 0.4, 0.8\}$

θ	*Title*	*Nominal*	*Robust*
	hub locations	$3, 7, 12$	$3, 6, 12$
0.1	ideal cost	21138.57	21802.00
	violated cost	21887.88	21802.00
	hub locations	$3, 7, 12$	$3, 6, 12$
0.2	ideal cost	21138.57	22434.53
	violated cost	22637.06	22434.53
	hub locations	$3, 7, 12$	$3, 6, 12$
0.3	ideal cost	21138.57	23067.06
	violated cost	23386.31	23067.06
	hub locations	$3, 7, 12$	$4, 6, 12$
0.4	ideal cost	21138.57	23585.54
	violated cost	24135.55	23585.54

In Table 3, the value for each θ, it can be seen that the hub locations in nominal model hold unchanged whereas it changes in robust model at $\theta = 0.4$. In addition, on one hand, because of ignoring uncertainty, the *Nominal* column keep constant value in *ideal cost* as θ varied whereas there has been a big perturbation in *violated cost*. On the other hand, the entries of the column *Robust* show that the cost just increases with the increase in the interval of uncertainty. The robust strategy gives the best worst-case objective value that guarantees 100% immunization against perturbations.

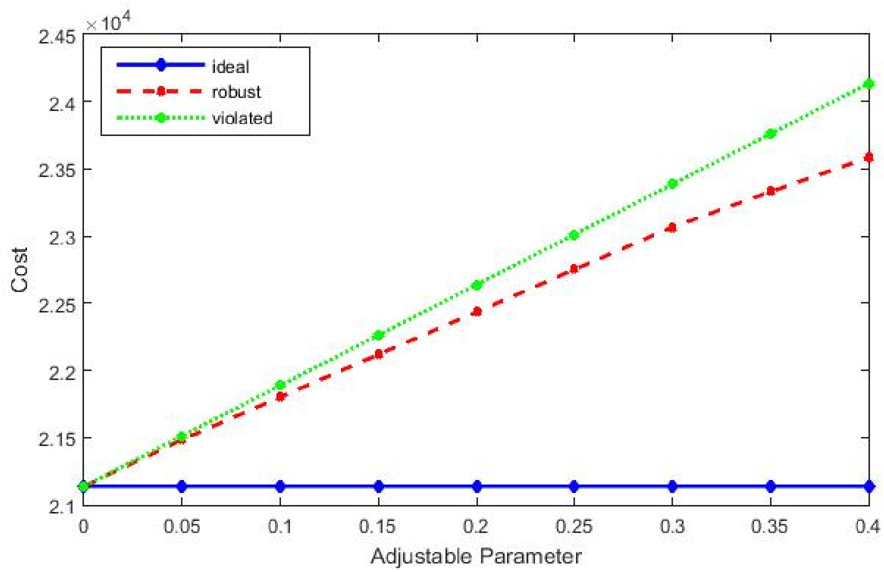


Figure 4: Relationship between the total cost of nominal and robust models

Figure 4 depicts the effect of the interval of uncertainty on the total costs for the nominal and robust models. The adjustable parameter θ is a protection parameter. As it increases, the value of robust cost is higher than that of nominal cost. However, we can observe that the violated cost suffer a substantial growth when the nominal model has a perturbation. It can be easily deducted from Figure 4 that it is necessary to set the interval of uncertainty at its higher possible level in order to gain a more valuable protection against the violation.

In summary, the uncertain parameters have a great influence on the optimal solution. Some unpredictable perturbations can make the solution infeasible even result in loss of life and property. It is suitable to accept the robust solution which is the best uncertainty immunized.

6 Extension: Discount Factor with Ellipsoid Distribution

As stated in the previous section, the formulation of robust counterpart can be described by different uncertainty sets. In this section, we will make an extension that introduce an ellipsoid uncertainty set $U_{ellipsoidal}$ as follows:

$$U_{ellipsoidal} = \{P_j | P_j = \hat{P}_j + \xi_j, e^T \xi_j = 0, \sqrt{\sum_{j\in J_i} \xi_j^2} \leq \Omega, \forall j \in J_i\}. \tag{13}$$

In the case of (9), the box uncertainty set guarantees every solution is feasible for the perturbation of probability distribution. However, in reality, the uncertainty set is not necessary defined to cover the uncertain space. And there allows some degrees of constraint violation. For instance, the ellipsoid uncertainty set, that is, the conic representation, guarantees less 100% immunization, and holds less conservative robust solutions than the box robust counterpart.

According to the robust counterpart formulation of (10), we transform the inner maximization problem into its conic dual. It is assumed that if the distribution of probability P_{kl}^s belongs to the ellipsoid uncertainty set defined in Eq. (13), then the inner maximization problem can be rewritten as follows:

$$\max_{\xi_{kl}^s \in U_{ellipsoid}} \left\{ \sum_{s\in S} \sum_{(i,k,l,j)\in N} C_0 \xi_{kl}^s \alpha^s d_{kl} X_{iklj} \middle| e^T \xi_{kl}^s = 0, \|\xi_{kl}^s\|_2 \leq \Omega, \forall s \in S, (k,l) \in N \right\}. \tag{14}$$

We define the Lagrangian L associated with the problem (14) as

$$L(\xi, \Omega, \mu) = \sum_{s\in S} \sum_{(i,k,l,j)\in N} C_0 \xi_{kl}^s \alpha^s d_{kl} X_{iklj} + \sum_{s\in S} \sum_{(k,l)\in N} \mu_{kl}^s \left(\Omega - \|\xi_{kl}^s\|_2\right) + \sum_{s\in S} \sum_{(k,l)\in N} \omega_{kl} e^T \xi_{kl}^s, \tag{15}$$

where the vectors ω_{kl} and μ_{kl}^s are called the dual variables.

Then the Lagrangian dual function g is defined as the maximum value of the Lagrangian over ξ (see [3], p221),

$$\begin{aligned} g(\omega,\mu) &= \max_{\xi} L(\xi,\omega,\mu) \\ &= \max_{\xi} \left\{ \sum_{s\in S} \sum_{(i,k,l,j)\in N} C_0 \xi_{kl}^s \alpha^s d_{kl} X_{iklj} + \sum_{s\in S} \sum_{(i,k,l,j)\in N} \mu_{kl}^s \left(\Omega - \|\xi_{kl}^s\|_2\right) + \sum_{s\in S} \sum_{(i,k,l,j)\in N} \omega_{kl} e^T \xi_{kl}^s \right\} \\ &= \sum_{s\in S} \sum_{(k,l)\in N} \mu_{kl}^s \Omega + \max_{\xi} \left\{ \sum_{s\in S} \sum_{(i,k,l,j)\in N} \left(C_0 \alpha^s d_{kl} X_{iklj} + \omega_{kl} e^T\right) \xi_{kl}^s - \sum_{s\in S} \sum_{(k,l)\in N} \mu_{kl}^s \|\xi_{kl}^s\|_2 \right\} \\ &= \sum_{s\in S} \sum_{(k,l)\in N} \mu_{kl}^s \Omega + \sum_{s\in S} \sum_{(i,k,l,j)\in N} f^* \left(C_0 \alpha^s d_{kl} X_{iklj} + \omega_{kl} e^T\right), \end{aligned} \tag{16}$$

where the conjugate function of $f(\xi) = \sum_{s\in S} \sum_{(k,l)\in N} \mu_{kl}^s \|\xi_{kl}^s\|_2$ is given by

$$f^*(y) = \begin{cases} 0 & \|y\|_2 \leq \mu \\ \infty & otherwise. \end{cases} \tag{17}$$

Since the Lagrange dual function yields upper bounds for any $\mu_{kl}^s \geq 0$, we have the equivalent formulation of (14)

$$\min_{\omega,\mu} g(\omega,\mu) = \min_{\omega,\mu} \left\{ \sum_{s\in S} \sum_{(k,l)\in N} \mu_{kl}^s \Omega \middle| \begin{array}{l} \|C_0 \alpha^s d_{kl} X_{iklj} + \omega_{kl} e^T\|_2 \leq \mu_{kl}^s \\ \mu_{kl}^s \geq 0, \forall s \in S, (k,l) \in N \end{array} \right\}. \tag{18}$$

Incorporation of the above conic dual into the objective function of (10) and removal of the min operator obtains the tractable formulation. We can also employ the above robust counterpart under ellipsoid uncertainty set to deal with the uncertainty.

7 Conclusions

In this paper, we study a single allocation p-hub median model under a novel uncertain parameter−−discount factor. The discount factor is an important parameter that represents the scale economies on the inter-hub linkage. Unpredictable perturbation of discount factor can make the nominal optimal solution completely meaningless in p-hub model, thus ultimately leading to loss of property. To solve it, we consider a robust optimization approach to deal with uncertain parameter and employ an interval of uncertainty to describe it. Computational experiments are presented to evaluate the performance of our model. The results show that the optimal solutions are very sensitive to adjustable parameter. The robust optimization can provide the solution robustness.

For future studies, we also propose to consider more uncertain parameters for the capacitated hub location problem. And we may employ some effective algorithms for the larger scale models.

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