Entropy of Dynamical Systems from the Observer’s Viewpoint, with Countable $\sigma$-algebras

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Abstract

In this paper, we introduce the notion of relative probability measure spaces by the concept of observer. The notions of relative entropy and relative conditional entropy of countable $\sigma$-algebras are defined and studied. Using the suggested concept of relative entropy we define the relative entropy of a dynamical system and we prove some of its ergodic properties. Finally it is proved that, the relative entropy of dynamical systems is invariant under conjugate relation.

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1 Introduction

The study of the concept entropy is very important in sciences nowadays. Entropy plays an important role in a variety of problem areas, including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and others. One of the applied branches of mathematics is the entropy of a dynamical system. The notion of observer has been applied in dynamical systems, topology, information theory and mathematical physics$^1, 2, 3, 4, 5, 6, 7$.

Klement in$^8$ studied the notions of fuzzy $\sigma$-algebra and $F$-probability measure space. Khare defined and studied the notions of entropy and conditional entropy of finite fuzzy $\sigma$-algebras$^9, 10$. The notions of entropy and conditional entropy of infinite fuzzy $\sigma$-algebras are defined and studied in$^11$. In the present paper, we define the notions of relative $\sigma$-algebra and relative probability measure space by using an observer and then we define the notions of entropy and conditional entropy of infinite relative $\sigma$-algebras. We prove some ergodic properties about them. After that we define relative measure preserving transformation and consider it as a relative dynamical system. Using the suggested concept of relative entropy, we define the relative entropy of a dynamical system and prove some theorems about the measure. It is noted that some investigations concerning entropy of dynamical systems and related notions in the above setup were carried in$^12, 13, 14, 15, 16$.

The paper is organized as follows. In the next section, we define the notions of relative $\sigma$-algebra, relative probability measure space and relative measure preserving transformation. In section 3, the entropy and conditional entropy of infinite relative $\sigma$-algebras are defined and basic properties of these measures are proved. In section 4, using the suggested concept of entropy of infinite relative $\sigma$-algebras, we define the entropy of a relative dynamical system. Finally, it is shown that conjugate dynamical systems have the same relative entropy. Our results are summarized in the final section.

2 Relative Measure Spaces

In this section, we define the notion of relative $\sigma$-algebra by using an observer and then we define the notion of relative probability measure space. Also, we define the notion of relative measure preserving transformation.

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Let \( X \) be a non-empty set and an observer be a fuzzy set \( \eta : X \rightarrow [0, 1] \).

**Definition 2.1** A \( \sigma \)-algebra from the point of view \( \eta \), is a collection \( \Sigma_\eta \) of subsets of \( \eta \) (\( \lambda \subseteq \eta \) i.e \( \forall x \in X \), \( \lambda(x) \leq \eta(x) \)) such that:

(i) \( \eta \in \Sigma_\eta \);
(ii) If \( \lambda \in \Sigma_\eta \) then \( \eta - \lambda \in \Sigma_\eta \);
(iii) If \( \{\lambda_i\}_{i=1}^\infty \subseteq \Sigma_\eta \) then \( \bigvee_{i=1}^\infty \lambda_i = \sup_i \lambda_i \in \Sigma_\eta \).

Note that if \( \eta = 1 \), then the \( \sigma \)-algebra from the point of view \( \eta = 1 \) is the fuzzy \( \sigma \)-algebra [13].

**Example 2.2** Let \( X = [-1, 1] \) and \( \eta : X \rightarrow [0, \frac{1}{2}] \) defined by \( \eta(x) = \frac{1}{2} \) for every \( x \in [-1, 1] \). Then \( \Sigma_\eta = \{\lambda_j : j \in \mathbb{N}\} \cup \{\mu_k : k = 2, 3, \ldots\} \) is a \( \sigma \)-algebra from the point of view \( \eta \), where \( \lambda_j : [-1, 1] \rightarrow [0, 1/2] \) by \( \lambda_j(x) = 1/2^j \) and \( \mu_k : [-1, 1] \rightarrow [0, 1/2] \) by \( \mu_k(x) = (2^{k-1} - 1)/2^k \). Let us check the properties of \( \Sigma_\eta \).

(i) \( \eta = \lambda_1 \in \Sigma_\eta \);
(ii) For every \( j \in \mathbb{N} \), \( \eta = \lambda_j = \sum \mu_j = \lambda_j \in \Sigma_\eta \);
(iii) Let \( \{\gamma_i\}_{i=1}^\infty \subseteq \Sigma_\eta \). Since the sequences \( \{\lambda_j : j \in \mathbb{N}\} \) and \( \{\mu_k\}_{k=2}^\infty \) are monotone, we get \( \sup_i \gamma_i : i \in \Sigma_\eta \).

**Definition 2.3** A function \( m : \Sigma_\eta \rightarrow [0, 1] \) is called \( \eta \)-relative probability measure when,

(i) \( m(\eta) = \sup_x \eta(x) \);
(ii) \( m(\eta - \lambda) = m(\eta) - m(\lambda) \) for every \( \lambda \in \Sigma_\eta \);
(iii) \( m(\lambda \lor \mu) + m(\lambda \land \mu) = m(\lambda) + m(\mu) \) for every \( \lambda, \mu \in \Sigma_\eta \);
(iv) If \( \{\lambda_i\}_{i=1}^\infty \subseteq \Sigma_\eta \) and \( \lambda_i \uparrow \lambda \), then \( m(\lambda) = \sup_i m(\lambda_i) \).

Now we say that \((X, \Sigma_\eta, m)\) is a relative probability measure space and each element of \( \Sigma_\eta \) is a relative measurable set.

In Definition 2.3 you observe that the \( \eta = 1 \)-relative probability measure is the \( F \)-probability measure and \((X, \Sigma_1, m)\) is the \( F \)-probability measure space which were defined in [13].

**Example 2.4** Let \( \Sigma_\eta \) be the relative \( \sigma \)-algebra defined in Example 2.3. Define \( m : \Sigma_\eta \rightarrow [0, 1/2] \) by \( m(\lambda_j) = 1/2^j \), \( m(\mu_k) = (2^{k-1} - 1)/2^k \). Also let \( \lor \) and \( \land \) mean respectively supremum and infimum. Then \((X, \Sigma_\eta, m)\) is a relative probability measure space. It is easy to see that the properties of Definition 2.3 hold.

**Definition 2.5** Let \((X, \Sigma_\eta, m)\) be a relative probability measure space. For \( \lambda, \mu \in \Sigma_\eta \), the relation \( = (\text{mod } m) \) is defined by

\[ \lambda = \mu \pmod{m} \iff m(\lambda) = m(\mu) = m(\lambda \lor \mu). \]

The relation \( = (\text{mod } m) \) is an equivalence relation on \( \Sigma_\eta \). The set of all equivalence classes induced by this relation is denoted by \( \Sigma_\eta^\equiv \), and \( \overline{\mu} \) denotes the equivalence class determined by \( \mu \). \( \lambda, \mu \in \Sigma_\eta \) is called \( m \)-disjoint if \( \lambda \land \mu = 0 \pmod{m} \), i.e. \( m(\lambda \land \mu) = 0 \).

**Definition 2.6** Let \((X, \Sigma_\eta, m)\) be a relative probability measure space and let \( N \) be a relative sub-\( \sigma \)-algebra of \( \Sigma_\eta \). An element \( \overline{\nu} \in \overline{N} \) is called an atom of \( N \) if \( m(\nu) > 0 \) and, for any \( \bar{\lambda} \in \overline{N} \),

\[ m(\lambda \land \mu) = m(\lambda) \neq m(\mu) \implies m(\lambda) = 0. \]

The set of all atoms of \( N \) is denoted by \( \overline{N} \), and \( F(\Sigma_\eta) \) denote the collection of relative sub-\( \sigma \)-algebras of \( \Sigma_\eta \) with countably many atoms.

**Definition 2.7** Let \((X_1, M_1, m_1)\) and \((X_2, M_2, m_2)\) be \( \eta \)-relative probability measure spaces. We say \( T : X_1 \rightarrow X_2 \) is a \((\eta, \eta)\) relative measure preserving transformation when:

(i) \( T^{-1}(\mu) \land \eta \in M_1 \), for every \( \mu \in M_2 \) where \( (T^{-1}(\mu))(x) = \mu(T(x)) \);
(ii) \( m_1(T^{-1}(\mu) \land \eta) = m_2(\mu) \) for all \( \mu \in M_2 \).

**Theorem 2.8** Let \( T : X \rightarrow X \) be a \((\eta, \eta)\) relative measure preserving transformation, and let \( N \in F(\Sigma_\eta) \). Then \( T^{-1}N \in F(\Sigma_{\eta,T^{-1}}) \), where

\[ T^{-1}N = \{\eta \land T^{-1} \mu : \mu \in \overline{N}\}. \]
Proof. It is necessary to check the axioms of Definition 2.1 for $T^{-1} \mathcal{N}$. Since $\eta \in \mathcal{N}$, we have $\eta \wedge T^{-1} \eta \in T^{-1} \mathcal{N}$. Let $\eta \wedge T^{-1} \mu \in T^{-1} \mathcal{N}$, $\mu \in \mathcal{N}$ implies that $\eta - \mu \in \mathcal{N}$. So $(\eta \wedge T^{-1} \eta) - (\eta \wedge T^{-1} \mu) = \eta \wedge T^{-1} (\eta - \mu) \in T^{-1} \mathcal{N}$. Now let $\{\eta \wedge T^{-1} \mu_i\}_{i=1}^{\infty} \subseteq T^{-1} \mathcal{N}$. Since $\mathcal{N} \in \mathcal{F}(\Sigma_{\eta})$, we can write
\[
\bigvee_{i=1}^{\infty} (\eta \wedge T^{-1} \mu_i) = \eta \wedge \bigvee_{i=1}^{\infty} T^{-1} \mu_i = \eta \wedge T^{-1} \left( \bigvee_{i=1}^{\infty} \mu_i \right) \in T^{-1} \mathcal{N}.
\]

Note that $m_{\eta}(\eta \wedge T^{-1}(\mu)) = m(\mu) > 0$ for all $\mu \in \mathcal{N}$ because $T$ is a $(\eta, \eta)$ relative measure preserving transformation.

Definition 2.9 Let $(X, M, m)$ be a $\eta$-relative probability measure space. We say $T : X \rightarrow X$ is relative ergodic if for every atom $\mu \in \mathcal{M}$ with $(T^{-1}\mu) \wedge \eta = \mu$ we deduce that $m(\mu) = 0$ or $m_{\eta}(\mu) = m(\eta)$. Then $m$ is called $T$-relative ergodic.

Theorem 2.10 Let $\Omega$ be the set of invariant relative probability measures on $X$ (i.e $T^{-1}(\mu) \wedge \eta = \mu$ for every $T \in \Omega$). Also suppose that $m_1, m_2 \in \Omega$, $0 < \lambda < m(\eta)$ and $m = \lambda m_1 + (m(\eta) - \lambda)m_2$ imply that $m_1 = m_2$. Then $m$ is relative ergodic.

Proof. Suppose $m$ is not relative ergodic. So there exists $\mu \in \mathcal{M}$ such that $0 < m(\mu) < m(\eta)$ and $T^{-1}(\mu) \wedge \eta = \mu$. For every atom $\xi \in \mathcal{M}$ we have $\xi = (\xi \wedge \mu) \vee (\xi \wedge (\eta - \mu))$.

Now we can write
\[
m(\xi) = m((\xi \wedge \mu) \vee (\xi \wedge (\eta - \mu))) = m(\mu)\left(\frac{m(\xi \wedge \mu)}{m(\mu)}\right) + m(\eta - \mu)\left(\frac{m(\xi \wedge (\eta - \mu))}{m(\eta - \mu)}\right)
\]
\[
= \lambda m_1(\xi) + (m(\eta) - \lambda)m_2(\xi),
\]
where $\lambda = m(\mu)$, $m_1(\xi) = m(\xi \wedge \mu)/m(\mu)$ and $m_2(\xi) = m(\xi \wedge (\eta - \mu))/m(\eta - \mu)$. This implies that $m(\lambda \wedge \mu) = m(\mu)$.

In the remainder of this paper, $M$ is a countable $\eta$-relative $\sigma$-algebra and $\mathcal{F}_\eta(M)$ is the set of all $\eta$-relative sub $\sigma$-algebras of $M$ with countably many atoms.

Definition 2.11 Let $M$ be a $\eta$-relative $\sigma$-algebra and $N_1, N_2 \in \mathcal{F}_\eta(M)$ such that $\{\lambda_i : i \in \mathbb{N}\}$ and $\{\mu_j : j \in \mathbb{N}\}$ be the atoms of $N_1, N_2$, respectively. Then their join is defined by $N_1 \vee N_2 = \{\lambda_i \wedge \mu_j : m(\lambda_i \wedge \mu_j) > 0, i, j \in \mathbb{N}\}$.

Definition 2.12 Let $(X, M, m)$ be a $\eta$-relative probability measure space and $N_1, N_2 \in \mathcal{F}_\eta(M)$. Then $N_2$ is called an $m$-refinement of $N_1$ and denoted by $N_1 \leq_m N_2$, if for every $\mu \in \mathcal{N}_2$, there exists $\lambda \in \mathcal{N}_1$ such that $m(\lambda \wedge \mu) = m(\mu)$.

3 Relative Entropy of $\sigma$-Algebras

In this section we define the notions of relative entropy and relative conditional entropy of relative $\sigma$-algebras and we prove some ergodic properties about them.

Definition 3.1 Let $(X, M, m)$ be a $\eta$-relative probability measure space and $N \in \mathcal{F}_\eta(M)$. The relative entropy of $N$ is defined by:
\[
H_\eta(N) = -\log \sup_{\mu_i \in \mathcal{N}} m(\mu_i).
\]

Theorem 3.2 Let $T : X \rightarrow X$ be a $(\eta, \eta)$ relative measure preserving transformation of relative probability space $(X, M, m)$. If $N \in \mathcal{F}_\eta(M)$, then $H_\eta(T^{-1}N) = H_\eta(N)$.
Proof. Suppose \( \tilde{N} = \{\mu_i : i \in \mathbb{N}\}. \) Since \( T \) is a \((\eta, \eta)\) relative measure preserving transformation, we have for each \( i \in \mathbb{N}, \)

\[
m_\eta(\eta \land T^{-1}\mu_i) = m(\mu_i).
\]

But \( T^{-1}N = \{\eta \land T^{-1}\mu_i : \mu_i \in N\}, \) thus

\[
H_\eta(T^{-1}N) = -\log \sup_i m(\eta \land T^{-1}\mu_i) = -\log \sup_i m(\mu_i) = H_\eta(N).
\]

**Definition 3.3** Let \((X, M, m)\) be a \(\eta\)-relative probability measure space and \(N_1, N_2 \in F_\eta(M).\) Also let \(\tilde{N}_1 = \{\mu_i : i \in \mathbb{N}\}, \tilde{N}_2 = \{\lambda_j : j \in \mathbb{N}\}. \) The conditional relative entropy of \(N_1\) given \(N_2\) is defined by:

\[
H_\eta(N_1|N_2) = -\log \frac{\sup_i m(\mu_i \land \lambda_j)}{\sup_j m(\lambda_j)}.
\]

**Example 3.4** Let \(\Sigma_\eta\) be the relative \(\sigma\)-algebra and \(m\) be the \(\eta\)-relative probability measure defined in Examples \(\ref{Example:3.4a} \) and \(\ref{Example:3.4b}.\) Suppose \(N_1 = \{\lambda_j : j \in \mathbb{N}\}\) and \(N_2 = \{\mu_k : k \in \mathbb{N}\}. \) Then \(F(\Sigma_\eta) = N_1 \cup N_2.\) Therefore

\[
H_\eta(N_1) = -\log \sup_{\lambda_j \in N_1} m(\lambda_j)
\]

\[
= -\log \sup_{\lambda_j \in N_1} \left(\frac{1}{2}\right)
\]

\[
= -\log \left(\frac{1}{2}\right) = \log 2,
\]

also,

\[
H_\eta(N_2) = -\log \sup_{\mu_k \in N_2} m(\mu_k)
\]

\[
= -\log \sup_{\mu_k \in N_2} \left(\frac{2^{k-1} - 1}{2^k}\right)
\]

\[
= -\log \sup_{\mu_k \in N_2} \left(\frac{1}{2} - \frac{1}{2^k}\right)
\]

\[
= -\log \left(\frac{1}{2}\right) = \log 2,
\]

and,

\[
H_\eta(N_2|N_1) = -\log \frac{\sup_{\lambda_j} m(\mu_k \land \lambda_j)}{\sup_{\lambda_j} m(\lambda_j)}
\]

\[
= -\log \left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = 0.
\]

**Theorem 3.5** Let \(T : X \rightarrow X\) be a \((\eta, \eta)\) relative measure preserving transformation of the relative probability space \((X, M, m).\) Also let \(N_1, N_2 \in F_\eta(M).\) Then

\[
H_\eta(T^{-1}N_1|T^{-1}N_2)) = H_\eta(N_1|N_2).
\]

**Proof.** Suppose \(\tilde{N}_1 = \{\mu_i : i \in \mathbb{N}\}\) and \(\tilde{N}_2 = \{\lambda_j : j \in \mathbb{N}\}. \) Since \(T\) is a \((\eta, \eta)\) relative measure preserving transformation and \(T^{-1}\tilde{N}_1 = \{\eta \land T^{-1}\mu_i : \mu_i \in \tilde{N}_1\}, T^{-1}\tilde{N}_2 = \{\eta \land T^{-1}\lambda_j : \lambda_j \in \tilde{N}_2\}, \) we have

\[
H_\eta(T^{-1}N_1|T^{-1}N_2)) = -\log \frac{\sup_{i,j} m((\eta \land T^{-1}\mu_i) \land (\eta \land T^{-1}\lambda_j))}{\sup_{j} m(\eta \land T^{-1}\lambda_j)}
\]

\[
= -\log \frac{\sup_{i,j} m(\eta \land T^{-1}\lambda_j)}{\sup_{j} m(\eta \land T^{-1}\lambda_j)}
\]

\[
= -\log \frac{\sup_{i,j} m(\mu_i \land \lambda_j)}{\sup_{j} m(\lambda_j)}
\]

\[
= H_\eta(N_1|N_2).
\]
Theorem 3.6 Let \((X, M, m)\) be a \(\eta\)-relative probability measure space and let \(N_1, N_2, N_3 \in F_\eta(M)\) with \(N_1 = \{\mu_i : i \in \mathbb{N}\}\), \(N_2 = \{\lambda_j : j \in \mathbb{N}\}\) and \(N_3 = \{\gamma_k : k \in \mathbb{N}\}\). Then

(i) \(\eta < 1\) iff \(H_\eta(N_1) > 0\);
(ii) \(H_\eta(N_1 \vee N_2) \geq H_\eta(N_1)\) and \(H_\eta(N_1 \vee N_2) \geq H_\eta(N_2)\);
(iii) If \(N_1\) and \(N_2\) are independent, then
\[
H_\eta(N_1 \vee N_2) = H_\eta(N_1) + H_\eta(N_2);
\]
(iv) \(H_\eta(N_1 \vee N_2) = H_\eta(N_2) + H_\eta(N_1|N_2)\);
(v) If \(N_1\) and \(N_2 \vee N_3\) are independent, then
\[
H_\eta(N_1 \vee N_2|N_3) = H_\eta(N_1) + H_\eta(N_2|N_3);
\]
(vi) If \(N_1 \leq m N_2\) then \(H_\eta(N_1) \leq H_\eta(N_2)\);
(vii) If \(N_1 \leq m N_2\) then \(H_\eta(N_1|N_3) \leq H_\eta(N_2|N_3)\).

Proof. (i) Obvious.
(ii) Since \(m(\mu_i) \geq m(\mu_i \wedge \lambda_j)\) for any \(i, j \in \mathbb{N}\), we have
\[
\sup_{i,j \in \mathbb{N}} m(\mu_i \wedge \lambda_j) \leq \sup_{i \in \mathbb{N}} m(\mu_i).
\]
So \(H_\eta(N_1 \vee N_2) \geq H_\eta(N_1)\). Similarly \(H_\eta(N_1 \vee N_2) \geq H_\eta(N_2)\).
(iii)
\[
H_\eta(N_1 \vee N_2) = -\log \sup_{i,j} m(\mu_i \wedge \lambda_j)
= -\log \sup_{i,j} m(\mu_i)m(\lambda_j)
= H_\eta(N_1) + H_\eta(N_2).
\]
(iv)
\[
\sup_{i,j} m(\mu_i \wedge \lambda_j) = \frac{\sup_{i,j} m(\mu_i \wedge \lambda_j)}{\sup_j m(\lambda_j)} \sup_j m(\lambda_j).
\]
Therefore
\[
H_\eta(N_1 \vee N_2) = -\log \sup_{i,j} m(\mu_i \wedge \lambda_j)
= -\log \frac{\sup_{i,j} m(\mu_i \wedge \lambda_j)}{\sup_j m(\lambda_j)} \log \sup_j m(\lambda_j)
= H_\eta(N_1|N_2) + H_\eta(N_2).
\]
(v) Since \(N_1\) and \(N_2 \vee N_3\) are independent, we have for each \(i,j,k \in \mathbb{N}\),
\[
m_\eta(\mu_i \wedge (\lambda_j \wedge \gamma_k)) = m(\mu_i)m(\lambda_j \wedge \gamma_k).
\]
So
\[
H_\eta(N_1 \vee N_2|N_3) = -\log \sup_{i,j,k \in \mathbb{N}} \frac{m(\mu_i \wedge \lambda_j \wedge \gamma_k)}{m(\gamma_k)}
= -\log \sup_{i,j,k \in \mathbb{N}} \frac{m(\mu_i)m(\lambda_j \wedge \gamma_k)}{m(\gamma_k)}
= -\log \sup_{i \in \mathbb{N}} m(\mu_i) \sup_{j,k \in \mathbb{N}} \frac{m(\lambda_j \wedge \gamma_k)}{m(\gamma_k)}
= -\log \sup_{i \in \mathbb{N}} m(\mu_i) - \log \sup_{j,k \in \mathbb{N}} \frac{m(\lambda_j \wedge \gamma_k)}{m(\gamma_k)}
= H_\eta(N_1) + H_\eta(N_2|N_3).
(vi) Obvious.
(vii) Since $N_1 \lor N_3 \leq m, N_2 \lor N_3$, by parts iv) and vi) we have

$$H_\eta(N_1|N_3) = H_\eta(N_1 \lor N_3) - H_\eta(N_3) \leq H_\eta(N_2 \lor N_3) - H_\eta(N_3) = H_\eta(N_2|N_3).$$

4 Relative Entropy of Dynamical Systems

In this section, using the suggested concept of relative entropy, we define the relative entropy of a dynamical system and prove some of its ergodic properties.

**Definition 4.1** Let $T : X \to X$ be a relative $(\eta, \eta)$ measure preserving transformation of $(X, M, m)$ and let $N \in F_\eta(M)$ with $N = \{\mu_i : i \in N\}$. The relative entropy of $T$ with respect to $N$ is defined by:

$$h_\eta(T, N) = \limsup_{n \to \infty} \frac{1}{n} H_\eta(\vee_{i=0}^{n-1} T^{-i}N).$$

**Theorem 4.2** If $T : X \to X$ is a relative measure preserving transformation of relative probability measure space $(X, M, m)$ and $N_1, N_2 \in F_\eta(M)$ with the property in Definition 4.2, then

(i) $h_\eta(T, N) \leq H_\eta(N)$;
(ii) $N_1 \leq m, N_2$ implies that $h_\eta(T, N_1) \leq h_\eta(T, N_2)$;
(iii) $h_\eta(T, T^{-1}N) = h_\eta(T, N)$.

**Proof.** The assertions follow from Theorems 4.1 (vi) and 4.2.

**Definition 4.3** Let $T : X \to X$ be a $(\eta, \eta)$ relative measure preserving transformation of $(X, M, m)$. The relative entropy of $T$ is defined by:

$$h_\eta(T) = \sup_N h_\eta(T, N),$$

where the supremum is taken over all $N \in F_\eta(M)$ with the property in Definition 4.1.

**Theorem 4.4** Let $T : X \to X$ be a $(\eta, \eta)$ relative measure preserving transformation of relative probability space $(X, M, m)$. Then

i) $h_\eta(id) = 0$;
ii) For $k \in N$, $h_\eta(T^k) = kh_\eta(T)$.

**Proof.** i) By the definition we have $\vee_{i=0}^{n-1} T^{-i}N = N$, for any $n \in N$. Therefore

$$h_\eta(id, N) = \limsup_{n \to \infty} \frac{1}{n} H_\eta(N) = 0.$$

ii) Let $N$ be an arbitrary countable relative sub $\sigma$-algebra of $M$. We have

$$h_\eta(T^k, \vee_{i=0}^{n-1} T^{-i}N) = \limsup_{n \to \infty} \frac{1}{n} H_\eta(\vee_{j=0}^{n-1} (T^k)^{-j}(\vee_{i=0}^{n-1} (T^{-i}N)))$$

$$= \limsup_{n \to \infty} \frac{1}{n} H_\eta(\vee_{j=0}^{n-1} \vee_{i=0}^{k-1} T^{-(kj+i)}N)$$

$$= \limsup_{n \to \infty} \frac{1}{n} H_\eta(\vee_{i=0}^{nk-1} T^{-i}N)$$

$$= \limsup_{n \to \infty} \frac{nk}{n} \frac{1}{nk} H_\eta(\vee_{i=0}^{nk-1} T^{-i}N)$$

$$= kh_\eta(T, N).$$

Therefore

$$kh_\eta(T) = \sup_N h_\eta(T, N) = \sup_N h_\eta(T^k, \vee_{i=0}^{k-1} T^{-i}N) \leq \sup_N h_\eta(T^k, N) = h_\eta(T^k).$$
Since \( N \leq m \lor k \ll 1 \), we have
\[
h_\eta(T^k, N) \leq h_\eta(T^k, \lor_{i=0}^{k-1} T^{-i} N) = kh_\eta(T, N).
\]

**Definition 4.5** Let \( T : X \to X \) and \( S : X \to X \) be two \((\eta, \eta)\) relative measure preserving transformations of \((X, M, m)\). We say that \( T \) and \( S \) are \( \eta \)-conjugate if there exists a bijective \((\eta, \eta)\) relative measure preserving transformation \( \varphi : X \to X \) such that \( \varphi o T = So \varphi \).

In the following theorem we prove that the relative entropy of relative dynamical systems is invariant under the relation of conjugate.

**Theorem 4.6** If \( T : X \to X \) and \( S : X \to X \) are \( \eta \)-conjugate, then
\[
h_\eta(T) = h_\eta(S).
\]

**Proof.** By Definition 4.5, there exists a bijective \((\eta, \eta)\) relative measure preserving transformation \( \varphi : X \to X \) such that \( \varphi o T = So \varphi \). We can write
\[
h_\eta(S, N) = \lim sup_{n \to \infty} \frac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} S^{-i} N)
\]
\[
= \lim sup_{n \to \infty} \frac{1}{n} H_\eta(\varphi^{-1}(\bigvee_{i=0}^{n-1} S^{-i} N))
\]
\[
= \lim sup_{n \to \infty} \frac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} \varphi^{-1}(S^{-i} N))
\]
\[
= \lim sup_{n \to \infty} \frac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} T^{-1}(\varphi^{-i} N))
\]
\[
= h_\eta(T, \varphi^{-1} N).
\]

So
\[
h_\eta(S) = \sup_{N} h_\eta(S, N)
\]
\[
= \sup_{N} h_\eta(T, \varphi^{-1} N)
\]
\[
\leq \sup_{N} h_\eta(T, N)
\]
\[
= h_\eta(T).
\]

Therefore \( h_\eta(S) \leq h_\eta(T) \). Similarly we obtain \( h_\eta(T) \leq h_\eta(S) \).

**5 Conclusion**

This contribution has defined the notions of relative \( \sigma \)-algebra, relative probability measure and relative probability measure spaces by the concept of observer. We defined entropy and conditional entropy of relative \( \sigma \)-algebras having countably many atoms. We proved some of their ergodic properties. Then, using the suggested concept of relative entropy of relative \( \sigma \)-algebras, the relative entropy of a dynamical system was defined. Finally, it was shown that isomorphic relative dynamical systems have the same relative entropy. Accordingly, this concept will be a new tool for distinction of non-isomorphic relative dynamical systems.

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References


