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# Entropy of Dynamical Systems from the Observer's Viewpoint, with Countable $\sigma$-algebras

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**Abstract**

In this paper, we introduce the notion of relative probability measure spaces by the concept of observer. The notions of relative entropy and relative conditional entropy of countable $\sigma$-algebras are defined and studied. Using the suggested concept of relative entropy we define the relative entropy of a dynamical system and we prove some of its ergodic properties. Finally it is proved that, the relative entropy of dynamical systems is invariant under conjugate relation.



**Keywords:** observer, countable relative $\sigma$-algebra, relative measure space, atom, relative entropy

## 1 Introduction

The study of the concept entropy is very important in sciences nowadays. Entropy plays an important role in a variety of problem areas, including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and others. One of the applied branches of mathematics is the entropy of a dynamical system. The notion of observer has been applied in dynamical systems, topology, information theory and mathematical physics [1, 6, 14, 15]. So the study of the concept of entropy of dynamical systems by the notion of observer is very important. The entropy of a fuzzy process is defined and studied in [2, 3, 9, 17, 18]. Klement in [13] studied the notions of fuzzy $\sigma$-algebra and $F$-probability measure space. Khare defined and studied the notions of entropy and conditional entropy of finite fuzzy $\sigma$-algebras [11, 12]. The notions of entropy and conditional entropy of infinite fuzzy $\sigma$-algebras are defined and studied in [3]. In the present paper, we define the notions of relative $\sigma$-algebra and relative probability measure space by using an observer and then we define the notions of entropy and conditional entropy of infinite relative $\sigma$-algebras. We prove some ergodic properties about them. After that we define relative measure preserving transformation and consider it as a relative dynamical system. Using the suggested concept of relative entropy, we define the relative entropy of a dynamical system and prove some theorems about the measure. It is noted that some investigations concerning entropy of dynamical systems and related notions in the above setup were carried in [4, 5, 7, 8, 10].

The paper is organized as follows. In the next section, we define the notions of relative $\sigma$-algebra, relative probability measure space and relative measure preserving transformation. In section 3, the entropy and conditional entropy of infinite relative $\sigma$-algebras are defined and basic properties of these measures are proved. In section 4, using the suggested concept of entropy of infinite relative $\sigma$-algebras, we define the entropy of a relative dynamical system. Finally, it is shown that conjugate dynamical systems have the same relative entropy. Our results are summarized in the final section.

## 2 Relative Measure Spaces

In this section, we define the notion of relative $\sigma$-algebra by using an observer and then we define the notion of relative probability measure space. Also, we define the notion of relative measure preserving transformation.

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Let $X$ be a non-empty set and an observer be a fuzzy set $\eta : X \longrightarrow [0,1]$.

**Definition 2.1** *A $\sigma$-algebra from the point of view $\eta$, is a collection $\Sigma_\eta$ of subsets of $\eta$, ($\lambda \subseteq \eta$ i.e $\forall x \in X$, $\lambda(x) \leq \eta(x)$) such that:*
*(i) $\eta \in \Sigma_\eta$;*
*(ii) If $\lambda \in \Sigma_\eta$ then $\eta - \lambda \in \Sigma_\eta$;*
*(iii) If $\{\lambda_i\}_{i=1}^{\infty} \subseteq \Sigma_\eta$ then $\bigvee_{i=1}^{\infty} \lambda_i = \sup_i \lambda_i \in \Sigma_\eta$.*

Note that if $\eta = 1$, then the $\sigma$-algebra from the point of view $\eta = 1$ is the fuzzy $\sigma$-algebra [13].

**Example 2.2** *Let $X = [-1,1]$ and $\eta : X \longrightarrow [0, \frac{1}{2}]$ defined by $\eta(x) = \frac{1}{2}$ for every $x \in [-1,1]$. Then $\sum_\eta = \{\lambda_j : j \in \mathbf{N}\} \cup \{\mu_k : k = 2,3,...\}$ is a $\sigma$-algebra from the point of view $\eta$, where $\lambda_j : [-1,1] \longrightarrow [0,1/2]$ by $\lambda_j(x) = 1/2^j$ and $\mu_k : [-1,1] \longrightarrow [0,1/2]$ by $\mu_k(x) = (2^{k-1}-1)/2^k$. Let us check the properties of $\sum_\eta$.*
*(i) $\eta = \lambda_1 \in \sum_\eta$;*
*(ii) For every $j \in \mathbf{N}$, $\eta - \lambda_j = \mu_j \in \sum_\eta$ and $\eta - \mu_j = \lambda_j \in \sum_\eta$;*
*(iii) Let $\{\gamma_i\}_{i=1}^{\infty} \subseteq \Sigma_\eta$. Since the sequences $\{\lambda_j : j \in \mathbf{N}\}$ and $\{\mu_k\}_{k=2}^{\infty}$ are monotone, we get $\sup_i\{\gamma_i : i \in \mathbf{N}\} \in \Sigma_\eta$.*

**Definition 2.3** *A function $m : \Sigma_\eta \longrightarrow [0,1]$ is called $\eta$-relative probability measure when,*
*(i) $m(\eta) = \sup_x \eta(x)$;*
*(ii) $m(\eta - \lambda) = m(\eta) - m(\lambda)$ for every $\lambda \in \Sigma_\eta$;*
*(iii) $m(\lambda \vee \mu) + m(\lambda \wedge \mu) = m(\lambda) + m(\mu)$ for every $\lambda, \mu \in \Sigma_\eta$;*
*(iv) If $\{\lambda_i\}_{i=1}^{\infty} \subseteq \Sigma_\eta$ and $\lambda_i \uparrow \lambda$, then $m(\lambda) = \sup_i m(\lambda_i)$.*

Now we say that $(X, \Sigma_\eta, m)$ is a relative probability measure space and each element of $\Sigma_\eta$ is a relative measurable set.

In Definition 2.3 you observe that the $\eta = 1$-relative probability measure is the $F$-probability measure and $(X, \Sigma_1, m)$ is the $F$-probability measure space which were defined in [13].

**Example 2.4** *Let $\Sigma_\eta$ be the relative $\sigma$-algebra defined in Example 2.2. Define $m : \Sigma_\eta \longrightarrow [0,1/2]$ by $m(\lambda_j) = 1/2^j$, $m(\mu_k) = \mu_k = (2^{k-1}-1)/2^k$. Also let $\vee$ and $\wedge$ mean respectively supremum and infimum. Then $(X, \Sigma_\eta, m)$ is a relative probability measure space. It is easy to see that the properties of Definition 2.3, hold.*

**Definition 2.5** *Let $(X, \Sigma_\eta, m)$ be a relative probability measure space. For $\lambda, \mu \in \Sigma_\eta$, the relation $=$ (mod $m$) is defined by*

$$\lambda = \mu \ (mod\ m) \iff m(\lambda) = m(\mu) = m(\lambda \vee \mu).$$

*The relation $=$ (mod $m$) is an equivalence relation on $\Sigma_\eta$. The set of all equivalence classes induced by this relation is denoted by $\tilde{\Sigma_\eta}$, and $\tilde{\mu}$ denotes the equivalence class determined by $\mu$. $\lambda, \mu \in \Sigma_\eta$ is called $m$-disjoint if $\lambda \wedge \mu = 0$ (mod $m$), i.e. $m(\lambda \wedge \mu) = 0$.*

**Definition 2.6** *Let $(X, \Sigma_\eta, m)$ be a relative probability measure space and let $N$ be a relative sub-$\sigma$-algebra of $\tilde{\Sigma_\eta}$. An element $\tilde{\mu} \in \tilde{N}$ is called an atom of $N$ if $m(\mu) > 0$ and, for any $\tilde{\lambda} \in \tilde{N}$,*

$$m(\lambda \wedge \mu) = m(\lambda) \neq m(\mu) \implies m(\lambda) = 0.$$

The set of all atoms of $N$ is denoted by $\tilde{N}$, and $F(\Sigma_\eta)$ denote the collection of relative sub-$\sigma$-algebras of $\Sigma_\eta$ with countably many atoms.

**Definition 2.7** *Let $(X_1, M_1, m_1)$ and $(X_2, M_2, m_2)$ be $\eta$-relative probability measure spaces. We say $T : X_1 \longrightarrow X_2$ is a $(\eta, \eta)$ relative measure preserving transformation when:*
*(i) $T^{-1}(\mu) \wedge \eta \in M_1$, for every $\mu \in M_2$ where $(T^{-1}(\mu))(x) = \mu(T(x))$;*
*(ii) $m_1(T^{-1}(\mu) \wedge \eta) = m_2(\mu)$ for all $\mu \in \tilde{M}_2$.*

**Theorem 2.8** *Let $T : X \longrightarrow X$ be a $(\eta, \eta)$ relative measure preserving transformation, and let $N \in F(\Sigma_\eta)$. Then $T^{-1}N \in F(\Sigma_{\eta \wedge T^{-1}\eta})$, where*

$$\widetilde{T^{-1}N} = \{\eta \wedge T^{-1}\mu : \mu \in \tilde{N}\}.$$

*Proof.* It is necessary to check the axioms of Definition 2.1 for $T^{-1}N$. Since $\eta \in \tilde{N}$, we have $\eta \wedge T^{-1}\eta \in \widetilde{T^{-1}N}$. Let $\eta \wedge T^{-1}\mu \in \widetilde{T^{-1}N}$. $\mu \in \tilde{N}$ implies that $\eta - \mu \in \tilde{N}$. So $(\eta \wedge T^{-1}\eta) - (\eta \wedge T^{-1}\mu) = \eta \wedge T^{-1}(\eta - \mu) \in \widetilde{T^{-1}N}$. Now let $\{\eta \wedge T^{-1}\mu_i\}_{i=1}^{\infty} \subseteq \widetilde{T^{-1}N}$. Since $N \in F(\Sigma_\eta)$, we can write

$$\bigvee_{i=1}^{\infty}(\eta \wedge T^{-1}\mu_i) = \eta \wedge \bigvee_{i=1}^{\infty} T^{-1}\mu_i = \eta \wedge T^{-1}(\bigvee_{i=1}^{\infty} \mu_i) \in \widetilde{T^{-1}N}.$$

Note that $m_\eta(\eta \wedge T^{-1}(\mu)) = m(\mu) > 0$ for all $\mu \in \tilde{N}$ because $T$ is a $(\eta, \eta)$ relative measure preserving transformation.

**Definition 2.9** *Let $(X, M, m)$ be a $\eta$-relative probability measure space. We say $T : X \longrightarrow X$ is relative ergodic if for every atom $\mu \in \tilde{M}$ with $(T^{-1}\mu) \wedge \eta = \mu$ we deduce that $m(\mu) = 0$ or $m_\eta(\mu) = m(\eta)$. Then $m$ is called $T$-relative ergodic.*

**Theorem 2.10** *Let $\Omega$ be the set of invariant relative probability measures on $X$ (i.e $T^{-1}(\mu) \wedge \eta = \mu$ for every $T \in \Omega$). Also suppose that $m_1, m_2 \in \Omega$, $0 < \lambda < m(\eta)$ and $m = \lambda m_1 + (m(\eta) - \lambda)m_2$ imply that $m_1 = m_2$. Then $m$ is relative ergodic.*

*Proof.* Suppose $m$ is not relative ergodic. So there exists $\mu \in \tilde{M}$ such that $0 < m(\mu) < m(\eta)$ and $T^{-1}(\mu) \wedge \eta = \mu$. For every atom $\xi \in \tilde{M}$ we have

$$\xi = (\xi \wedge \mu) \vee (\xi \wedge (\eta - \mu)).$$

Now we can write

$$\begin{aligned} m(\xi) &= m((\xi \wedge \mu) \vee (\xi \wedge (\eta - \mu)) \\ &= m(\mu)(\frac{m(\xi \wedge \mu)}{m(\mu)}) + m(\eta - \mu)(\frac{m(\xi \wedge (\eta - \mu))}{m(\eta - \mu)}) \\ &= \lambda m_1(\xi) + (m(\eta) - \lambda)m_2(\xi), \end{aligned}$$

where $\lambda = m(\mu)$, $m_1(\xi) = m(\xi \wedge \mu)/m(\mu)$ and $m_2(\xi) = m(\xi \wedge (\eta - \mu))/m(\eta - \mu)$. This implies that $m_\eta = \lambda m_1 + (m(\eta) - \lambda)m_2$.

In the remainder of this paper, $M$ is a countable $\eta$-relative $\sigma$-algebra and $F_\eta(M)$ is the set of all $\eta$-relative sub $\sigma$-algebras of $M$ with countably many atoms.

**Definition 2.11** *Let $M$ be a $\eta$-relative $\sigma$-algebra and $N_1, N_2 \in F_\eta(M)$ such that $\{\lambda_i : i \in \mathbf{N}\}$ and $\{\mu_j : j \in \mathbf{N}\}$ be the atoms of $N_1, N_2$, respectively. Then their join is defined by*

$$N_1 \bigvee N_2 = \{\lambda_i \wedge \mu_j : m(\lambda_i \wedge \mu_j) > 0, i, j \in \mathbf{N}\}.$$

**Definition 2.12** *Let $(X, M, m)$ be a $\eta$-relative probability measure space and $N_1, N_2 \in F_\eta(M)$. Then $N_2$ is called an $m$-refinement of $N_1$ and denoted by $N_1 \leq_m N_2$, if for every $\mu \in \tilde{N_2}$, there exists $\lambda \in \tilde{N_1}$ such that $m(\lambda \wedge \mu) = m(\mu)$.*

## 3 Relative Entropy of $\sigma$-Algebras

In this section we define the notions of relative entropy and relative conditional entropy of relative $\sigma$-algebras and we prove some ergodic properties about them.

**Definition 3.1** *Let $(X, M, m)$ be a $\eta$-relative probability measure space and $N \in F_\eta(M)$. The relative entropy of $N$ is defined by:*

$$H_\eta(N) = -\log \sup_{\mu_i \in \tilde{N}} m(\mu_i).$$

**Theorem 3.2** *Let $T : X \longrightarrow X$ be a $(\eta, \eta)$ relative measure preserving transformation of relative probability space $(X, M, m)$. If $N \in F_\eta(M)$, then*

$$H_\eta(T^{-1}N) = H_\eta(N).$$

*Proof.* Suppose $\tilde{N} = \{\mu_i : i \in \mathbf{N}\}$. Since $T$ is a $(\eta, \eta)$ relative measure preserving transformation, we have for each $i \in \mathbf{N}$,

$$m_\eta(\eta \wedge T^{-1}\mu_i) = m(\mu_i).$$

But $T^{-1}N = \{\eta \wedge T^{-1}\mu_i : \mu_i \in N\}$, thus

$$H_\eta(T^{-1}N) = -\log \sup_i m(\eta \wedge T^{-1}\mu_i) = -\log \sup_i m(\mu_i) = H_\eta(N).$$

**Definition 3.3** *Let $(X, M, m)$ be a $\eta$-relative probability measure space and $N_1, N_2 \in F_\eta(M)$. Also let $\tilde{N}_1 = \{\mu_i : i \in \mathbf{N}\}$, $\tilde{N}_2 = \{\lambda_j : j \in \mathbf{N}\}$. The conditional relative entropy of $N_1$ given $N_2$ is defined by:*

$$H_\eta(N_1|N_2) = -\log \frac{\sup_{i,j} m(\mu_i \wedge \lambda_j)}{\sup_j m(\lambda_j)}.$$

**Example 3.4** *Let $\Sigma_\eta$ be the relative $\sigma$-algebra and $m$ be the $\eta$-relative probability measure defined in Examples 2.2 and 2.4. Suppose $N_1 = \{\lambda_j : j \in \mathbf{N}\}$ and $N_2 = \{\mu_k : k \in \mathbf{N}\}$. Then $F(\Sigma_\eta) = N_1 \cup N_2$. Therefore*

$$\begin{aligned} H_\eta(N_1) &= -\log \sup_{\lambda_j \in \tilde{N}_1} m(\lambda_j) \\ &= -\log \sup_{\lambda_j \in \tilde{N}_1} (\frac{1}{2^j}) \\ &= -\log(\frac{1}{2}) = \log 2, \end{aligned}$$

*also,*

$$\begin{aligned} H_\eta(N_2) &= -\log \sup_{\mu_k \in \tilde{N}_2} m(\mu_k) \\ &= -\log \sup_{\mu_k \in \tilde{N}_2} (\frac{2^{k-1} - 1}{2^k}) \\ &= -\log \sup_{\mu_k \in \tilde{N}_2} (\frac{1}{2} - \frac{1}{2^k}) \\ &= -\log(\frac{1}{2}) = \log 2, \end{aligned}$$

*and,*

$$\begin{aligned} H_\eta(N_2|N_1) &= -\log \frac{\sup_{k,j} m(\mu_k \wedge \lambda_j)}{\sup_j m(\lambda_j)} \\ &= -\log(\frac{\frac{1}{2}}{\frac{1}{2}}) = 0. \end{aligned}$$

**Theorem 3.5** *Let $T : X \longrightarrow X$ be a $(\eta, \eta)$ relative measure preserving transformation of the relative probability space $(X, M, m)$. Also let $N_1, N_2 \in F_\eta(M)$. Then*

$$H_\eta(T^{-1}N_1|T^{-1}N_2)) = H_\eta(N_1|N_2).$$

*Proof.* Suppose $\tilde{N}_1 = \{\mu_i : i \in \mathbf{N}\}$ and $\tilde{N}_2 = \{\lambda_j : j \in \mathbf{N}\}$. Since $T$ is a $(\eta, \eta)$ relative measure preserving transformation and $T^{-1}\tilde{N}_1 = \{\eta \wedge T^{-1}\mu_i : \mu_i \in \tilde{N}_1\}$, $T^{-1}\tilde{N}_2 = \{\eta \wedge T^{-1}\lambda_j : \lambda_j \in \tilde{N}_2\}$, we have

$$\begin{aligned} H_\eta(T^{-1}N_1|T^{-1}N_2)) &= -\log \frac{\sup_{i,j} m((\eta \wedge T^{-1}\mu_i) \wedge (\eta \wedge T^{-1}\lambda_j))}{\sup_j m(\eta \wedge T^{-1}\lambda_j)} \\ &= -\log \frac{\sup_{i,j} m(\eta \wedge T^{-1}(\mu_i \wedge \lambda_j))}{\sup_j m(\eta \wedge T^{-1}\lambda_j)} \\ &= -\log \frac{\sup_{i,j} m(\mu_i \wedge \lambda_j)}{\sup_j m(\lambda_j)} \\ &= H_\eta(N_1|N_2). \end{aligned}$$

**Theorem 3.6** *Let $(X, M, m)$ be a $\eta$-relative probability measure space and let $N_1, N_2, N_3 \in F_\eta(M)$ with $\tilde{N_1} = \{\mu_i : i \in \mathbf{N}\}$, $\tilde{N_2} = \{\lambda_j : j \in \mathbf{N}\}$ and $\tilde{N_3} = \{\gamma_k : k \in \mathbf{N}\}$. Then*
*(i) $\eta < 1$ iff $H_\eta(N_1) > 0$;*
*(ii) $H_\eta(N_1 \vee N_2) \geq H_\eta(N_1)$ and $H_\eta(N_1 \vee N_2) \geq H_\eta(N_2)$;*
*(iii) If $N_1$ and $N_2$ are independent, then*

$$H_\eta(N_1 \vee N_2) = H_\eta(N_1) + H_\eta N_2);$$

*(iv) $H_\eta(N_1 \vee N_2) = H_\eta(N_2) + H_\eta(N_1|N_2)$;*
*(v) If $N_1$ and $N_2 \vee N_3$ are independent, then*

$$H_\eta(N_1 \vee N_2|N_3) = H_\eta(N_1) + H_\eta N_2|N_3);$$

*(vi) If $N_1 \leq_m N_2$ then $H_\eta(N_1) \leq H_\eta(N_2)$;*
*(vii) If $N_1 \leq_m N_2$ then $H_\eta(N_1|N_3) \leq H_\eta(N_2|N_3)$.*

*Proof.* (i) Obvious.
(ii) Since $m(\mu_i) \geq m(\mu_i \wedge \lambda_j)$ for any $i, j \in \mathbf{N}$, we have

$$\sup_{i,j\in\mathbf{N}} m(\mu_i \wedge \lambda_j) \leq \sup_{i\in\mathbf{N}} m(\mu_i).$$

So $H_\eta(N_1 \vee N_2) \geq H_\eta(N_1)$. Similarly $H_\eta(N_1 \vee N_2) \geq H_\eta(N_2)$.
(iii)

$$\begin{aligned} H_\eta(N_1 \vee N_2) &= -\log \sup_{i,j} m(\mu_i \wedge \lambda_j) \\ &= -\log \sup_{i,j} m(\mu_i)m(\lambda_j) \\ &= H_\eta(N_1) + H_\eta(N_2). \end{aligned}$$

(iv)

$$\sup_{i,j} m(\mu_i \wedge \lambda_j) = \frac{\sup_{i,j} m(\mu_i \wedge \lambda_j)}{\sup_j m(\lambda_j)} \sup_j m(\lambda_j).$$

Therefore

$$\begin{aligned} H_\eta(N_1 \vee N_2) &= -\log \sup_{i,j} m(\mu_i \wedge \lambda_j) \\ &= -\log \frac{\sup_{i,j} m(\mu_i \wedge \lambda_j)}{\sup_j m(\lambda_j)} - \log \sup_j m(\lambda_j) \\ &= H_\eta(N_1|N_2) + H_\eta(N_2). \end{aligned}$$

(v) Since $N_1$ and $N_2 \vee N_3$ are independent, we have for each $i.j, k \in \mathbf{N}$,

$$m_\eta(\mu_i \wedge (\lambda_j \wedge \gamma_k)) = m(\mu_i)m(\lambda_j \wedge \gamma_k).$$

So

$$\begin{aligned} H_\eta(N_1 \vee N_2|N_3) &= -\log \sup_{i,j,k\in\mathbf{N}} \frac{m(\mu_i \wedge \lambda_j \wedge \gamma_k)}{m(\gamma_k)} \\ &= -\log \sup_{i,j,k\in\mathbf{N}} \frac{m(\mu_i)m(\lambda_j \wedge \gamma_k)}{m(\gamma_k)} \\ &= -\log \sup_{i\in\mathbf{N}} m(\mu_i) \sup_{j,k\in\mathbf{N}} \frac{m(\lambda_j \wedge \gamma_k)}{m(\gamma_k)} \\ &= -\log \sup_{i\in\mathbf{N}} m(\mu_i) - \log \sup_{j,k\in\mathbf{N}} \frac{m(\lambda_j \wedge \gamma_k)}{m(\gamma_k)} \\ &= H_\eta(N_1) + H_\eta(N_2|N_3). \end{aligned}$$

(vi) Obvious.
(vii) Since $N_1 \vee N_3 \leq_m N_2 \vee N_3$, by parts iv) and vi) we have

$$\begin{aligned} H_\eta(N_1|N_3) &= H_\eta(N_1 \vee N_3) - H_\eta(N_3) \\ &\leq H_\eta(N_2 \vee N_3) - H_\eta(N_3) \\ &= H_\eta(N_2|N_3). \end{aligned}$$

# 4 Relative Entropy of Dynamical Systems

In this section, using the suggested concept of relative entropy, we define the relative entropy of a dynamical system and prove some of its ergodic properties.

**Definition 4.1** *Let $T : X \longrightarrow X$ be a relative $(\eta, \eta)$ measure preserving transformation of $(X, M, m)$ and let $N \in F_\eta(M)$ with $\tilde{N} = \{\mu_i : i \in \mathbf{N}\}$. The relative entropy of $T$ with respect to $N$ is defined by:*

$$h_\eta(T, N) = \limsup_{n\to\infty} \tfrac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} T^{-i}N).$$

**Theorem 4.2** *If $T : X \longrightarrow X$ is a relative measure preserving transformation of relative probability measure space $(X, M, m)$ and $N_1, N_2 \in F_\eta(M)$ with the property in Definition 4.1, then*
*(i) $h_\eta(T, N) \leq H_\eta(N)$;*
*(ii) $N_1 \leq_m N_2$ implies that $h_\eta(T, N_1) \leq h_\eta(T, N_2)$;*
*(iii) $h_\eta(T, T^{-1}N) = h_\eta(T, N)$.*

*Proof.* The assertions follow from Theorems 3.6 (vi) and 3.2.

**Definition 4.3** *Let $T : X \longrightarrow X$ be a $(\eta, \eta)$ relative measure preserving transformation of $(X, M, m)$. The relative entropy of $T$ is defined by:*

$$h_\eta(T) = \sup_N h_\eta(T, N),$$

*where the supremum is taken over all $N \in F_\eta(M)$ with the property in Definition 4.1.*

**Theorem 4.4** *Let $T : X \longrightarrow X$ be a $(\eta, \eta)$ relative measure preserving transformation of relative probability space $(X, M, m)$. Then*
*i) $h_\eta(id) = 0$;*
*ii) For $k \in \mathbf{N}$, $h_\eta(T^k) = kh_\eta(T)$.*

*Proof.* i) By the definition we have $\vee_{i=0}^{n-1} T^{-i}N = N$, for any $n \in \mathbf{N}$. Therefore

$$h_\eta(id, N) = \limsup_{n\to\infty} \frac{1}{n} H_\eta(N) = 0.$$

ii) Let $N$ be an arbitrary countable relative sub $\sigma$-algebra of $M$. We have

$$\begin{aligned} h_\eta(T^k, \vee_{i=0}^{n-1} T^{-i}N) &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\vee_{j=0}^{n-1} (T^k)^{-j} (\vee_{i=0}^{n-1} (T^{-i}N)) \\ &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\vee_{j=0}^{n-1} \vee_{i=0}^{k-1} T^{-(kj+i)}N) \\ &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\vee_{i=0}^{nk-1} T^{-i}N) \\ &= \limsup_{n\to\infty} \frac{nk}{n} \frac{1}{nk} H_\eta(\vee_{i=0}^{nk-1} T^{-i}N) \\ &= kh_\eta(T, N). \end{aligned}$$

Therefore

$$kh_\eta(T) = k \sup_N h_\eta(T, N) = \sup_N h_\eta(T^k, \vee_{i=0}^{k-1} T^{-i}N) \leq \sup_N h_\eta(T^k, N) = h_\eta(T^k).$$

Since $N \leq_m \vee_{i=0}^{k-1} T^{-i}N$, we have

$$h_\eta(T^k, N) \leq h_\eta(T^k, \vee_{i=0}^{k-1} T^{-i}N) = kh_\eta(T, N).$$

**Definition 4.5** *Let $T : X \longrightarrow X$ and $S : X \longrightarrow X$ be two $(\eta, \eta)$ relative measure preserving transformations of $(X, M, m)$. We say that $T$ and $S$ are $\eta$-conjugate if there exists a bijective $(\eta, \eta)$ relative measure preserving transformation $\varphi : X \longrightarrow X$ such that $\varphi o T = S o \varphi$.*

In the following theorem we prove that the relative entropy of relative dynamical systems is invariant under the relation of conjugate.

**Theorem 4.6** *If $T : X \longrightarrow X$ and $S : X \longrightarrow X$ are $\eta$-conjugate, then*

$$h_\eta(T) = h_\eta(S).$$

*Proof.* By Definition 4.5, there exists a bijective $(\eta, \eta)$ relative measure preserving transformation $\varphi : X \longrightarrow X$ such that $\varphi o T = S o \varphi$. We can write

$$\begin{aligned} h_\eta(S, N) &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} S^{-i}N) \\ &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\varphi^{-1}(\bigvee_{i=0}^{n-1} S^{-i}N)) \\ &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} \varphi^{-1}(S^{-i}N)) \\ &= \limsup_{n\to\infty} \frac{1}{n} H_\eta(\bigvee_{i=0}^{n-1} T^{-1}(\varphi^{-i}N)) \\ &= h_\eta(T, \varphi^{-1}N). \end{aligned}$$

So

$$\begin{aligned} h_\eta(S) &= \sup_N h_\eta(S, N) \\ &= \sup_N h_\eta(T, \varphi^{-1}N) \\ &\leq \sup_N h_\eta(T, N) \\ &= h_\eta(T). \end{aligned}$$

Therefore $h_\eta(S) \leq h_\eta(T)$. Similarly we obtain $h_\eta(T) \leq h_\eta(S)$.

# 5 Conclusion

This contribution has defined the notions of relative $\sigma$-algebra, relative probability measure and relative probability measure spaces by the concept of observer. We defined entropy and conditional entropy of relative $\sigma$-algebras having countably many atoms. We proved some of their ergodic properties. Then, using the suggested concept of relative entropy of relative $\sigma$-algebras, the relative entropy of a dynamical system was defined. Finally, it was shown that isomorphic relative dynamical systems have the same relative entropy. Accordingly, this concept will be a new tool for distinction of non-isomorphic relative dynamical systems.

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