Operations on Single-Valued Neutrosophic Graphs

Muhammad Akram, Gulfam Shahzadi
Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

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Abstract

The concept of neutrosophic sets can be utilized as a mathematical tool to deal with imprecise and unspecified information. In this paper, we apply the concept of single-valued neutrosophic sets to graphs. We introduce the notion of single-valued neutrosophic graphs, and present some fundamental operations on single-valued neutrosophic graphs. We explore some interesting properties of single-valued neutrosophic graphs by level graphs. We highlight some flaws in the definitions of Broumi et al. [10] and Shah-Hussain [16]. We also present an application of single-valued neutrosophic graphs in social network.

Keywords: level graphs, single-valued neutrosophic graphs, characterizations

1 Introduction

Graph theory has been highly successful in certain academic fields, including natural sciences and engineering. Graph theoretic models can sometimes provide a useful structure upon which analytical techniques can be used. It is often convenient to depict the relationships between pairs of elements of a system by means of a graph or a digraph. The vertices of the graph represent the system elements and its edges or arcs represent the relationships between the elements. This approach is especially useful for transportation, scheduling, sequencing, allocation, assignment, and other problems which can be modelled as networks. Such a graph theoretic model is often useful as an aid in communicating.

Zadeh [23] introduced the concept of fuzzy set. Atanassov [8] introduced the intuitionistic fuzzy sets which is a generalization of fuzzy sets. Fuzzy set theory and intuitionistic fuzzy sets theory are useful models for dealing with uncertainty and incomplete information. But they may not be sufficient in modeling of indeterminate and inconsistent information encountered in real world. In order to cope with this issue, neutrosophic set theory was proposed by Smarandache [17] as a generalization of fuzzy sets and intuitionistic fuzzy sets. However, since neutrosophic sets are identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval [0, 1]. There are some difficulties in modeling of some problems in engineering and sciences. To overcome these difficulties, in 2010, concept of single-valued neutrosophic sets and its operations defined by Wang et al. [20] as a generalization of intuitionistic fuzzy sets. Yang et al. introduced concept of single-valued neutrosophic relation based on single-valued neutrosophic set.

making method using aggregation operators. In this research article, we apply the concept of single-valued neutrosophic sets to graphs. We introduce the notion of single-valued neutrosophic graphs, and present its fundamental operations. We explore some interesting properties of single-valued neutrosophic graphs by level graphs. We highlight some flaws in the definitions of Broumi et al. [11] and Shah-Hussain [10]. We also present an application of single-valued neutrosophic graphs in social network. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [3, 14, 23, 28].

2 Operations on Single-Valued Neutrosophic Graphs

**Definition 2.1.** [17] Let \( X \) be a space of points (objects). A neutrosophic set \( A \) in \( X \) is characterized by a truth-membership function \( T_A(x) \), an indeterminacy-membership function \( I_A(x) \) and a falsity-membership function \( F_A(x) \). The functions \( T_A(x), I_A(x), \) and \( F_A(x) \) are real standard or non-standard subsets of \([0^-1^+]\). That is, \( T_A(x) : X \to [0^-1^+] \), \( I_A(x) : X \to [0^-1^+] \) and \( F_A(x) : X \to [0^-1^+] \). To apply neutrosophic sets in real-life problems more conveniently, Wang et al. [21] defined single-valued neutrosophic sets which takes the value from the subset of \([0,1]\).

**Definition 2.2.** A single-valued neutrosophic graph is a pair \( G = (A,B) \), where \( A : V \to [0,1] \) is single-valued neutrosophic set in \( V \) and \( B : V \times V \to [0,1] \) is single-valued neutrosophic relation on \( V \) such that

\[
T_B(xy) \leq \min\{T_A(x),T_A(y)\},
\]

\[
I_B(xy) \leq \min\{I_A(x),I_A(y)\},
\]

\[
F_B(xy) \leq \max\{F_A(x),F_A(y)\}
\]

for all \( x, y \in V \). \( A \) is called single-valued neutrosophic vertex set of \( G \) and \( B \) is called single-valued neutrosophic edge set of \( G \), respectively. We note that \( B \) is symmetric single-valued neutrosophic relation on \( A \). If \( B \) is not symmetric single-valued neutrosophic relation on \( A \), then \( G = (A,B) \) is called a single-valued neutrosophic directed graph.

**Example 2.1.** Consider a crisp graph \( G^* = (V,E) \) such that \( V = \{a,b,c,d,e,f\} \), \( E = \{ab,ac,bd,ce,cf,ef,be\} \). Let \( A \) and \( B \) be the single-valued neutrosophic sets of \( V \) and \( E \), respectively, as shown in following Tables. By simple calculations, it is easy to see that \( G = (A,B) \) is a single-valued neutrosophic graph as shown in Fig. 44.

<table>
<thead>
<tr>
<th>A</th>
<th>a</th>
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<tr>
<td>F</td>
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</tbody>
</table>

**Definition 2.3.** A single-valued neutrosophic graph \( G = (A,B) \) is called complete if the following conditions are satisfied:

\[
T_B(xy) = \min\{T_A(x),T_A(y)\},
\]

\[
I_B(xy) = \min\{I_A(x),I_A(y)\},
\]

\[
F_B(xy) = \max\{F_A(x),F_A(y)\}
\]

for all \( x, y \in V \).
Example 2.2. Consider a single-valued neutrosophic \( G = (A, B) \) on the nonempty set \( V = \{a, b, c, d, \} \) as shown in Fig. 2.2. By direct calculations, it is easy to see that \( G \) is a complete.

Definition 2.4. Let \( A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in V \} \) be a single-valued neutrosophic set of the set \( V \). For \( \alpha \in [0, 1] \), the \( \alpha \)-cut of \( A \) is the crisp set \( A_\alpha \) defined by

\[
A_\alpha = \{x \in V: \text{either } (T_A(x), I_A(x) \geq \alpha) \text{ or } F_A(x) \leq 1 - \alpha \}.
\]

Let \( B = \{< xy, T_B(xy), I_B(xy), F_B(xy) >\} \) be a neutrosophic set on \( E \subseteq V \times V \). For \( \alpha \in [0, 1] \), the \( \alpha \)-cut is the crisp set \( B_\alpha \) defined by

\[
B_\alpha = \{xy \in E: \text{either } (T_B(xy), I_B(xy) \geq \alpha) \text{ or } F_B(xy) \leq 1 - \alpha \}.
\]

Example 2.3. Consider a single-valued neutrosophic graph \( G = (A, B) \) on non-empty set \( V = \{a, b, c, d, e\} \) as shown in Fig. 2.2.

Take \( \alpha = 0.4 \). We have \( A_{0.4} = \{b, c, d\} \), \( B_{0.4} = \{bc, cd, bd\} \). Clearly, the 0.4-level graph \( G_{0.4} = (A_{0.4}, B_{0.4}) \) is a subgraph of crisp graph \( G^* = (V, E) \).
Proposition 2.4. The level graph $G_\alpha = (A_\alpha, B_\alpha)$ is a crisp graph.

Theorem 2.5. $G = (A, B)$ is a single-valued neutrosophic graph if and only if $G_\alpha = (A_\alpha, B_\alpha)$ is a crisp graph for each $\alpha \in [0, 1]$.

Proof. Let $G = (A, B)$ be a single-valued neutrosophic graph. For each $\alpha \in [0, 1]$, take $xy \in B_\alpha$. Then $\alpha \leq T_B(xy), \alpha \leq I_B(xy)$ or $1 - \alpha \geq F_B(xy)$. Since $G$ is a single-valued neutrosophic graph, it follows that

$$\alpha \leq T_B(xy) \leq \min \{T_A(x), T_A(y)\},$$

$$\alpha \leq I_B(xy) \leq \min \{I_A(x), I_A(y)\},$$

$$1 - \alpha \geq F_B(xy) \leq \max \{F_A(x), F_A(y)\}.$$ 

This shows that $\alpha \leq T_A(x), \alpha \leq T_A(y), \alpha \leq I_A(x), \alpha \leq I_A(y)$ and $1 - \alpha \geq F_A(x), 1 - \alpha \geq F_A(y)$, that is, $x, y \in A_\alpha$. Therefore, $G_\alpha = (A_\alpha, B_\alpha)$ is a graph for each $\alpha \in [0, 1]$.

Conversely, let $G_\alpha = (A_\alpha, B_\alpha)$ be a graph for each $\alpha \in [0, 1]$. For every $xy \in V \times V$, let $T_B(xy) = \alpha, I_B(xy) = \alpha$ and $F_B(xy) \leq 1 - \alpha$. Then $xy \in B_\alpha$. Since $G_\alpha = (A_\alpha, B_\alpha)$ is a graph, we have $x, y \in A_\alpha$; $T_A(x) \geq \alpha, I_A(x) \geq \alpha$, or $F_A(x) \leq 1 - \alpha$ and $T_A(y) \geq \alpha, I_A(y) \geq \alpha$, or $F_A(y) \leq 1 - \alpha$, $\min \{T_A(x), T_A(y)\} \geq \alpha, \min \{I_A(x), I_A(y)\} \geq \alpha$, and $\max \{F_A(x), F_A(y)\} \leq 1 - \alpha$.

Thus

$$T_B(xy) = \alpha \leq \min \{T_A(x), T_A(y)\},$$

$$I_B(xy) = \alpha \leq \min \{I_A(x), I_A(y)\},$$

$$F_B(xy) \leq \max \{F_A(x), F_A(y)\} \leq 1 - \alpha,$$

that is, $G = (A, B)$ is a single-valued neutrosophic graph.

\[\square\]

Definition 2.5. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The Cartesian product $G_1 \times G_2$ is defined as a pair $(A, B)$ such that

(i) $T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,

$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,

$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $T_B((x, x_2)(x, y_2)) = \min(T_{B_1}(x), T_{B_2}(x_2y_2))$,

$I_B((x, x_2)(x, y_2)) = \min(I_{B_1}(x), I_{B_2}(x_2y_2))$,

$F_B((x, x_2)(x, y_2)) = \max(F_{B_1}(x), F_{B_2}(x_2y_2))$ for all $x \in V_1$ and all $x_2y_2 \in E_2$,

(iii) $T_B((x_1, z)(y_1, z)) = \min(T_{B_1}(x_1y_1), T_{B_2}(z))$,

$I_B((x_1, z)(y_1, z)) = \min(I_{B_1}(x_1y_1), I_{B_2}(z))$,

$F_B((x_1, z)(y_1, z)) = \max(F_{B_1}(x_1y_1), F_{B_2}(z))$ for all $z \in V_2$ and all $x_1y_1 \in E_1$.

Proposition 2.6. The Cartesian product of single-valued neutrosophic graphs is a single-valued neutrosophic graph.
Theorem 2.7. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the Cartesian product of $G_1$ and $G_2$ if and only if for each $\alpha \in [0, 1]$, the $\alpha$-level graph $G_\alpha$ is the Cartesian product of $(G_1)_\alpha$ and $(G_2)_\alpha$.

Proof. Let $G = (A, B)$ be the Cartesian product of single-valued neutrosophic graphs $G_1$ and $G_2$. For each $\alpha \in [0, 1]$, if $(x, y) \in A_\alpha$, we have
\[
\min(T_{A_1}(x), T_{A_2}(y)) = T_A(x, y) \geq \alpha, \\
\min(I_{A_1}(x), I_{A_2}(y)) = I_A(x, y) \geq \alpha, \\
\max(F_{A_1}(x), F_{A_2}(y)) = F_A(x, y) \leq 1 - \alpha,
\]
so $x \in (A_1)_\alpha$ and $y \in (A_2)_\alpha$, that is, $(x, y) \in (A_1)_\alpha \times (A_2)_\alpha$. Therefore, $A_\alpha \subseteq (A_1)_\alpha \times (A_2)_\alpha$. Let $(x, y) \in (A_1)_\alpha \times (A_2)_\alpha$, then $x \in (A_1)_\alpha$ and $y \in (A_2)_\alpha$. It follows that $\min(T_{A_1}(x), T_{A_2}(y)) \geq \alpha$, $\min(I_{A_1}(x), I_{A_2}(y)) \geq \alpha$, or $\max(F_{A_1}(x), F_{A_2}(y)) \leq 1 - \alpha$. Since $(A, B)$ is the Cartesian product of $G_1$ and $G_2$, $T_A(x, y) \geq \alpha$, $I_A(x, y) \geq \alpha$, or $F_A(x, y) \leq 1 - \alpha$, that is, $(x, y) \in A_\alpha$. Therefore, $(A_1)_\alpha \times (A_2)_\alpha \subseteq A_\alpha$ and so $(A_1)_\alpha \times (A_2)_\alpha = A_\alpha$. We now prove $B_\alpha = E$, where $E$ is the edge set of the Cartesian product $(G_1)_\alpha \times (G_2)_\alpha$ for each $\alpha \in [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_\alpha$. Then, $T_B((x_1, x_2)(y_1, y_2)) \geq \alpha$, $I_B((x_1, x_2)(y_1, y_2)) \geq \alpha$, or $F_B((x_1, x_2)(y_1, y_2)) \leq 1 - \alpha$. Since $(A, B)$ is the Cartesian product of $G_1$ and $G_2$, one of the following cases hold:

(i) $x_1 = y_1$ and $x_2 y_2 \in E$.

(ii) $x_2 = y_2$ and $x_1 y_1 \in E$.

For the case (i), we have
\[
T_B((x_1, x_2)(y_1, y_2)) = \min(T_{A_1}(x_1), T_{B_2}(x_2y_2)) \geq \alpha, \\
I_B((x_1, x_2)(y_1, y_2)) = \min(I_{A_1}(x_1), I_{B_2}(x_2y_2)) \geq \alpha, \\
F_B((x_1, x_2)(y_1, y_2)) = \max(F_{A_1}(x_1), F_{B_2}(x_2y_2)) \leq 1 - \alpha,
\]
so $T_{A_1}(x_1) \geq \alpha$, $I_{A_1}(x_1) \geq \alpha$ or $F_{A_1}(x_1) \leq 1 - \alpha$ and $T_{B_2}(x_2y_2) \geq \alpha$, $I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1 - \alpha$. It follows that $x_1 = y_1 \in (A_1)_\alpha$, $x_2 y_2 \in (B_2)_\alpha$, that is, $(x_1, x_2)(y_1, y_2) \in E$. Similarly, for the case (ii), we conclude that $(x_1, x_2)(y_1, y_2) \in E$. Therefore, $B_\alpha \subseteq E$. For $(x, x_2)(x_1, y_2) \in E$, $T_A(x, y) \geq \alpha$, $I_A(x, y) \geq \alpha$ or $F_A(x, y) \leq 1 - \alpha$ and $T_{B_2}(x_2y_2) \geq \alpha$, $I_{B_2}(x_2y_2) \geq \alpha$ or $F_{B_2}(x_2y_2) \leq 1 - \alpha$. Since $(A, B)$ is the Cartesian product of $G_1$ and $G_2$, we have
\[
T_B((x, x_2)(x_2, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2y_2)) \geq \alpha, \\
I_B((x, x_2)(x_2, y_2)) = \min(I_{A_1}(x), I_{B_2}(x_2y_2)) \geq \alpha, \\
F_B((x, x_2)(x_2, y_2)) = \max(F_{A_1}(x), F_{B_2}(x_2y_2)) \leq 1 - \alpha.
\]
Therefore, $(x, x_2)(x_2, y_2) \in B_\alpha$. Similarly for every $(x_1, z)(y_1, z) \in E$, we have $(x_1, z)(y_1, z) \in B_\alpha$. Therefore, $E \subseteq B_\alpha$ and so $E = B_\alpha$.

Conversely, suppose that $G_\alpha = (A_\alpha, B_\alpha)$ is the Cartesian product of $(G_1)_\alpha = ((A_1)_\alpha, (B_1)_\alpha)$ and $(G_2)_\alpha = ((A_2)_\alpha, (B_2)_\alpha)$ for each $\alpha \in [0, 1]$. Let $\min(T_{A_1}(x_1), T_{A_2}(x_2)) = \alpha$, $\min(I_{A_1}(x_1), I_{A_2}(x_2)) = \alpha$ or $\max(F_{A_1}(x_1), F_{A_2}(x_2)) = 1 - \alpha$ for some $(x_1, x_2) \in V_1 \times V_2$. Then $x_1 \in (A_1)_\alpha$ and $x_2 \in (A_2)_\alpha$. By hypothesis, $(x_1, x_2) \in A_\alpha$, hence
\[
T_A(x_1, x_2) \geq \alpha = \min(T_{A_1}(x_1), T_{A_2}(x_2)), \\
I_A(x_1, x_2) \geq \alpha = \min(I_{A_1}(x_1), I_{A_2}(x_2)), \\
F_A(x_1, x_2) \leq 1 - \alpha = \max(F_{A_1}(x_1), F_{A_2}(x_2)).
\]
Take $T_A(x_1, x_2) = \beta$, $I_A(x_1, x_2) = \beta$ or $F_A(x_1, x_2) = 1 - \beta$, then $(x_1, x_2) \in A_\beta$. Since $(A_\beta, B_\beta)$ is the Cartesian product of $((A_1)_\beta, (B_1)_\beta)$ and $((A_2)_\beta, (B_2)_\beta)$, then $x_1 \in (A_1)_\beta$ and $x_2 \in (A_2)_\beta$. Hence $T_A(x_1, x_2) \geq \beta$ or $F_A(x_1, x_2) \leq 1 - \beta$ and $T_A(x_1, x_2) \geq \beta$, $I_A(x_2) \geq \beta$ or $F_A(x_2) \leq 1 - \beta$. It follows that
\[
\min(T_{A_1}(x_1), T_{A_2}(x_2)) \geq T_A(x_1, x_2), \\
\min(I_{A_1}(x_1), I_{A_2}(x_2)) \geq I_A(x_1, x_2),
\]
Therefore,
\[ T_A(x_1, x_2) = \min(T_A(x_1), T_A(x_2)), \]
\[ I_A(x_1, x_2) = \min(I_A(x_1), I_A(x_2)), \]
\[ F_A(x_1, x_2) = \max(F_A(x_1), F_A(x_2)) \]
for all \((x_1, x_2) \in V_1 \times V_2\). Similarly, for every \(x \in V_1\) and every \(x_2y_2 \in E_2\), let
\[ \min(T_A(x), T_B(x_2y_2)) = \alpha, \]
\[ \min(I_A(x), I_B(x_2y_2)) = \alpha, \]
\[ \max(F_A(x), F_B(x_2y_2)) = 1 - \alpha, \]
and
\[ T_B((x, x_2)(x_2, y_2)) = \beta, \]
\[ I_B((x, x_2)(x_2, y_2)) = \beta, \]
\[ F_B((x, x_2)(x_2, y_2)) = 1 - \beta. \]
Then we have \(T_A(x) \geq \alpha, I_A(x) \geq \alpha\) or \(F_A(x) \leq 1 - \alpha\) and \(T_B(x_2y_2) \geq \alpha, I_B(x_2y_2) \geq \alpha\) or \(F_B(x_2y_2) \leq 1 - \alpha\), that is, \(x \in (A_1)_\alpha, x_2y_2 \in (B_2)_\alpha\) where \(\alpha \in [0, 1]\) and \((x, x_2)(x_2, y_2) \in B_\beta\) where \(\beta \in [0, 1]\). Since \((A_\alpha, B_\alpha)\) resp. \((A_\beta, B_\beta)\) is the Cartesian product of \(((A_1)_\alpha, (B_1)_\alpha)\) and \(((A_2)_\alpha, (B_2)_\alpha)\) resp. \(((A_1)_\beta, (B_1)_\beta)\) and \(((A_2)_\beta, (B_2)_\beta)\), we have \((x, x_2)(x_2, y_2) \in B_\beta, x \in (A_1)_\beta\) and \(x_2y_2 \in (B_2)_\beta\) which implies \(T_A(x) \geq \beta, I_A(x) \geq \beta\) or \(F_A(x) \leq 1 - \beta\) and \(T_B(x_2y_2) \geq \beta, I_B(x_2y_2) \geq \beta\) or \(F_B(x_2y_2) \leq 1 - \beta\). It follows that
\[ T_B((x, x_2)(x_2, y_2)) \geq \alpha = \min(T_A(x), T_B(x_2y_2)), \]
\[ I_B((x, x_2)(x_2, y_2)) \geq \alpha = \min(I_A(x), I_B(x_2y_2)), \]
\[ F_B((x, x_2)(x_2, y_2)) \leq 1 - \alpha = \max(F_A(x), F_B(x_2y_2)), \]
and
\[ \min(T_A(x), T_B(x_2y_2)) \geq \beta = T_B((x, x_2)(x_2, y_2)), \]
\[ \min(I_A(x), I_B(x_2y_2)) \geq \beta = I_B((x, x_2)(x_2, y_2)), \]
\[ \max(F_A(x), F_B(x_2y_2)) \leq 1 - \beta = F_B((x, x_2)(x_2, y_2)). \]
Therefore,
\[ T_B((x, x_2)(x_2, y_2)) = \min(T_A(x), T_B(x_2y_2)), \]
\[ I_B((x, x_2)(x_2, y_2)) = \min(I_A(x), I_B(x_2y_2)), \]
\[ F_B((x, x_2)(x_2, y_2)) = \max(F_A(x), F_B(x_2y_2)) \]
for all \(x \in V_1\) and \(x_2y_2 \in E_2\). Similarly, we can show that
\[ T_B((x, y_2)(x_2, z)) = \min(T_B((x_1, y_1), T_A(z)), \]
\[ I_B((x, y_2)(x_2, z)) = \min(I_B((x_1, y_1), I_A(z)), \]
\[ F_B((x, y_2)(x_2, z)) = \max(F_B((x_1, y_1), F_A(z))) \]
for all \(z \in V_2\) and \(x_1y_1 \in E_1\). This ends the proof.

**Definition 2.6.** Let \(G_1 = (A_1, B_1)\) and \(G_2 = (A_2, B_2)\) be single-valued neutrosophic graphs of \(G_1^* = (V_1, E_1)\) and \(G_2^* = (V_2, E_2)\), respectively. The composition \(G_1[G_2]\) is defined as a pair \((A, B)\) such that

(i) \(T_A(x_1, x_2) = \min(T_A(x_1), T_A(x_2)), \)
\(I_A(x_1, x_2) = \min(I_A(x_1), I_A(x_2)), \)
\(F_A(x_1, x_2) = \max(F_A(x_1), F_A(x_2)) \)
for all \((x_1, x_2) \in V_1 \times V_2\).
Therefore, \((x; x) \in G\) and for all \(x_2 y_2 \in E_2\),

\[(iii) \quad T_B((x_1, z)(y_1, z)) = \min(T_{B_1}(x_1 y_1), T_{A_3}(z)),
\]

\(I_B((x_1, z)(y_1, z)) = \min(I_{B_1}(x_1 y_1), I_{A_3}(z)),
\]

\(F_B((x_1, z)(y_1, z)) = \max(F_{B_1}(x_1 y_1), F_{A_3}(z)) \) for all \(z \in V_2\) and for all \(x_1 y_1 \in E_1\),

\[(iv) \quad T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1 y_1), T_{A_2}(x_2), T_{A_2}(y_2)),
\]

\(I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1 y_1), I_{A_2}(x_2), I_{A_3}(y_2)),
\]

\(F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1 y_1), F_{A_2}(x_2), F_{A_3}(y_2)) \) for all \(x_2, y_2 \in V_2\), where \(x_2 \neq y_2\) and for all \(x_1 y_1 \in E_1\).

**Proposition 2.8.** The composition of single-valued neutrosophic graphs is a single-valued neutrosophic graph.

**Theorem 2.9.** Let \(G_1 = (A_1, B_1)\) and \(G_2 = (A_2, B_2)\) be single-valued neutrosophic graphs of \(G_1^* = (V_1, E_1)\) and \(G_2^* = (V_2, E_2)\), respectively. Then \(G = (A, B)\) is the composition of \(G_1\) and \(G_2\) if and only if for each \(\alpha \in [0, 1]\), the \(\alpha\)-level graph \(G\) is the composition of \((G_1)_\alpha\) and \((G_2)_\alpha\).

**Proof.** Let \(G = (A, B)\) be the composition of single-valued neutrosophic graphs \(G_1\) and \(G_2\). By the definition of \(G_1[G_2]\) and in the same way as in the proof of Theorem 2.7, we have \(A_\alpha = (A_1)_\alpha \times (A_2)_\alpha\) and \(B_\alpha \subseteq E\), where \(E\) is the edge set of the composition \((G_1)_\alpha(G_2)_\alpha\) for each \(\alpha \in [0, 1]\). Let \((x_1, x_2)(y_1, y_2) \in B_\alpha\) Then \(T_B((x_1, x_2)(y_1, y_2)) \geq \alpha, I_B((x_1, x_2)(y_1, y_2)) \geq \alpha\) or \(F_B((x_1, x_2)(y_1, y_2)) \leq 1 - \alpha\). Since \(G = (A, B)\) is the composition \(G_1[G_2]\), one of the following cases hold:

(i) \(x_1 = y_1\) and \(x_2 y_2 \in E_2\).

(ii) \(x_2 = y_2\) and \(x_1 y_1 \in E_1\).

(iii) \(x_2 \neq y_2\) and \(x_1 y_1 \in E_1\).

For the cases (i) and (ii), similarly as in the cases (i) and (ii) in the proof of Theorem 2.7, we obtain \((x_1, x_2)(y_1, y_2) \in E\). For the case (iii), we have

\[T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1 y_1), T_{A_2}(x_2), T_{A_2}(y_2)) \geq \alpha,
\]

\[I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1 y_1), I_{A_2}(x_2), I_{A_2}(y_2)) \geq \alpha,
\]

\[F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1 y_1), F_{A_2}(x_2), F_{A_2}(y_2)) \leq 1 - \alpha.
\]

Thus, \(T_{B_1}(x_1 y_1) \geq \alpha, I_{B_1}(x_1 y_1) \geq \alpha\) or \(T_{B_1}(x_1 y_1) \leq 1 - \alpha\) and \(T_{A_2}(x_2) \geq \alpha, I_{A_2}(x_2) \geq \alpha\) or \(F_{B_1}(x_1 y_1) \leq 1 - \alpha\). It follows that \(x_2, y_2 \in (A_2)_\alpha\) and \(x_1 y_1 \in (B_1)_\alpha\), that is, \((x_1, x_2)(y_1, y_2) \in E\). Therefore, \(B_\alpha \subseteq E\). For every \((x, x_2)(y, y_2) \in E, T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha\) or \(F_{B_1}(x_1 y_1) \leq 1 - \alpha\) and \(T_{B_2}(x_2 y_2) \geq \alpha, I_{B_2}(x_2 y_2) \geq \alpha\) or \(F_{B_2}(x_2 y_2) \leq 1 - \alpha\). Since \(G = (A, B)\) is the composition \(G_1[G_2]\), we have

\[T_B((x, x_2)(y, x_2)) = \min(T_{A_1}(x), T_{B_2}(x_2 y_2)) \geq \alpha,
\]

\[I_B((x, x_2)(y, x_2)) = \min(I_{A_1}(x), I_{B_2}(x_2 y_2)) \geq \alpha,
\]

\[F_B((x, x_2)(y, x_2)) = \max(F_{A_1}(x), F_{B_2}(x_2 y_2)) \leq 1 - \alpha.
\]

Therefore, \((x, x_2)(y, x_2) \in B_\alpha\).

Similarly, for every \((x_1, z)(y_1, z) \in E\), we have \((x_1, z)(y_1, z) \in B_\alpha\). For every \((x_1, x_2)(y_1, y_2) \in E\) where \(x_2 \neq y_2, x_1 \neq y_1, T_{B_1}(x_1 y_1) \geq \alpha, I_{B_1}(x_1 y_1) \geq \alpha\) or \(F_{B_1}(x_1 y_1) \leq 1 - \alpha\) and \(T_{A_2}(x_2) \geq \alpha, I_{A_2}(x_2) \geq \alpha\) or \(F_{A_2}(x_2) \leq 1 - \alpha\) and \(T_{A_2}(y_2) \geq \alpha, I_{A_2}(y_2) \geq \alpha\) or \(F_{A_2}(y_2) \leq 1 - \alpha\). Since \(G = (A, B)\) is the composition \(G_1[G_2]\), we have

\[T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1 y_1), T_{A_2}(x_2), T_{A_2}(y_2)) \geq \alpha,
\]

\[I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1 y_1), I_{A_2}(x_2), I_{A_3}(y_2)) \geq \alpha,
\]

\[F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1 y_1), F_{A_2}(x_2), F_{A_3}(y_2)) \leq 1 - \alpha.
\]

Thus, \((x_1, x_2)(y_1, y_2) \in B_\alpha\). Therefore, \(E \subseteq B_\alpha\) and so \(E = B_\alpha\). The converse part is obvious, hence we omit its proof.
Definition 2.7. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (A_2, B_2) \), respectively. The union \( G_1 \cup G_2 \) is defined as a pair \((A, B)\) such that

\[
(i) \quad T_A(x) = \begin{cases} 
T_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \not\in V_2, \\
T_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \not\in V_1, \\
\max(T_{A_1}(x), T_{A_2}(x)) & \text{if } x \in V_1 \cap V_2.
\end{cases}
\]

\[
(ii) \quad I_A(x) = \begin{cases} 
I_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \not\in V_2, \\
I_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \not\in V_1, \\
\max(I_{A_1}(x), I_{A_2}(x)) & \text{if } x \in V_1 \cap V_2.
\end{cases}
\]

\[
(iii) \quad F_A(x) = \begin{cases} 
F_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \not\in V_2, \\
F_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \not\in V_1, \\
\min(F_{A_1}(x), F_{A_2}(x)) & \text{if } x \in V_1 \cap V_2.
\end{cases}
\]

\[
(iv) \quad T_B(xy) = \begin{cases} 
T_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \not\in E_2, \\
T_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \not\in E_1, \\
\max(T_{B_1}(xy), T_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2.
\end{cases}
\]

\[
(v) \quad I_B(xy) = \begin{cases} 
I_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \not\in E_2, \\
I_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \not\in E_1, \\
\max(I_{B_1}(xy), I_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2.
\end{cases}
\]

\[
(vi) \quad F_B(xy) = \begin{cases} 
F_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \not\in E_2, \\
F_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \not\in E_1, \\
\min(F_{B_1}(xy), F_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2.
\end{cases}
\]

Proposition 2.10. The union of single-valued neutrosophic graphs is a single-valued neutrosophic graph.

Theorem 2.11. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \), respectively, and \( V_1 \cap V_2 = \emptyset \). Then \( G = (A, B) \) is the union of \( G_1 \) and \( G_2 \) if and only if each alpha-level graph \( G_\alpha \) is the union of \((G_1)_\alpha\) and \((G_2)_\alpha\).

Proof. Let \( G = (A, B) \) be the union of single-valued neutrosophic graphs \( G_1 \) and \( G_2 \). We have to show that \( A_\alpha = (A_1)_\alpha \cup (A_2)_\alpha \) for each \( \alpha \in [0, 1] \). Let \( x \in A_\alpha \), then \( x \in V_1 \setminus V_2 \) or \( x \in V_2 \setminus V_1 \). If \( x \in V_1 \setminus V_2 \), then \( T_{A_1}(x) = T_A(x) \geq \alpha, I_{A_1}(x) = I_A(x) \geq \alpha \) or \( F_{A_1}(x) = F_A(x) \leq 1 - \alpha \), which implies \( x \in (A_1)_\alpha \). Analogously, \( x \in V_2 \setminus V_1 \) implies \( x \in (A_2)_\alpha \). Therefore, \( x \in (A_1)_\alpha \cup (A_2)_\alpha \), and so \( A_\alpha \subseteq (A_1)_\alpha \cup (A_2)_\alpha \). Now let \( x \in (A_1)_\alpha \cup (A_2)_\alpha \). Then \( x \in (A_1)_\alpha \), \( x \notin (A_2)_\alpha \) or \( x \in (A_2)_\alpha \), \( x \notin (A_1)_\alpha \). For the first case, we have \( T_{A_1}(x) = T_A(x) \geq \alpha, I_{A_1}(x) = I_A(x) \geq \alpha \) or \( F_{A_1}(x) = F_A(x) \leq 1 - \alpha \), which implies \( x \in A_\alpha \). For the second case, we have \( T_{A_2}(x) = T_A(x) \geq \alpha, I_{A_2}(x) = I_A(x) \geq \alpha \) or \( F_{A_2}(x) = F_A(x) \leq 1 - \alpha \). Hence \( x \in A_\alpha \).

Consequently, \( (A_1)_\alpha \cup (A_2)_\alpha \subseteq A_\alpha \).

To prove that \( B_\alpha = (B_1)_\alpha \cup (B_2)_\alpha \), for each \( \alpha \in [0, 1] \), consider \( xy \in B_\alpha \). Then \( xy \in E_1 \setminus E_2 \) or \( xy \in E_2 \setminus E_1 \). For \( xy \in E_1 \setminus E_2 \) we have \( T_{B_1}(xy) = T_B(xy) \geq \alpha, I_{B_1}(xy) = I_B(xy) \geq \alpha \) or \( F_{B_1}(xy) = F_B(xy) \leq 1 - \alpha \). Thus \( xy \in (B_1)_\alpha \). Similarly, \( xy \in E_2 \setminus E_1 \) gives \( xy \in (B_2)_\alpha \). Therefore, \( B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha \). For \( xy \notin (B_1)_\alpha \cup (B_2)_\alpha \), then \( xy \in (B_1)_\alpha \setminus (B_2)_\alpha \) or \( xy \in (B_2)_\alpha \setminus (B_1)_\alpha \) for the first case \( T_{B_1}(xy) = T_B(xy) \geq \alpha, I_{B_1}(xy) = I_B(xy) \geq \alpha \) or \( F_{B_1}(xy) = F_B(xy) \leq 1 - \alpha \), hence \( xy \in B_\alpha \). In the second case we obtain \( xy \in B_\alpha \). Therefore, \( (B_1)_\alpha \cup (B_2)_\alpha \subseteq B_\alpha \). The converse part is obvious, hence we omit its proof.

Definition 2.8. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \), respectively. The join \( G_1 + G_2 \) is defined as a pair \((A, B)\) such that

\[
(i) \quad T_A(x) = \begin{cases} 
T_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \not\in V_2, \\
T_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \not\in V_1, \\
\max(T_{A_1}(x), T_{A_2}(x)) & \text{if } x \in V_1 \cap V_2.
\end{cases}
\]

\[
(ii) \quad I_A(x) = \begin{cases} 
I_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \not\in V_2, \\
I_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \not\in V_1, \\
\max(I_{A_1}(x), I_{A_2}(x)) & \text{if } x \in V_1 \cap V_2.
\end{cases}
\]

(iii) \( F_A(x) = \begin{cases} F_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ F_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ \min(F_{A_1}(x), F_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases} \)

(iv) \( T_B(xy) = \begin{cases} T_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ T_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \max(T_{B_1}(xy), T_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2, \\ \min(T_{A_1}(x), T_{A_2}(y)) & \text{if } xy \in E'. \end{cases} \)

(v) \( I_B(xy) = \begin{cases} I_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ I_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \max(I_{B_1}(xy), I_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2, \\ \min(I_{A_1}(x), I_{A_2}(y)) & \text{if } xy \in E'. \end{cases} \)

(vi) \( F_B(xy) = \begin{cases} F_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ F_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ \min(F_{B_1}(xy), F_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2, \\ \max(F_{A_1}(x), F_{A_2}(y)) & \text{if } xy \in E'. \end{cases} \)

Theorem 2.13. The join of single-valued neutrosophic graphs is a single-valued neutrosophic graph.

Proposition 2.12. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (A_2, B_2) \), respectively, and \( V_1 \cap V_2 = \emptyset \). Then \( G = (A, B) \) is the join of \( G_1 \) and \( G_2 \) if and only if each \( \alpha \)-level graph \( G_\alpha \) is the join of \( (G_1)_\alpha \) and \( (G_2)_\alpha \).

Proof. Let \( G = (A, B) \) be the join of single-valued neutrosophic graphs \( G_1 \) and \( G_2 \). By the definition of union and the proof of Theorem 2.11, \( A_\alpha = (A_1)_\alpha \cup (A_2)_\alpha \), for each \( \alpha \in [0, 1] \). We show that \( B_\alpha = (B_1)_\alpha \cup (B_2)_\alpha \cup E'_\alpha \) for each \( \alpha \in [0, 1] \), where \( E'_\alpha \) is the set of all edges joining the vertices of \( (A_1)_\alpha \) and \( (A_2)_\alpha \).

From the proof of Theorem 2.11, it follows that \( (B_1)_\alpha \cup (B_2)_\alpha \subseteq B_\alpha \). If \( xy \in E'_\alpha \), then \( T_{A_1}(x) \geq \alpha, I_{A_1}(x) \geq \alpha \) or \( F_{A_1}(x) \leq 1 - \alpha \), and \( T_{A_2}(y) \geq \alpha, I_{A_2}(y) \geq \alpha \) or \( F_{A_2}(y) \leq 1 - \alpha \). Hence

\[ T_B(xy) = \min(T_{A_1}(x), T_{A_2}(y)) \geq \alpha, \]
\[ I_B(xy) = \min(I_{A_1}(x), I_{A_2}(y)) \geq \alpha, \]

or

\[ F_B(xy) = \max(F_{A_1}(x), F_{A_2}(y)) \leq 1 - \alpha. \]

It follows that \( xy \in B_\alpha \). Therefore, \( (B_1)_\alpha \cup (B_2)_\alpha \cup E'_\alpha \subseteq B_\alpha \). For every \( xy \in B_\alpha \), if \( xy \in E_1 \cup E_2 \), then \( xy \in (B_1)_\alpha \cup (B_2)_\alpha \), by the proof of Theorem 2.11. Therefore, \( B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha \). If \( x \in V_1 \) and \( y \in V_2 \), then

\[ \min(T_{A_1}(x), T_{A_2}(y)) = T_B(xy) \geq \alpha, \]
\[ \min(I_{A_1}(x), I_{A_2}(y)) = I_B(xy) \geq \alpha, \]

or

\[ \max(F_{A_1}(x), F_{A_2}(y)) = F_B(xy) \leq 1 - \alpha, \]

so \( x \in (A_1)_\alpha \) and \( y \in (A_2)_\alpha \). Thus \( xy \in E'_\alpha \). Therefore, \( B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha \cup E'_\alpha \). The converse part is obvious, hence we omit its proof.

\[ \square \]

Definition 2.9. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (A_2, B_2) \), respectively. The cross product \( G_1 \ast G_2 \) is defined as a pair \( (A, B) \) such that

(i) \( T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2)), \)
\( I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2)), \)
\( F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2)) \) for all \( (x_1, x_2) \in V_1 \times V_2, \)

(ii) \( T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)), \)
\( I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)), \)
\( F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) \) for all \( x_1y_1 \in E_1 \) and for all \( x_2y_2 \in E_2. \)
Example 2.14. Consider $G_1$ and $G_2$ are two single-valued neutrosophic graphs as shown in Figure 2.4 such that $A_1 = \{a(0.4, 0.6, 0.7), b(0.9, 0.3, 0.8)\}$, $A_2 = \{c(0.5, 0.7, 0.9), d(0.2, 0.9, 0.3), e(0.8, 0.7, 0.6)\}$, $B_1 = \{(ab, 0.3, 0.2, 0.7)\}$, and $B_2 = \{(cd, 0.1, 0.6, 0.8), (de), 0.1, 0.6, 0.5\}$. Then, we have cross product of $G_1$ and $G_2$, defined as $G_1 \ast G_2 = (A, B)$, where $A = A_1 \ast A_2$ and $B = B_1 \ast B_2$.

![Figure 2.4: (1) $G_1$ (2) $G_2$](image)

Table 1: $T_A(x_1, x_2)$, $I_A(x_1, x_2)$, $F_A(x_1, x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$

<table>
<thead>
<tr>
<th>$(x_1, x_2)$</th>
<th>$T_A(x_1, x_2)$</th>
<th>$I_A(x_1, x_2)$</th>
<th>$F_A(x_1, x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a, c)$</td>
<td>0.4</td>
<td>0.6</td>
<td>0.9</td>
</tr>
<tr>
<td>$(a, d)$</td>
<td>0.2</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$(a, e)$</td>
<td>0.4</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$(b, c)$</td>
<td>0.5</td>
<td>0.3</td>
<td>0.9</td>
</tr>
<tr>
<td>$(b, d)$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.8</td>
</tr>
<tr>
<td>$(b, e)$</td>
<td>0.8</td>
<td>0.3</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 2: $T_B((x_1, x_2)(y_1, y_2))$, $I_B((x_1, x_2)(y_1, y_2))$, $F_B((x_1, x_2)(y_1, y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$

<table>
<thead>
<tr>
<th>$(x_1, x_2)(y_1, y_2)$</th>
<th>$T_B((x_1, x_2)(y_1, y_2))$</th>
<th>$I_B((x_1, x_2)(y_1, y_2))$</th>
<th>$F_B((x_1, x_2)(y_1, y_2))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a, c)(b, d)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>$(a, c)(b, e)$</td>
<td>0</td>
<td>0</td>
<td>0.7</td>
</tr>
<tr>
<td>$(a, d)(b, c)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>$(a, d)(b, e)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.7</td>
</tr>
<tr>
<td>$(a, e)(b, c)$</td>
<td>0</td>
<td>0</td>
<td>0.7</td>
</tr>
<tr>
<td>$(a, e)(b, d)$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.7</td>
</tr>
</tbody>
</table>

According to Definition 2.9 the degrees of truth, indeterminacy and falsity memberships of vertices and edges are calculated as,

\[
T_A(a, c) = \min(T_{A_1}(a), T_{A_2}(c)) = \min(0.4, 0.5) = 0.4,
\]

\[
I_A(a, c) = \min(I_{A_1}(a), I_{A_2}(c)) = \min(0.6, 0.7) = 0.6,
\]

\[
F_A(a, c) = \max(F_{A_1}(a), F_{A_2}(c)) = \max(0.7, 0.9) = 0.9,
\]

and

\[
T_B((a, c)(b, d)) = \min(T_{B_1}(a, b), T_{B_2}(c, d)) = \min(0.3, 0.1) = 0.1,
\]
\[ I_B((a, c)(b, d)) = \min(I_{B_1}(a, b), I_{B_2}(c, d)) = \min(0.2, 0.6) = 0.2, \]
\[ T_B((a, c)(b, d)) = \max(F_{B_1}(a, b), F_{B_2}(c, d)) = \max(0.7, 0.8) = 0.8. \]

All the truth, indeterminacy and falsity membership degrees of vertices and edges of \( G_1 \ast G_2 \) are given in Table A and Table B, respectively. Thus, we have the following graph representing the cross product \( G_1 \ast G_2 \) of \( G_1 \) and \( G_2 \).

![Figure 2.5: \( G_1 \ast G_2 \)](image)

**Proposition 2.15.** The cross product of single-valued neutrosophic graphs is a single-valued neutrosophic graph.

**Theorem 2.16.** Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (A_2, B_2) \), respectively. Then \( G = (A, B) \) is the cross product of \( G_1 \) and \( G_2 \) if and only if each level graph \( G_\alpha \) is the cross product of \( (G_1)_\alpha \) and \( (G_2)_\alpha \).

**Proof.** Let \( G = (A, B) \) be the cross product of \( G_1 \) and \( G_2 \). By the definition of the Cartesian product and the proof of Theorem 2.17, we have \( A_\alpha = (A_1)_\alpha \times (A_2)_\alpha \), for each \( \alpha \in [0, 1] \). We show that

\[ B_\alpha = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha\} \]

for each \( \alpha \in [0, 1] \). In fact, if \( (x_1, x_2)(y_1, y_2) \in B_\alpha \), then

\[ T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) \geq \alpha, \]
\[ I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) \geq \alpha, \]

or

\[ F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) \leq 1 - \alpha, \]

so \( T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha \) or \( F_{B_1}(x_1y_1) \leq 1 - \alpha \) and \( T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha \) or \( F_{B_2}(x_2y_2) \leq 1 - \alpha \). So, \( x_1y_1 \in (B_1)_\alpha \) and \( x_2y_2 \in (B_2)_\alpha \). Therefore, \( B_\alpha \subseteq \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha\} \). Now if \( x_1y_1 \in (B_1)_\alpha \) and \( x_2y_2 \in (B_2)_\alpha \), then \( T_{B_1}(x_1y_1) \geq \alpha, I_{B_1}(x_1y_1) \geq \alpha \) or \( F_{B_1}(x_1y_1) \leq 1 - \alpha \) and \( T_{B_2}(x_2y_2) \geq \alpha, I_{B_2}(x_2y_2) \geq \alpha \) or \( F_{B_2}(x_2y_2) \leq 1 - \alpha \). It follows that

\[ T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1y_1), T_{B_2}(x_2y_2)) \geq \alpha, \]
\[ I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1y_1), I_{B_2}(x_2y_2)) \geq \alpha, \]

or

\[ F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1y_1), F_{B_2}(x_2y_2)) \leq 1 - \alpha. \]
Since $G = (A, B)$ is the cross product of $G_1 \times G_2$. Therefore, $(x_1, x_2)(y_1, y_2) \in B_\alpha$, this implies $\{(x_1, x_2)(y_1, y_2) \mid x_1 y_1 \in (B_1)_\alpha, x_2 y_2 \in (B_2)_\alpha\} \subseteq B_\alpha$.

Conversely, let each $\alpha$-level graph $G_\alpha = (A, B_\alpha)$ be the cross product of $(G_1)_\alpha = ((A_1)_\alpha, (B_1)_\alpha)$ and $(G_2)_\alpha = ((A_2)_\alpha, (B_2)_\alpha)$. In view of the fact that the cross product $(A_\alpha, B_\alpha)$ has the same vertex set as the Cartesian product of $((A_1)_\alpha, (B_1)_\alpha)$ and $((A_2)_\alpha, (B_2)_\alpha)$, and by the proof of Theorem 2.4, we have

$$T_A((x_1, x_2)) = \min(T_{A_1}(x_1), T_{A_2}(x_2)),$$

$$I_A((x_1, x_2)) = \min(I_{A_1}(x_1), I_{A_2}(x_2)),$$

$$F_A((x_1, x_2)) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$$

for all $(x_1, x_2) \in V_1 \times V_2$. Let

$$\min(T_{B_1}(x_1 y_1), T_{B_2}(x_2 y_2)) = \alpha,$$

$$\min(I_{B_1}(x_1 y_1), I_{B_2}(x_2 y_2)) = \alpha,$$

or

$$\max(F_{B_1}(x_1 y_1), F_{B_2}(x_2 y_2)) = 1 - \alpha,$$

and

$$T_B((x_1, x_2)(y_1, y_2)) = \beta,$$

$$I_B((x_1, x_2)(y_1, y_2)) = \beta,$$

or

$$F_B((x_1, x_2)(y_1, y_2)) = 1 - \beta$$

for $x_1 y_1 \in E_1$, $x_2 y_2 \in E_2$. Then $T_{B_1}(x_1 y_1) \geq \alpha, I_{B_1}(x_1 y_1) \geq \alpha$ or $F_{B_1}(x_1 y_1) \leq 1 - \alpha$ and $T_{B_2}(x_2 y_2) \geq \alpha, I_{B_2}(x_2 y_2) \geq \alpha$ or $F_{B_2}(x_2 y_2) \leq 1 - \alpha$, hence $x_1 y_1 \in (B_1)_\alpha$, $x_2 y_2 \in (B_2)_\alpha$, where $\alpha \in [0, 1]$ and $(x_1, x_2)(y_1, y_2) \in B_\beta$ where $\beta \in [0, 1]$ and consequently $x_1 y_1 \in (B_1)_\beta$, $x_2 y_2 \in (B_2)_\beta$, since $B_\beta = \{(x_1, x_2)(y_1, y_2) \mid x_1 y_1 \in (B_1)_\beta, x_2 y_2 \in (B_2)_\beta\}$. It follows that $(x_1, x_2)(y_1, y_2) \in B_\beta$, $T_{B_1}(x_1 y_1) \geq \beta, I_{B_1}(x_1 y_1) \geq \beta$ or $F_{B_1}(x_1 y_1) \leq 1 - \beta$ and $T_{B_2}(x_2 y_2) \geq \beta, I_{B_2}(x_2 y_2) \geq \beta$ or $F_{B_2}(x_2 y_2) \leq 1 - \beta$. Therefore,

$$T_B((x_1, x_2)(y_1, y_2)) = \beta \leq \min(T_{B_1}(x_1 y_1), T_{B_2}(x_2 y_2)) = \alpha \leq T_B((x_1, x_2)(y_1, y_2)),$$

$$I_B((x_1, x_2)(y_1, y_2)) = \beta \leq \min(I_{B_1}(x_1 y_1), I_{B_2}(x_2 y_2)) = \alpha \leq I_B((x_1, x_2)(y_1, y_2)),$$

$$F_B((x_1, x_2)(y_1, y_2)) = 1 - \beta \geq \max(F_{B_1}(x_1 y_1), F_{B_2}(x_2 y_2)) = 1 - \alpha \geq F_B((x_1, x_2)(y_1, y_2)).$$

Hence

$$T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1 y_1), T_{B_2}(x_2 y_2)),$$

$$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1 y_1), I_{B_2}(x_2 y_2)),$$

$$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1 y_1), F_{B_2}(x_2 y_2)).$$

This ends the proof.

**Definition 2.10.** Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be single-valued neutrosophic graphs. The lexicographic product $G_1 \bullet G_2$ is the pair $(A, B)$ of single-valued neutrosophic sets defined on the lexicographic product $G_1 \bullet G_2$ such that

(i) $T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2))$,

$I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2))$,

$F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$.

(ii) $T_B((x, x_2)(y, y_2)) = \min(T_{B_1}(x), T_{B_2}(x_2 y_2))$,

$I_B((x, x_2)(y, y_2)) = \min(I_{B_1}(x), I_{B_2}(x_2 y_2))$,

$F_B((x, x_2)(y, y_2)) = \max(F_{B_1}(x), F_{B_2}(x_2 y_2))$ for all $(x) \in V_1$ and for all $x_2 y_2 \in E_2$.

(iii) $T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1 y_1), T_{B_2}(x_2 y_2))$,

$I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1 y_1), I_{B_2}(x_2 y_2))$,

$F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1 y_1), F_{B_2}(x_2 y_2))$ for all $x_1 y_1 \in E_1$ and for all $x_2 y_2 \in E_2$. 
Example 2.17. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs with underlying crisp graphs \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \), respectively as shown in Figure 2.7. The lexicographic product \( G_1 \times G_2 = (A, B) \) of \( G_1 \) and \( G_2 \) is shown in Figure 2.7.

\[
\begin{align*}
\text{Figure 2.6: Single-valued neutrosophic graphs } G_1 \text{ and } G_2.
\end{align*}
\]

\[
\begin{align*}
\text{Figure 2.7: Lexicographic product of } G_1 \text{ and } G_2.
\end{align*}
\]

Proposition 2.18. The lexicographic product of single-valued neutrosophic graphs is a single-valued neutrosophic graph.

Theorem 2.19. Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (A_2, B_2) \), respectively. Then \( G = (A, B) \) is the lexicographic product of \( G_1 \) and \( G_2 \) if and only if \( G_\alpha = (G_1)_\alpha \bowtie (G_2)_\alpha \) for each \( \alpha \in [0, 1] \).

Proof. Let \( G = (A, B) = G_1 \bowtie G_2 \). By the definition of Cartesian product \( G_1 \times G_2 \) and the proof of Theorem 2.17, we have \( A_\alpha = (A_1)_\alpha \times (A_2)_\alpha \) for each \( \alpha \in [0, 1] \). We show that \( B_\alpha = E_\alpha \cup E_\alpha' \) for each \( \alpha \in [0, 1] \), where \( E_\alpha = \{(x, x_2)(y_2, y) \mid x \in V_1, x_2y_2 \in (B_2)_\alpha \} \) is the subset of the edge set of the Cartesian product \( (G_1)_\alpha \times (G_2)_\alpha \), and \( E_\alpha' = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in (B_1)_\alpha, x_2y_2 \in (B_2)_\alpha \} \) is the edge set of the cross product \( (G_1)_\alpha \bowtie (G_2)_\alpha \).

For every \( (x_1, x_2)(y_1, y_2) \in B_\alpha \), \( x_1 = y_1, x_2y_2 \in E_2 \) or \( x_1y_1 \in E_1, x_2y_2 \in E_2 \). If \( x_1 = y_1, x_2y_2 \in E_2 \), then \( (x_1, x_2)(y_1, y_2) \in E_\alpha \), by the definition of the Cartesian product and the proof of Theorem 2.17. If \( x_1y_1 \in E_1, x_2y_2 \in E_2 \), then \( (x_1, x_2)(y_1, y_2) \in E_\alpha' \), by the definition of cross product and the proof Theorem 2.17. Therefore, \( B_\alpha \subseteq E_\alpha \cup E_\alpha' \). From the definition of the Cartesian product and the proof of Theorem 2.17, we
conclude that \( E_\alpha \subseteq B_\alpha \), and also from the definition of cross product and the proof Theorem 2.16, we obtain \( E' \subseteq B_\alpha \). Therefore, \( E \cup E' \subseteq B_\alpha \).

Conversely, let \( G_\alpha = (A_\alpha, B_\alpha) = (G_1)_\alpha \bullet (G_2)_\alpha \) for each \( \alpha \in [0,1] \). We know that \((G_1)_\alpha \bullet (G_2)_\alpha\) has the same vertex set as the Cartesian product \((G_1)_\alpha \times (G_2)_\alpha\). Now by the proof of Theorem 2.16, we have

\[
T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2)),
\]
\[
I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2)),
\]
\[
F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2)).
\]

for all \((x_1, x_2) \in V_1 \times V_2\). Let for \( x \in V_1 \) and \( x_2 y_2 \in E_2 \) will be \( \min(T_{A_1}(x), T_{B_2}(x_2 y_2)) = \alpha, \min(I_{A_1}(x), I_{B_2}(x_2 y_2)) = 1 - \alpha \) and \( T_B((x_2)(x)(x_2)(y_2)) = \beta, I_B((x)(x_2)(x)(y_2)) = \beta \) or \( F_B((x)(x_2)(x)(y_2)) = 1 - \beta \). Then, in view of the definitions of the Cartesian product and lexicographic product, we have

\[
(x, x_2)(x, y_2) \in (B_1)_\alpha \bullet (B_2)_\alpha \iff (x, x_2)(x, y_2) \in (B_1)_\alpha \times (B_2)_\alpha,
\]
\[
(x, x_2)(x, y_2) \in (B_1)_\beta \bullet (B_2)_\beta \iff (x, x_2)(x, y_2) \in (B_1)_\beta \times (B_2)_\beta.
\]

From this, by the same way as in the proof of Theorem 2.16, we conclude

\[
T_B((x)(x_2)(x)(x_2)) = \min(T_{A_1}(x), T_{B_2}(x_2 y_2)),
\]
\[
I_B((x)(x_2)(x)(x_2)) = \min(I_{A_1}(x), I_{B_2}(x_2 y_2)),
\]
\[
F_B((x)(x_2)(x)(x_2)) = \max(F_{A_1}(x), F_{B_2}(x_2 y_2)).
\]

Now let \( T_B((x)(x_2)(y_1, y_2)) = \alpha, I_B((x)(x_2)(y_1, y_2)) = \alpha \) or \( F_B((x)(x_2)(y_1, y_2)) = 1 - \alpha \) and \( T_B((x)(x_2)(y_1, y_2)) = \beta, \min(I_{B_1}(x_1 y_1), I_{B_2}(x_2 y_2)) = \beta \) or \( \max(F_{B_1}(x_1 y_1), F_{B_2}(x_2 y_2)) = 1 - \beta \) for \( x_1 y_1 \in I_1 \) and \( x_2 y_2 \in E_2 \). Then in view of the definitions of cross product and the lexicographic product, we have

\[
(x, x_2)(y_1, y_2) \in (B_1)_\alpha \bullet (B_2)_\alpha \iff (x, x_2)(y_1, y_2) \in (B_1)_\alpha \bullet (B_2)_\alpha,
\]
\[
(x, x_2)(y_1, y_2) \in (B_1)_\beta \bullet (B_2)_\beta \iff (x, x_2)(y_1, y_2) \in (B_1)_\beta \bullet (B_2)_\beta.
\]

By the same way as in the proof of Theorem 2.16, we can conclude

\[
T_B((x)(x_2)(y_1, y_2)) = \min(T_{B_1}(x_1 y_1), T_{B_2}(x_2 y_2)),
\]
\[
I_B((x)(x_2)(y_1, y_2)) = \min(I_{B_1}(x_1 y_1), I_{B_2}(x_2 y_2)),
\]
\[
F_B((x)(x_2)(y_1, y_2)) = \max(F_{B_1}(x_1 y_1), F_{B_2}(x_2 y_2)).
\]

This ends the proof.

**Proposition 2.20.** Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), respectively, such that \( V_1 = V_2 \), \( A_1 = A_2 \) and \( E_1 \cap E_2 = \emptyset \). Then \( G = (A, B) \) is the union of \( G_1 \) and \( G_2 \) if and only if \( G_\alpha \) is the union of \((G_1)_\alpha\) and \((G_2)_\alpha\) for each \( \alpha \in [0,1] \).

**Proof.** Let \( G = (A, B) \) be the union of single-valued neutrosophic graphs \( G_1 \) and \( G_2 \). Then by the definition of the union and the fact that \( V_1 = V_2 \), \( A_1 = A_2 \), we have \( A = A_1 = A_2 \), hence \( A_\alpha = (A_1)_\alpha \cup (A_2)_\alpha \). We now show that \( B_\alpha = (B_1)_\alpha \cup (B_2)_\alpha \) for each \( \alpha \in [0,1] \). For every \( xy \in (B_1)_\alpha \) we have \( T_B(xy) = T_B(xy) \geq \alpha, I_B(xy) = I_B(xy) \geq \alpha \) or \( F_B(xy) = F_B(xy) \leq \alpha \), hence \( xy \in B_\alpha \). Therefore, \( (B_1)_\alpha \subseteq B_\alpha \). Similarly we obtain \( (B_2)_\alpha \subseteq B_\alpha \). Thus, \( (B_1)_\alpha \cup (B_2)_\alpha \subseteq B_\alpha \). For every \( xy \in B_\alpha \), \( xy \in E_1 \) or \( xy \in E_2 \). If \( xy \in E_1 \), \( T_B(xy) = T_B(xy) \geq \alpha, I_B(xy) = I_B(xy) \geq \alpha \) or \( F_B(xy) = F_B(xy) \leq \alpha \) and hence \( xy \in (B_1)_\alpha \). If \( xy \in E_2 \), we have \( xy \in (B_2)_\alpha \). Therefore, \( B_\alpha \subseteq (B_1)_\alpha \cup (B_2)_\alpha \).

Conversely, suppose that the \( \alpha \)-level graph \( G_\alpha = (A_\alpha, B_\alpha) \) be the union of \((G_1)_\alpha\) and \((G_2)_\alpha\). Let \( T_A(x) = \alpha, I_A(x) = \alpha \) or \( F_A(x) = 1 - \alpha \) and \( T_A(x) = \beta, I_A(x) = \beta \) or \( F_A(x) = 1 - \beta \) for some \( x \in V_1 = V_2 \). Then \( x \in (A_1)_\alpha \) where \( \alpha \in [0,1] \) and \( x \in (A_2)_\beta \) where \( \beta \in [0,1] \) so \( x \in (A_1)_\alpha \) and \( x \in (A_2)_\beta \), because \( A_\alpha = (A_1)_\alpha \) and \( A_\beta = (A_2)_\beta \). It follows that \( T_A(x) \geq \alpha, I_A(x) \geq \alpha \) or \( F_A(x) \leq 1 - \alpha \) and \( T_A(x) \geq \beta, I_A(x) \geq \beta \) or \( F_A(x) \leq 1 - \beta \). Therefore, \( T_A(x) \geq T_A(x), I_A(x) \geq I_A(x) \) or \( F_A(x) \leq F_A(x) \) and \( T_A(x) \geq T_A(x), I_A(x) \geq I_A(x) \) or \( F_A(x) \leq F_A(x) \). Since \( A_1 = A_2 \), \( V_1 = V_2 \), then \( A = A_1 = A_2 \).

By a similar method, we conclude that
This ends the proof.

**Definition 2.21.** Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \), respectively. The strong product \( G_1 \boxtimes G_2 \) is defined as a pair \( (A, B) \) such that

(i) \( T_A(x_1, x_2) = \min(T_{A_1}(x_1), T_{A_2}(x_2)) \),
\( I_A(x_1, x_2) = \min(I_{A_1}(x_1), I_{A_2}(x_2)) \),
\( F_A(x_1, x_2) = \max(F_{A_1}(x_1), F_{A_2}(x_2)) \) for all \( (x_1, x_2) \in V_1 \times V_2 \),

(ii) \( T_B((x, y_2)(x, y_2)) = \min(T_{B_1}(x, y_1), T_{B_2}(x, y_2)) \),
\( I_B((x, y_2)(x, y_2)) = \min(I_{B_1}(x, y_1), I_{B_2}(x, y_2)) \),
\( F_B((x, y_2)(x, y_2)) = \max(F_{B_1}(x, y_1), F_{B_2}(x, y_2)) \) for all \( x \in V_1 \) and for all \( y_2 y_2 \in E_2 \),

(iii) \( T_B((x_1, z)(y_1, z)) = \min(T_{B_1}(x_1 y_1), T_{B_2}(z)) \),
\( I_B((x_1, z)(y_1, z)) = \min(I_{B_1}(x_1 y_1), I_{B_2}(z)) \),
\( F_B((x_1, z)(y_1, z)) = \max(F_{B_1}(x_1 y_1), F_{B_2}(z)) \) for all \( z \in V_2 \) and for all \( x_1 y_1 \in E_1 \),

(iv) \( T_B((x_1, x_2)(y_1, y_2)) = \min(T_{B_1}(x_1, y_1), T_{B_2}(x_2 y_2)) \),
\( I_B((x_1, x_2)(y_1, y_2)) = \min(I_{B_1}(x_1, y_1), I_{B_2}(x_2 y_2)) \),
\( F_B((x_1, x_2)(y_1, y_2)) = \max(F_{B_1}(x_1, y_1), F_{B_2}(x_2 y_2)) \) for all \( x_1 y_1 \in E_1 \) and for all \( x_2 y_2 \in E_2 \).

**Example 2.21.** Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be two single-valued neutrosophic graphs of the crisp graphs \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \), respectively as shown in Fig. 2.8. The strong product \( G_1 \boxtimes G_2 = (A, B) \) of \( G_1 \) and \( G_2 \) is given in Fig. 2.8.

**Proposition 2.22.** The strong product single-valued neutrosophic graphs is a single-valued neutrosophic graph.

The following theorem is given by without proof.

**Theorem 2.23.** Let \( G_1 = (A_1, B_1) \) and \( G_2 = (A_2, B_2) \) be single-valued neutrosophic graphs of \( G_1^* = (V_1, E_1) \) and \( G_2^* = (V_2, E_2) \), respectively. Then \( G \) is the strong product of \( G_1 \) and \( G_2 \) if and only if \( G_\alpha \), where \( \alpha \in [0, 1] \), is the strong product of \( (G_1)_\alpha \) and \( (G_2)_\alpha \).
Strong product of SVNGs

Definition 2.12. The complement of a single-valued neutrosophic graph \( G = (A, B) \) is a single-valued neutrosophic graph \( \overline{G} = (\overline{A}, \overline{B}) \), where

1. \( \overline{V} = V \),

2. \( \overline{T}_A(v_i) = T_A(v_i), \overline{I}_A(v_i) = I_A(v_i), \overline{F}_A(v_i) = F_A(v_i) \), for all \( v_i \in V \),

3. \[
\overline{T}_B(v_i, v_j) = \begin{cases} 
\min[T_A(v_i), T_A(v_j)] & \text{if } T_B(v_i, v_j) = 0, \\
\min[T_A(v_i), T_A(v_j)] - T_B(v_i, v_j) & \text{if } T_B(v_i, v_j) > 0,
\end{cases}
\]

\[
\overline{I}_B(v_i, v_j) = \begin{cases} 
\min[I_A(v_i), I_A(v_j)] & \text{if } I_B(v_i, v_j) = 0, \\
\min[I_A(v_i), I_A(v_j)] - I_B(v_i, v_j) & \text{if } I_B(v_i, v_j) > 0,
\end{cases}
\]

\[
\overline{F}_B(v_i, v_j) = \begin{cases} 
\max[F_A(v_i), F_A(v_j)] & \text{if } F_B(v_i, v_j) = 0, \\
\max[F_A(v_i), F_A(v_j)] - F_B(v_i, v_j) & \text{if } F_B(v_i, v_j) > 0,
\end{cases}
\]

for all \( v_i, v_j \in V \).

Example 2.24. Consider a single-valued neutrosophic graph \( G = (A, B) \) on a non-empty set \( V = \{v_1, v_2, v_3, v_4\} \). Single-valued neutrosophic graph \( G = (A, B) \) and complement of single-valued neutrosophic graph \( \overline{G} = (\overline{A}, \overline{B}) \) are shown in Figure 2.10.

Figure 2.9: Strong product of SVNGs

Figure 2.10: Single-valued neutrosophic graph \( G \) and its complement \( \overline{G} \).
3 Remark on Definitions of Broumi et al. [10] and Shah-Hussain [16]

Broumi et al. [10] proposed single-valued neutrosophic graphs as follows.

**Definition 3.1.** [10] A single-valued neutrosophic graph is a pair $G = (A, B)$, where $A$ and $B$ are single-valued neutrosophic sets on $V$ and $E$, respectively, such that

\[
T_B(xy) \leq \min\{T_A(x), T_A(y)\},
\]
\[
I_B(xy) \geq \max\{I_A(x), I_A(y)\},
\]
\[
F_B(xy) \geq \max\{F_A(x), F_A(y)\},
\]
\[
0 \leq T_B(xy) + I_B(xy) + F_B(xy) \leq 3 \text{ for all } xy \in E.
\]

There are some flaws in Definition 3.1. Definition 3.1 violates the definitions of complement and join of single-valued neutrosophic graphs as it can be seen in the following examples.

**Example 3.1.** Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two single-valued neutrosophic graphs. When we apply above definition of join of single-valued neutrosophic graphs $G_1$ and $G_2$ then it is easy to note that the indeterminacy-membership values do not satisfy the condition, $I_B(v_i, v_j) \geq \max\{I_A(v_i), I_A(v_j)\}$ as it can be seen in Fig. [17]. This contradict the definition of single-valued neutrosophic graph.

![Figure 3.1: Join of a single-valued neutrosophic graph](image)

**Example 3.2.** Let $G = (A, B)$ be a single-valued neutrosophic graph. When we apply the above definition of complement of a single-valued neutrosophic graph then we see that $\overline{G}$ is not a single-valued neutrosophic
graph as it can be seen in Fig. 3.2. Since the indeterminacy-membership and the falsity-membership do not satisfy the conditions, \( I_B(v_i, v_j) \geq \max(I_A(v_i), I_A(v_j)) \) and \( F_B(v_i, v_j) \geq \max(F_A(v_i), F_A(v_j)) \), respectively. This contradict the definition of single-valued neutrosophic graph.

\[
\begin{align*}
&G = (V, E), \\
&\text{where } V = \{a, b, c, d\} \\
&\text{and } E = \left\{(a, b), (b, c), (c, d), (d, a)\right\}.
\end{align*}
\]

\[\text{Figure 3.2: Complement of a single-valued neutrosophic graph}\]

Shah and Hussain [15] defined single-valued neutrosophic graphs as follows.

**Definition 3.2.** [15] A neutrosophic graph is a pair \( G = (A, B) \), where \( A \) and \( B \) are neutrosophic sets in \( V \) and \( B \), respectively, such that

\[
\begin{align*}
&T_B(xy) \leq \min\{T_A(x), T_A(y)\}, \\
&I_B(xy) \leq \min\{I_A(x), I_A(y)\}, \\
&F_B(xy) \geq \max\{F_A(x), F_A(y)\}
\end{align*}
\]

for all \( xy \in V \times V \).

There are some flaws in Definition 3.4 as it can be seen in the following example.

**Example 3.3.** Consider a single-valued neutrosophic graph \( G \) on a nonempty set \( V = \{a, b, c, d\} \) as shown in the Fig. 3.3.

\[\text{Figure 3.3: A single-valued neutrosophic graph } G\]

\[\text{Figure 3.4: Complement of single-valued neutrosophic graph } G\]

\[\text{Figure 3.4: Complement of single-valued neutrosophic graph } G\]
It is easy to see that complement of the single-valued neutrosophic graph $G$ shown in Fig. 3.2 is not a single-valued neutrosophic graph. Because in single-valued neutrosophic graph falsity value can never be negative.

Thus, we conclude that our Definition 2.2 on single-valued neutrosophic graphs is more suitable for further study of neutrosophic graphs.

4 Application in Social Network

Graphical models have many applications in our daily life problems. Man is the most adjustable and adapting creature. When human beings interact with each other, more or less they leave an impact (good or bad) on each other. Naturally a human being has influence on others. We can use single-valued neutrosophic digraph to examine the influence of the people on each other’s thinking in a group. We can investigate a person’s good influence, bad influence on the thinking of others. We can also examine the percentage of uncertain influence of that person. Single-valued neutrosophic digraph will also tell us about dominating person and about highly influenced person. We consider a social group on whatsapp.

Consider $I = \{\text{Malik}, \text{Haider}, \text{Imran}, \text{Razi}, \text{Ali}, \text{Hamza}, \text{Aziz}\}$ set of seven persons in a social group on whatsapp.

Let $A = \{(\text{Malik}, 0.6, 0.4, 0.5), (\text{Haider}, 0.5, 0.6, 0.3), (\text{Imran}, 0.4, 0.3, 0.2), (\text{Razi}, 0.7, 0.6, 0.4), (\text{Ali}, 0.4, 0.1, 0.2), (\text{Hamza}, 0.6, 0.4, 0.1), (\text{Aziz}, 0.7, 0.3, 0.5)\}$ be the single-valued neutrosophic set on the set $I$, where, truth value of each person represents his good influence on others, falsity value represents his bad influence on others, and indeterminacy value represents uncertainty in his influence.

Let $J = \{(\text{Hamza, Malik}), (\text{Hamza, Haider}), (\text{Hamza, Razi}), (\text{Hamza, Aziz}), (\text{Malik, Haider}), (\text{Imran, Haider}), (\text{Aziz, Malik}), (\text{Razi, Imran}), (\text{Razi, Ali}), (\text{Ali, Aziz})\}$ be the set of edges. Let $B$ be the single-valued neutrosophic set on the set $J$ as shown in Table 3.

Table 3: Single-valued neutrosophic set $B$ of edges

<table>
<thead>
<tr>
<th>Edge</th>
<th>T</th>
<th>I</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Hamza, Malik)</td>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>(Hamza, Haider)</td>
<td>0.5</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>(Hamza, Razi)</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>(Hamza, Aziz)</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>(Malik, Haider)</td>
<td>0.5</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>(Imran, Haider)</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>(Aziz, Malik)</td>
<td>0.5</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>(Razi, Imran)</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>(Razi, Ali)</td>
<td>0.4</td>
<td>0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>(Ali, Aziz)</td>
<td>0.3</td>
<td>0.1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The truth, indeterminacy and falsity values of each edge are calculated using:

$$T_B(xy) \leq T_A(x) \land T_A(y), \quad I_B(xy) \leq I_A(x) \land I_A(y), \quad F_B(xy) \leq F_A(x) \lor F_A(y).$$

The single-valued neutrosophic digraph $G = (A, B)$ is shown in Fig. 4.1. This single-valued neutrosophic digraph shows that Hamza has influence on Malik, Haider, Razi and Aziz. We can see that Hamza’s good influence on Haider is 50%, on Malik is 60%, on Razi is 30%, and on Aziz is 30%. His bad influence on Haider, Malik, Razi and Aziz is 30%, 40%, 40%, and 40%, respectively. Similarly his uncertain influence on Haider, Malik, Razi and Aziz is 30%, 40%, 30%, and 30%, respectively. We can investigate that out-degree of vertex Hamza is highest, that is, four. This shows that Hamza is dominating person in this social group. On the other hand, Haider has highest in-degree, that is, three. It tells us that Haider is highly influenced by others in this social group.
We now explain general procedure of this applications through following algorithm.

**Step 1.** Input the set of vertices $I = \{I_1, I_2, \cdots, I_n\}$ and a single-valued neutrosophic set $A$ which is defined on set $I$.

**Step 2.** Input the set of edges $J = \{J_1, J_2, \cdots, J_n\}$.

**Step 3.** Compute the truth-membership degree, indeterminacy degree and falsity-membership degree of each edge using:

$$T_B(xy) \leq T_A(x) \land T_A(y), \quad I_B(xy) \leq I_A(x) \land I_A(y), \quad F_B(xy) \leq F_A(x) \lor F_A(y).$$

**Step 4.** Compute the single-valued neutrosophic set $B$ of edges.

**Step 5.** Obtain a single-valued neutrosophic diagraph $G = (A, B)$.

## 5 Conclusion

Graph theory is an extremely useful tool in studying and modeling several applications in different areas. A single-valued neutrosophic graph is a generalization of intuitionistic fuzzy graph that is very useful to solve real-life problems. In this research article, we have presented certain characterization of single-valued neutrosophic graphs by level graphs. We have aim to extend our work to (1) single-valued neutrosophic soft graphs, (2) single-valued neutrosophic rough fuzzy graphs, (3) single-valued neutrosophic rough fuzzy soft graphs, and (4) single-valued neutrosophic fuzzy soft graphs.

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### References


