Journal of Uncertain Systems
Vol.11, No.3, pp.221-229, 2017
Online at: www.jus.org.uk

Full Averaging of Fuzzy Hyperbolic Differential Inclusions

Andrej V. Plotnikov[1,*], Tatyana A. Komleva[2], Irina V. Molchanyuk[1]
[1]*Department of Applied Mathematics, Odessa State Academy of Civil Engineering and Architecture*
4 Didrihsona str., 65029 Odessa, Ukraine
[2]*Department of Mathematics, Odessa State Academy of Civil Engineering and Architecture*
4 Didrihsona str., 65029 Odessa, Ukraine

Received 20 October 2016; Revised 30 October 2016

Abstract

Proper design for engineering applications requires detailed information of the system-property distributions such as temperature, velocity, density, etc., in the space and time domain. This information can be obtained by either experimental measurement or mathematics simulation. Although experimental measurement is reliable, it needs a lot of effort and time. Therefore, the mathematics simulation has become a more and more popular method. Frequently, those engineering design problems deal with a set of partial differential equations, which are to be numerically solved such as heat transfor, solid and fluid mechanics. When a physical problem is transformed into a deterministic partial differential equation, we cannot usually be sure that this modeling is perfect. Also, the initial and boundary value may not be known exactly. If the nature of errors is random, then instead of a deterministic problem, we get a random partial differential equation with random initial and boundary values. But if the underlying structure is not probabilistic, e.g. because of subjective choice then it may be appropriate to use fuzzy numbers instead of real random variables. As a result, we get systems, which are described by fuzzy partial differential equations or inclusions. In this article, we considered the fuzzy hyperbolic differential inclusions (fuzzy Darboux problem), introduced the concept of R-solution and proved the existence of such a solution. Also the substantiation of a possibility of application of full averaging method for hyperbolic differential inclusions with the fuzzy right-hand side with the small parameters is considered.

Keywords: hyperbolic differential inclusion, fuzzy, averaging, R-solution

1 Introduction

In 1990 Aubin [6] and Baidosov [7, 8] introduced differential inclusions with the fuzzy right-hand side. Their approach is based on usual differential inclusions. Hüllermeier [20, 21] introduced the concept of R-solution similarly how it has been done in [33]. Later, the various properties of fuzzy solutions of differential inclusions, and their use in modeling various natural science processes were considered (see [1, 3, 4, 18, 19, 26, 27] and the references therein).

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. After the systematic researches done by Krylov and Bogoliubov [25], in 1930s, the averaging method gradually became one of the classical methods in analyzing nonlinear oscillations (see [10, 12, 38, 41] and the references therein). In works Plotnikov et al. [35, 36, 37] the possibility of application of schemes of full and partial averaging for fuzzy differential inclusions with a small parameter was proved.

Lately, Abdul Rahman and Ahmad [2], Arara et al. [5], Bertone et al. [9], Buckley and Feuring [11], Chen et al. [14], Jafelice et al. [22], Long et al. [30], Mehmood and Akbar Azam [31], Pirzada and Vakaskar [34], Rahman and Ahmad [40] studied classical models of partial differential equations with uncertain parameters, considering the parameters as fuzzy numbers. It was an obvious step in the mathematical modeling of physical processes. Study of fuzzy partial differential equations means the generalization of partial differential equations in fuzzy sense. While doing modelling of real situation in terms of partial differential equation, we see that the variables and parameters involved in the equations are uncertain (in the sense that they are not completely known or inexact or imprecise). Many times common initial or boundary condition of ambient temperature is a fuzzy condition since ambient temperature is prone

*Corresponding author.
Email: a-plotnikov@ukr.net (A.V. Plotnikov).

to variation in a range. We express this impreciseness and uncertainties in terms of fuzzy numbers. So we come across with fuzzy partial differential equations. Also obviously, these equations can be written in as fuzzy partial differential inclusions.

In this work we consider fuzzy hyperbolic differential inclusions (fuzzy Darboux problem). The paper is organized as follows. In Section 2, some basic definitions and results related to fuzzy calculus are presented. In Section 3, we introduce the concept of fuzzy hyperbolic differential inclusions (fuzzy Darboux problem) and R-solution similarly how it has been done in [36, 38, 49, 51]. Also, we establish conditions for the existence and uniqueness of R-solution for fuzzy Darboux problem. In Section 4, we ground the possibility of application of full averaging method for fuzzy Darboux problem. This result generalize the results of Vityuk [38, 51] for the ordinary hyperbolic differential inclusions and Kiselevich [23], Korenevskii [24] for the ordinary hyperbolic differential equations. Concluding remarks are given in the final section of the paper.

2 Preliminaries

Let $comp(R^n)(conv(R^n))$ be a family of all nonempty (convex) compact subsets from the space R^n with the Hausdorff metric

$$h\left(A,B\right)=\min_{r\geq 0}\left\{B\subset S_r(A),\ A\subset S_r(B)\right\},$$

where $A,B\in comp(R^n)$, $S_r(A)$ is r-neighborhood of set A.

Let E^n be a family of all $u:R^n\to[0,1]$ such that u satisfies the following conditions:

1) u is normal, i.e. there exists an $x_0\in R^n$ such that $u(x_0)=1$;

2) u is fuzzy convex, i.e. $u\left(\lambda x+(1-\lambda)y\right)\geq\min\left\{u(x),u(y)\right\}$ for any $x,y\in R^n$ and $0\leq\lambda\leq 1$;

3) u is upper semicontinuous, i.e. for any $x_0\in R^n$ and $\varepsilon>0$ where exists $\delta(x_0,\varepsilon)>0$ such that $u(x)<u(x_0)+\varepsilon$ whenever $\parallel x-x_0\parallel<\delta(x_0,\varepsilon),\ x\in R^n$;

4) the closure of the set $\{x\in R^n\ :\ u(x)>0\}$ is compact.

If $u\in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space.

Definition 2.1. The set $\{x\in R^n\ :\ u(x)\geq\alpha\}$ is called the *α-level* $[u]^\alpha$ of a fuzzy number $u\in E^n$ for $0<\alpha\leq 1$. The closure of the set $\{x\in R^n\ :\ u(x)>0\}$ is called the 0-level $[u]^0$ of a fuzzy number $u\in E^n$.

It is clearly that the set $[u]^\alpha\in conv\left(R^n\right)$ for all $0\leq\alpha\leq 1$.

Theorem 2.1. [Stacking Theorem [32]] If $u\in E^n$ then

1) $[u]^\alpha\in conv(R^n)$ for all $\alpha\in[0,1]$;

2) $[u]^{\alpha_2}\subset[u]^{\alpha_1}$ for all $0\leq\alpha_1\leq\alpha_2\leq 1$;

3) if $\{\alpha_k\}$ is a nondecreasing sequence converging to $\alpha>0$, then $[u]^\alpha=\bigcap\limits_{k\geq 1}[u]^{\alpha_k}$.

Conversely, if $\{A_\alpha\ :\ \alpha\in[0,1]\}$ is the family of subsets of R^n satisfying conditions 1) - 3) then there exists $u\in E^n$ such that $[u]^\alpha=A_\alpha$ for $0<\alpha\leq 1$ and $[u]^0=\overline{\bigcup\limits_{0<\alpha\leq 1}A_\alpha}\subset A_0$.

Let θ be the fuzzy number defined by $\theta\left(x\right)=0$ if $x\neq 0$ and $\theta\left(0\right)=1$.

Define $D:\ E^n\times E^n\to[0,\infty)$ by the relation

$$D(u,v)=\sup_{0\leq\alpha\leq 1}h\left([u]^\alpha,[v]^\alpha\right).$$

Then D is a metric in E^n. Further we know that [39]:

i) (E^n,D) is a complete metric space,

ii) $D\left(u+w,v+w\right)=D\left(u,v\right)$ for all $u,v,w\in E^n$,

iii) $D\left(\lambda u,\lambda v\right)=|\lambda|\,D\left(u,v\right)$ for all $u,v\in E^n$ and $\lambda\in R$.

3 Fuzzy Hyperbolic Differential Inclusion R-solution

Consider the fuzzy hyperbolic differential inclusion (or in other words, fuzzy Darboux problem)

$$\begin{gathered}u_{xy}(x,y)\in F(x,y,u(x,y)),\\ u(x,0)=\varphi(x),\ x\in[0,a],\\ u(0,y)=\psi(y),\ y\in[0,b],\\ \varphi(0)=\psi(0),\end{gathered}\tag{1}$$

where $u \in R^n$, $u_{xy}(x,y) = \frac{\partial^2 u(x,y)}{\partial x \partial y}$, $F:[0,a]\times[0,b]\times R^n \to E^n$, $\varphi:[0,a]\to R^n$, $\psi:[0,b]\to R^n$.

We interpret fuzzy Darboux problem (1) as a family of set-valued Darboux problems

$$u_{xy}^{\alpha}(x,y) \in [F(x,y,u^{\alpha}(x,y))]^{\alpha}, \tag{2a}$$
$$u^{\alpha}(x,0) = \varphi(x),\ x\in[0,a], \tag{2b}$$
$$u^{\alpha}(0,y) = \psi(y),\ y\in[0,b], \tag{2c}$$
$$\varphi(0) = \psi(0),\ \alpha\in[0,1]. \tag{2d}$$

Qualitative properties and structure of the set of solutions of the set-valued Darboux problem have been studied by many authors, for instance De Blasi et al. [15, 16, 17], Cernea [13], Lažar [28], Vityuk [38, 48, 49, 50, 51, 52], Staicu [43], Teodoru [44, 45, 46, 47], etc.

Definition 3.1. [29, 42] A function $u\colon[0,a]\times[0,b]\to R^n$ is said to be absolutely continuous on $[0,a]\times[0,b]$ ($u(\cdot,\cdot)\in AC([0,a]\times[0,b])$) if there exist absolutely continuous functions $\varphi\colon[0,a]\to R^n$ and $\psi\colon[0,b]\to R^n$ and Lebesgue integrable function $g\colon[0,a]\times[0,b]\to R^n$ such that

$$u(x,y) = \varphi(x)+\psi(y)-c+\int_0^x\int_0^y g(\xi,\varsigma)d\varsigma d\xi,\ u(0,0)=\varphi(0)=\psi(0)=c.$$

Definition 3.2. An $\alpha-$solution $u^{\alpha}(\cdot,\cdot)$ of (1) is understood to be an absolutely continuous function $u^{\alpha}:[0,a]\times[0,b]\to R^n$ that satisfies (2a) for almost every $(x,y)\in[0,a]\times[0,b]$ and the boundary conditions (2b) and (2c) for any $x\in[0,a]$ and $y\in[0,b]$.

Let U^{α} denote the $\alpha-$solution set of (2) and $U^{\alpha}(x,y)=\{\,u^{\alpha}(x,y)\ :\ u^{\alpha}(\cdot,\cdot)\in U^{\alpha}\}$. Clearly a family of subsets $U(x,y)=\{\,U^{\alpha}(x,y)\ :\ \alpha\in[0,1]\,\}$ may not satisfy to conditions of Theorem 2.1, i.e. $U(x,y)\notin E^n$. For example, $U^{\alpha}(x,y)\in comp(R^n)$ and $U^{\alpha}(x,y)\notin conv(R^n)$ for any $\alpha\in[0,1]$.

Therefore, we introduce the definition of R-solutions for fuzzy Darboux problem (1).

Definition 3.3. The upper semicontinuous fuzzy mapping $R:[0,a]\times[0,b]\to E^n$ that satisfies to the following system

$$\sup_{\alpha\in[0,1]} h\left([R(t+\sigma,s+\eta)]^{\alpha},\ \bigcup_{u\in[R(t,s)]^{\alpha}}\{v(t+\sigma)+\tau(s+\eta)-u+\sigma\eta[F(t,s,u)]^{\alpha}\}\right)=o(\sigma\eta), \tag{3}$$

is called the R-solution of fuzzy Darboux problem (1), where

$$v(x)\in AC(t,t+\sigma),\tau(y)\in AC(s,s+\eta),v(x)\in[R(x,s)]^{\alpha},x\in[t,t+\sigma],$$

$$\tau(y)\in[R(t,y)]^{\alpha},y\in[s,s+\eta],v(t)=\tau(s)=u,\ \lim_{\substack{\sigma\to0_+\\ \eta\to0_+}}\frac{o(\sigma\eta)}{\sigma\eta}=0.$$

Now we are interested in the following question: Under what conditions, there exists a unique R-solution to (1). In the next theorem we find the existence result for a unique R-solution of fuzzy Darboux problem (1).

Theorem 3.1. Suppose the following conditions hold:

1) fuzzy mapping $F(\cdot,\cdot,u)$ is continuous, for all $u\in R^n$;

2) there exists $\lambda>0$ such that $D(F(x,y,u'),F(x,y,u"))\le\lambda\,\|u'-u"\|$ for all $u',u''\in R^n$ and every $(x,y)\in[0,a]\times[0,b]$;

3) there exists $\gamma>0$ such that $D(F(x,y,u),\theta)\le\gamma$ for every $(x,y,u)\in[0,a]\times[0,b]\times R^n$;

4) for all $\beta\in[0,1]$, $u',u"\in R^n$ and every $(x,y)\in[0,a]\times[0,b]$,

$$\beta\,F(x,y,u')+(1-\beta)\,F(x,y,u")\subset F(x,y,\ \beta\,u'+(1-\beta)\,u");$$

5) functions $\varphi(\cdot)$ and $\psi(\cdot)$ are absolutely continuous functions on $[0,a]$ and $[0,b]$.

Then there exists a unique R-solution $R(\cdot,\cdot)$ of fuzzy Darboux problem (1) defined on the set $[0,a]\times[0,b]$.

Proof. By [28, 48], every set-valued Darboux problem of family (2) has solution $u^{\alpha}(\cdot,\cdot)$ on the set $[0,a]\times[0,b]$, i.e. $U^{\alpha}(x,y)\ne\emptyset$ for every $(x,y)\in[0,a]\times[0,b]$ and $\alpha\in[0,1]$.

By [38, Theorem 4.1.2.6] and [49], $U^{\alpha}(x,y)\in comp(R^n)$ for every $(x,y)\in[0,a]\times[0,b]$ and $\alpha\in[0,1]$.

By [38, Theorem 4.1.2.5] and [50], if $u^\alpha(\cdot,\cdot) \in U^\alpha$ then

$$u^\alpha(x_2,y_2) - u^\alpha(x_2,y_1) - u^\alpha(x_1,y_2) + u^\alpha(x_1,y_1) \in \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}[F(x,y,u^\alpha(x,y))]^\alpha dydx$$

for every $[x_1,x_2]\times[y_1,y_2] \subset [0,a]\times[0,b]$.

Consider any solutions $u_1^\alpha(\cdot,\cdot),\ u_2^\alpha(\cdot,\cdot) \in U^\alpha$ and any $\beta \in [0,1]$. Let $u_\beta^\alpha(\cdot,\cdot)$ be such that

$$u_\beta^\alpha(x,y) = \beta u_1^\alpha(x,y) + (1-\beta)u_2^\alpha(x,y)$$

for every $(x,y) \in [0,a]\times[0,b]$.

Then

$$\begin{aligned}
&u_\beta^\alpha(x_2,y_2) - u_\beta^\alpha(x_2,y_1) - u_\beta^\alpha(x_1,y_2) + u_\beta^\alpha(x_1,y_1)\\
&= \beta u_1^\alpha(x_2,y_2) - \beta u_1^\alpha(x_2,y_1) - \beta u_1^\alpha(x_1,y_2) + \beta u_1^\alpha(x_1,y_1)\\
&\quad +(1-\beta)u_2^\alpha(x_2,y_2) - (1-\beta)u_2^\alpha(x_2,y_1) - (1-\beta)u_2^\alpha(x_1,y_2) + (1-\beta)u_2^\alpha(x_1,y_1)\\
&\in \beta\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}[F(x,y,u_1^\alpha(x,y))]^\alpha dydx + (1-\beta)\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}[F(x,y,u_2^\alpha(x,y))]^\alpha dydx\\
&\subset \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}[F(x,y,\beta u_1^\alpha(x,y) + (1-\beta)u_2^\alpha(x,y))]^\alpha dydx = \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}[F(x,y,u_\beta^\alpha(x,y))]^\alpha dydx,
\end{aligned}$$

i.e. $u_\beta^\alpha(x_2,y_2) - u_\beta^\alpha(x_2,y_1) - u_\beta^\alpha(x_1,y_2) + u_\beta^\alpha(x_1,y_1) \in \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}[F(x,y,u_\beta^\alpha(x,y))]^\alpha dydx$ for every $(x,y) \in [0,a]\times[0,b]$ and $\beta \in [0,1]$.

By [38, Theorem 4.1.2.5] and [50], $u_\beta^\alpha(x,y) \in U^\alpha(x,y)$ for every $(x,y) \in [0,a]\times[0,b]$, i.e. $U^\alpha(x,y) \in conv(R^n)$ for every $(x,y) \in [0,a]\times[0,b]$ and $\alpha \in [0,1]$.

Since, $[F(x,y,u)]^{\alpha_2} \subset [F(x,y,u)]^{\alpha_1}$ for all $0 \le \alpha_1 < \alpha_2 \le 1$ and $(x,y,u) \in [0,a]\times[0,b]\times R^n$, then $U^{\alpha_2}(x,y) \subset U^{\alpha_1}(x,y)$ for all $0 \le \alpha_1 < \alpha_2 \le 1$ and $(x,y) \in [0,a]\times[0,b]$.

By [49, 51], every Darboux problem of family (2) has one R-solution $R^\alpha(\cdot,\cdot)$ on the set $[0,a]\times[0,b]$ and we have $R^\alpha(x,y) = U^\alpha(x,y)$ for every $\alpha \in [0,1]$ and $(x,y) \in [0,a]\times[0,b]$.

By [20, 52], we get that a family of subsets $R(x,y) = \{\,R^\alpha(x,y)\ :\ \alpha \in [0,1]\,\}$ satisfies to conditions of Theorem 2.1, i.e. $R(x,y) \in E^n$ for every $(x,y) \in [0,a]\times[0,b]$. This concludes the proof.

4 The Method of Full Averaging

Now consider fuzzy Darboux problem with the small parameters

$$\begin{gathered}
u_{xy}(x,y) \in \varepsilon_1\varepsilon_2 F(x,y,u(x,y)),\\
u(x,0) = \varphi(x),\ x \in R_+,\\
u(0,y) = \psi(y),\ y \in R_+,\\
\varphi(0) = \psi(0),
\end{gathered} \tag{4}$$

where $\varepsilon_1 > 0,\ \varepsilon_2 > 0$ - small parameters, $R_+ = [0,+\infty)$.

In this work, we associate with the problem (4) the following full averaged fuzzy Darboux problem

$$\begin{gathered}
z_{xy}(x,y) \in \varepsilon_1\varepsilon_2 G(z(x,y)),\\
z(x,0) = \varphi(x),\ x \in R_+,\\
z(0,y) = \psi(y),\ y \in R_+,\\
\varphi(0) = \psi(0),
\end{gathered} \tag{5}$$

where $G : R^n \to E^n$ such that

$$\lim_{\substack{T_1\to\infty\\ T_2\to\infty}} D\left(\frac{1}{T_1T_2}\int\limits_0^{T_1}\int\limits_0^{T_2} F(x,y,z)dydx, G(z)\right) = 0. \tag{6}$$

The main theorem of this section is on averaging for fuzzy Darboux problem with the small parameters. It establishes nearness of R-solutions of (4) and (5), and reads as follows.

Theorem 4.1. Let in the domain $Q = \{ (x, y, u) \ : \ x \in R_+, y \in R_+, \ u \in B \subset R^n \}$ the following conditions hold:

1) fuzzy mapping $F(\cdot, \cdot, u)$ is continuous on $R_+ \times R_+$;

2) fuzzy mapping $F(x, y, \cdot)$ satisfied a Lipschitz condition $D(F(x, y, u'), F(x, y, u"))\ \le\ \lambda \|u' - u"\|$, with a Lipschitz constant $\lambda > 0$;

3) there exists $\gamma > 0$ such that $D(F(x, y, u), \theta) \le \gamma$ for every $(x, y) \in R_+ \times R_+$ and every $u \in R^n$;

4) for all $\beta \in [0, 1],\ u', u" \in R^n$ and every $(x, y) \in R_+ \times R_+$,

$$\beta\, F(x, y, u') + (1 - \beta)\, F(x, y, u") \subset F(x, y,\ \beta\, u' + (1 - \beta)\, u");$$

5) limit (6) exists uniformly with respect to u in the domain B;

6) functions $\varphi(\cdot)$ and $\psi(\cdot)$ are absolutely continuous functions on R_+ and $\varphi(x) \in B',\ \psi(y) \in B'$ for all $x, y \in R_+$, where $B' + S_\rho(0) \subset B$;

7) the R-solution of the Darboux problem

$$\begin{aligned}
&u^1_{xy}(x, y) \in \varepsilon_1\varepsilon_2[F(x, y, u^1(x, y))]^1,\\
&u^1(x, 0) = \varphi(x),\ x \in [0, \infty),\\
&u^1(0, y) = \psi(y),\ y \in [0, \infty),\\
&\varphi(0) = \psi(0),
\end{aligned}$$

together with a $\rho-$neighborhood belong to the domain B for $\varepsilon_1, \varepsilon_2 \in (0, \bar\varepsilon]$.

Then for any $\eta \in (0, \rho]$ and L > 0 there exists $\varepsilon_0(\eta, L) \in (0, \bar\varepsilon]$ such that for all $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0]$ and $(x, y) \in [0, L\varepsilon_1^{-1}] \times [0, L\varepsilon_2^{-1}]$ the following inequality holds

$$D(R(x, y), \bar R(x, y)) < \eta \tag{7}$$

where $R(\cdot, \cdot),\ \bar R(\cdot, \cdot)$ are the R-solutions of initial and full averaged Darboux problems.

Proof. By theorem 3.1, we have unit R-solution of Darboux problem (4) on $R_+ \times R_+$.

By condition 2)-5) of theorem, we obtain

$$\begin{aligned}
D(G(u^1), G(u^2)) \le{}& D\left(G(u^1), \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^1)dydx\right) + D\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^2)dydx, G(u^2)\right)\\
&+D\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^1)dydx, \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^2)dydx\right)\\
&\le 2\mu(T_1, T_2) + \lambda \left\|u^1 - u^2\right\|,\\
D(G(u), \theta) \le{}& D\left(G(u), \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^1)dydx\right) + D\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u)dydx, \theta\right)\\
&< \mu(T_1, T_2) + \gamma,\\
\beta G(u^1) + (1 - \beta)G(u^2) ={}& \beta \lim_{\substack{T_1\to\infty\\ T_2\to\infty}} \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^1)dydx + (1 - \beta) \lim_{\substack{T_1\to\infty\\ T_2\to\infty}} \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, u^2)dydx\\
&= \lim_{\substack{T_1\to\infty\\ T_2\to\infty}} \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} (\beta F(x, y, u^1) + (1 - \beta)F(x, y, u^2)dydx\\
&\subset \lim_{\substack{T_1\to\infty\\ T_2\to\infty}} \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2} F(x, y, \beta u^1 + (1 - \beta)u^2)dydx = G(\beta u^1 + (1 - \beta)u^2),
\end{aligned}$$

where $(x,y)\in R_+\times R_+, u,\ u^1,\ u^2\ \in B, \beta\in[0,1]$, $\lim\limits_{\substack{T_1\to\infty\\T_2\to\infty}}\mu(T_1,T_2)=0$, i.e. fuzzy mapping $G(\cdot)$ satisfied conditions 2)-4) of theorem. By theorem 3.1, we have unit R-solution of Darboux problem (5) on $R_+\times R_+$.

Let $\delta_1=L\varepsilon_1^{-1}$, $\delta_2=L\varepsilon_2^{-1}$,

$$K^n=\left\{(x_i,y_j)\ :\ x_i=ih,\ y_j=jl,\ i,j=0,1,2,...,n,\ h=\frac{\delta_1}{n},\ l=\frac{\delta_2}{n}\right\},$$

$K^n_{ij}=[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ and $K=[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$. We denote fuzzy mappings $P^m(\cdot,\cdot)$ and $Q^m(\cdot,\cdot)$ such that

$$[P^n(x,y)]^\alpha=\bigcup_{u\in[P^n(x_i,y_j)]^\alpha}\left\{v_{ij}(x)+\tau_{ij}(y)-u+\varepsilon_1\varepsilon_2(x-x_i)(y-y_j)[F(x_i,y_j,u)]^\alpha\right\},$$

$$[Q^n(x,y)]^\alpha=\bigcup_{z\in[Q^n(x_i,y_j)]^\alpha}\left\{\bar v_{ij}(x)+\bar\tau_{ij}(y)-z+\varepsilon_1\varepsilon_2(x-x_i)(y-y_j)[G(z)]^\alpha\right\},$$

$$P^n(x,0)=Q^n(x,0)=\varphi(x),P^n(0,y)=Q^n(0,y)=\psi(y),$$

where $v_{ij}(x)\in AC(x_i,x_{i+1})$, $\tau_{ij}(y)\in AC(y_j,y_{j+1})$, $v_{ij}(x)\in[P^n(x,y_j)]^\alpha$, $x\in[x_i,x_{i+1}]$, $\tau_{ij}(y)\in[P^n(x_i,y)]^\alpha$, $y\in[y_j,y_{j+1}]$, $v_{ij}(x_i)=\tau_{ij}(y_j)=u$, $\bar v_{ij}(x)\in AC(x_i,x_{i+1})$, $\bar\tau_{ij}(y)\in AC(y_j,y_{j+1})$, $\bar v_{ij}(x)\in[Q^n(x,y_j)]^\alpha$, $x\in[x_i,x_{i+1}]$, $\bar\tau_{ij}(y)\in[Q^n(x_i,y)]^\alpha$, $y\in[y_j,y_{j+1}]$, $\bar v_{ij}(x_i)=\bar\tau_{ij}(y_j)=z$.

By [51], it follows that the sequences $\{[P^n(\cdot,\cdot)]^\alpha\}_{n=1}^\infty$, and $\{[Q^n(\cdot,\cdot)]^\alpha\}_{n=1}^\infty$ are equicontinuous and fundamental and their limits are $\alpha-$levels of R-solutions $[R(\cdot,\cdot)]^\alpha$ and $[\bar R(\cdot,\cdot)]^\alpha$ of the problems (4) and (5).

Consequently, the sequences $\{P^n(\cdot,\cdot)\}_{n=1}^\infty$ and $\{Q^n(\cdot,\cdot)\}_{n=1}^\infty$ meet by $R(\cdot,\cdot)$ and $\bar R(\cdot,\cdot)$.

By [51], for any $\eta_1>0$ there exists $0<\varepsilon_0\le\bar\varepsilon$ such that

$$h([P^n(x,y)]^\alpha,[Q^n(x,y)]^\alpha)\le\eta_1\lambda^{-1}\exp(\lambda L^2),\tag{8}$$

$$h([R(x,y)]^\alpha,[P^n(x,y)]^\alpha)\le\frac{3\gamma L^2}{n}(1+\exp(\lambda L^2)),\tag{9}$$

$$h([Q^n(x,y)]^\alpha,[\bar R(x,y)]^\alpha)\le\frac{3\gamma L^2}{n}(1+\exp(\lambda L^2))\tag{10}$$

for any $\alpha\in[0,1]$, $(x,y)\in K$ and $\varepsilon_1,\varepsilon_2\in(0,\varepsilon_0]$.

Combining (8), (9) and (10), choosing

$$n\ge\frac{9\gamma L^2(1+\exp(\lambda L^2))}{\eta}\ \text{ and }\ \eta_1<\frac{\eta\,\lambda}{3\exp(\lambda L^2)},$$

we obtain

$$\begin{aligned}D(R(x,y),\bar R(x,y))&\le D(R(x,y),P^n(x,y))+D(P^n(x,y),Q^n(x,y))+D(Q^n(x,y),\bar R(x,y))\\&=\sup_{\alpha\in[0,1]}h([R(x,y)]^\alpha,[P^n(x,y)]^\alpha)+\sup_{\alpha\in[0,1]}h([P^n(x,y)]^\alpha,[Q^n(x,y)]^\alpha)\\&\quad+\sup_{\alpha\in[0,1]}h([Q^n(x,y)]^\alpha,[\bar R(x,y)]^\alpha)\\&<\frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3}=\eta.\end{aligned}$$

The theorem is proved.

5 Conclusion

We conclude with a few remarks:

Remark 5.1. In this work, we considered the fuzzy differential inclusion, when fuzzy mapping $F(\cdot,\cdot,u)$ is continuous on $[0,a]\times[0,b]$. If $F(\cdot,\cdot,u)$ is measurable on $[0,a]\times[0,b]$ then instead of the equation (1) it is possible to consider the following more simple equation

$$\sup_{\alpha\in[0,1]} h\left([R(t+\sigma,s+\eta)]^\alpha, \bigcup_{u\in[R(t,s)]^\alpha}\left\{v(t+\sigma)+\tau(s+\eta)-u+\int\limits_t^{t+\sigma}\int\limits_s^{s+\eta}[F(\varrho,\varsigma,u)]^\alpha d\varsigma d\varrho\right\}\right)=o(\sigma\eta),$$

and similarly we can prove all results received earlier.

Remark 5.2. If the limit (6) does not exist but there is a limit

$$\limsup_{\substack{T_1\to\infty\\ T_2\to\infty}} D\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}F(x,y,z)dydx, G(z)\right)=0 \tag{11}$$

or limit

$$\liminf_{\substack{T_1\to\infty\\ T_2\to\infty}} D\left(\frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}F(x,y,z)dydx, G(z)\right)=0, \tag{12}$$

it will be the condition $R(x,y)\subset\bar{R}(x,y)+S_\eta(\theta)$ or the condition $\bar{R}(x,y)\subset R(x,y)+S_\eta(\theta)$ for all $(x,y)\in[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ respectively.

Remark 5.3. If the condition 4) of Theorem 4.1 is not true, then the R-solutions can not exist. But there are valid the following conditions:

1) for any α-solution $u^\alpha(\cdot,\cdot)$ of inclusion (4) there exists a α-solution $z^\alpha(\cdot,\cdot)$ of inclusion (5) such that

$$\|u^\alpha(x,y)-z^\alpha(x,y)\|<\eta$$

for all $(x,y)\in[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ and $\alpha\in[0,1]$;

2) for any α-solution $z^\alpha(\cdot,\cdot)$ of inclusion (5) there exists a α-solution $u^\alpha(\cdot,\cdot)$ of inclusion (4) such that

$$\|u^\alpha(x,y)-z^\alpha(x,y)\|<\eta$$

for all $(x,y)\in[0,L\varepsilon_1^{-1}]\times[0,L\varepsilon_2^{-1}]$ and $\alpha\in[0,1]$.

References

[1] Abbasbandy, S., Viranloo, T.A., Lopez-Pouso, O., and J.J. Nieto, Numerical methods for fuzzy differential inclusions, *Computers & Mathematics with Applications*, vol.48, pp.1633–1641, 2004.

[2] Abdul Rahman, N.A., and M.Z. Ahmad, Applications of the fuzzy Sumudu transform for the solution of first order fuzzy differential equations, *Entropy*, vol.17, pp.4582–4601, 2015.

[3] Agarwal, R.P., O'Regan, D., and V. Lakshmikantham, A stacking theorem approach for fuzzy differential equations, *Nonlinear Analysis*, vol.55, pp.299–312, 2003.

[4] Agarwal, R.P., O'Regan, D., and V. Lakshmikantham, Maximal solutions and existence theory for fuzzy differential and integral equations, *Journal of Applied Analysis*, vol.11, no.2, pp.171–186, 2005.

[5] Arara, A., Benchohra, M., Ntouyas, S.K., and A. Ouahab, Fuzzy solutions for hyperbolic partial differential equations, *International Journal of Applied Mathematical Sciences*, vol.2, no.2, pp.181–195, 2005.

[6] Aubin, J.-P., Fuzzy differential inclusions, *Problems of Control and Information Theory*, vol.19, no.1, pp.55–67, 1990.

[7] Baidosov, V.A., Differential inclusions with fuzzy right-hand side, *Soviet Mathematics*, vol.40, no.3, pp.567–569, 1990.

[8] Baidosov, V.A., Fuzzy differential inclusions, *Journal of Applied Mathematics and Mechanics*, vol.54, no.1, pp.8–13, 1990.

[9] Bertone, A.M., Jafelice, R.M., de Barros, L.C., and R.C. Bassanezi, On fuzzy solutions for partial differential equations, *Fuzzy Sets and Systems*, vol.219, pp.68–80, 2013.

[10] Bogoliubov, N.N., and Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Non-linear Oscillations*, Gordon and Breach, New York, 1961.

[11] Buckley, J.J., and T. Feuring, Introduction to fuzzy partial differential equations, *Fuzzy Sets and Systems*, vol.105, no.2, pp.241–248, 1999.

[12] Burd, V., *Method of Averaging for Differential Equations on an Infinite Interval: Theory and Applications*, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[13] Cernea, A., On the set of solutions of some nonconvex nonclosed hyperbolic differential inclusions, *Czechoslovak Mathematical Journal*, vol.52, no.1, pp.215–224, 2002.

[14] Chen, Y.Y., Chang, Y.T., and B.S. Chen, Fuzzy solutions to partial differential equations: adaptive approach, *IEEE Transactions on Fuzzy Systems*, vol.17, no.1, pp.116–127, 2009.

[15] De Blasi, F.S., and J. Myjak, On the structure of the set of solutions of the Darboux problem of hyperbolic equations, *Proceedings of the Edinburgh Mathematical Society. Series II*, vol.29, no.1, pp.7–14, 1986.

[16] De Blasi, F.S., and J. Myjak, On the set of solutions of a differential inclusion, *Bulletin of the Institute of Mathematics. Academia Sinica*, vol.14, no.3, pp.271–275, 1986.

[17] De Blasi, F.S., Pianigiani, G., and V. Staicu, On the solution sets of some nonconvex hyperbolic differential inclusions, *Czechoslovak Mathematical Journal*, vol.45, no.1, pp.107–116, 1995.

[18] Dutta Majumder, D., and K.K. Majumdar, Complexity analysis, uncertainty management and fuzzy dynamical systems: a cybernetic approach, *Kybernetes*, vol.33, no.7, pp.1143–1184, 2004.

[19] Guo, M., Xue, X., and R. Li, Impulsive functional differential inclusions and fuzzy population models, *Fuzzy Sets and Systems*, vol.138, pp.601–615, 2003.

[20] Hüllermeier, E., An approach to modelling and simulation of uncertain dynamical system, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.5, no.2, pp.117–137, 1997.

[21] Hüllermeier, E., Numerical methods for fuzzy initial value problems, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.7, no.5, pp.439–461, 1999.

[22] Jafelice, R.M., Almeida, C.G., Meyer, J.F., and H.L. Vasconcelos, Fuzzy parameter in a partial differential equation model for population dispersal of leaf-cutting ants, *Nonlinear Analysis. Real World Applications*, vol.12, no.6, pp.3397–3412, 2011.

[23] Kiselevich, M., A Bogolyubov type theorem for hyperbolic equations, *Ukrainian Mathematical Journal*, vol.22, no.3, pp.326–331, 1970.

[24] Korenevskii, D.G., On the principle of averaging for second-order hyperbolic equations with functionally perturbed argument, *Ukrainian Mathematical Journal*, vol.23, no.2, pp.125–132, 1971.

[25] Krylov, N.M., and N.N. Bogoliubov, *Introduction to Nonlinear Mechanics*, Princeton University Press, Princeton, 1947.

[26] Lakshmikantham, V., Granna Bhaskar, T., and J. Vasundhara Devi, *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publishers, Cambridge, UK, 2006.

[27] Lakshmikantham, V., and R.N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor & Francis, London, UK, 2003.

[28] Lažar, V.L., Ulam-Hyers stability for partial differential inclusions, *Electronic Journal of Qualitative Theory of Differential Equations*, vol.21, pp.1–19, 2012.

[29] Lomtatidze, A., and J. Šremr, Caratheodory solutions to a hyperbolic differential inequality with a non-positive coefficient and delayed arguments, *Boundary Value Problems*, vol.2014, aritcle ID 52, 13 pages, 2014.

[30] Long, H.V., Son, N.T.K., Ha, N.T.M., and L.H. Son, The existence and uniqueness of fuzzy solutions for hyperbolic partial differential equations, *Fuzzy Optimization and Decision Making*, vol.13, no.4, pp.435–462, 2014.

[31] Mehmood, N., and A. Azam, Existence results for fuzzy partial differential inclusions, *Journal of Function Spaces*, article ID 6759294, 8 pages, 2016.

[32] Negoita, C.V., and D.A. Ralescu, *Application of Fuzzy Sets to Systems Analysis*, Wiley, New York, USA, 1975.

[33] Panasyuk, A.I., and V.I. Panasyuk, An equation generated by a differential inclusion, *Mathematical Notes of the Academy of Sciences of the USSR*, vol.27, no.3, pp.213–218, 1980.

[34] Pirzada, U.M., and D.C. Vakaskar, Solution of fuzzy heat equations using adomian decomposition method, *International Journal of Advances in Applied Mathematics and Mechanics*, vol.3, no.1, pp.87-91, 2015.

[35] Plotnikov, A.V., Komleva, T.A., and L.I. Plotnikova, The partial averaging of differential inclusions with fuzzy right-hand side, *Journal of Advanced Research in Dynamical and Control Systems*, vol.2, no.2, pp.26–34, 2010.

[36] Plotnikov, A.V., and T.A. Komleva, The partial averaging of fuzzy differential inclusions on finite interval, *International Journal of Differential Equations*, article ID 307941, 5 pages, 2014.

[37] Plotnikov, A.V., A procedure of complete averaging for fuzzy differential inclusions on a finite segment, *Ukrainian Mathematical Journal*, vol.67, no.3, pp.421–430, 2015.

[38] Plotnikov, V.A., Plotnikov, A.V., and A.N. Vityuk, *Differential Equations with a Multivalued Right-hand Side: Asymptotic Methods*, AstroPrint, Odessa, Ukraine, 1999.

[39] Puri, M.L., and D.A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, vol.114, pp.409–422, 1986.

[40] Rahman, N.A.A., and M.Z. Ahmad, Fuzzy Sumudu transform for solving fuzzy partial differential equations, *Journal of Nonlinear Science and Applications*, vol.9, pp.3226–3239, 2016.

[41] Sanders, J.A., and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*,Springer-Verlag, New York, 1985.

[42] Šremr, J., Absolutely continuous functions of two variables in the sense of Caratheodory, *Electronic Journal of Differential Equations*, vol.2010, no.154, pp.1–11, 2010.

[43] Staicu, V., On a non-convex hyperbolic differential inclusion, *Proceedings of the Edinburgh Mathematical Society, Series II*, vol.35, no.3, pp.375–382, 1992.

[44] Teodoru, G., Approximation of the solution of Darboux problem for the equation $\partial^2 z/\partial x \partial y \in F(x, y, z)$, *Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1985)*, pp.215–220, 1985.

[45] Teodoru, G., On the Darboux problem for the equation $\partial^2 z/\partial x \partial y \in F(x, y, z)$, *Analele Ştiinţifice ale Universităţii Al. I. Cuza din Iaşi. Matematică*, vol.32, no.3, pp.41–49, 1986.

[46] Teodoru, G., Continuous selections for multifunctions satisfying the Caratheodory type conditions. The Darboux problem associated to a multivalued equation, *Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1987)*, pp.281–286, 1987.

[47] Teodoru, G., A characterization of the solutions of the Darboux problem for the equation $\partial^2 z/\partial x \partial y \in F(x, y, z)$, *Analele Ştiinţifice ale Universităţii Al. I. Cuza din Iaşi. Matematică*, vol.33, no.1, pp.33–38, 1987.

[48] Vityuk, A.N., Properties of solutions of hyperbolic differential equations with many-valued right-hand sides, *Matematicheskaya Fizika i Nelineinaya Mekhanika*, vol.15, pp.59–62, 1991.

[49] Vityuk, A.N., Equation of the integral funnel of a partial differential inclusion, *Dopovidi Akademii Nauk Ukrainy*, vol.9, pp.19–20, 1992.

[50] Vityuk, A.N., On solutions of hyperbolic differential inclusions with nonconvex right-hand side, *Ukrainian Mathematical Journal*, vol.47, no.4, pp.617–621, 1995.

[51] Vityuk, A.N., On an R-solution generated by a differential inclusion of hyperbolic type, *Differential Equations*, vol.30, no.10, pp.1578–1586, 1994.

[52] Vityuk, A.N., Continuous dependence of the R-solution generated by a differential hyperbolic inclusion on parameters, *Ukrainian Mathematical Journal*, vol.47, no.10, pp.1625–1631, 1995.