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Arithmetic about Linear Combinations of GPIV Fuzzy Variables

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Abstract

Generalized Parametric Interval-Valued (GPIV) fuzzy variables is a natural extension of normalized Interval-Valued (IV) fuzzy variables, its arithmetic about linear combinations is an important research issue. This paper first defines GPIV normal fuzzy variables, Gamma fuzzy variables, Erland fuzzy variable and exponential fuzzy variables, then discusses the secondary possibility distributions of their linear combinations. The obtained results have potential applications in practical decision-making problems.

Keywords: interval-valued fuzzy variable, secondary possibility distribution, linear combination, decision making

1 Introduction

In 1975, Zadeh [27] introduced the concept of a type-2 (T2) fuzzy set as an extension of an ordinary fuzzy set. T2 fuzzy sets have been applied successfully to T2 fuzzy logic systems to handle linguistic and numerical uncertainties [20]. To further develop the theory of T2 fuzziness, Liu and Liu [16] proposed an axiomatic framework called fuzzy possibility theory. Since then, fuzzy possibility theory has been well-developed [2, 5, 17].

Based on T2 fuzzy theory, some interesting applications have been documented in the literature. For example, Bai and Liu [3] presented a new robust optimization method for supply chain network design problem by employing variable possibility distributions, while the variable possibility distributions are obtained by using the method of possibility critical value reduction to the secondary possibility distributions of uncertain demands and costs [1]. Kundu et al. [12, 13] employed T2 fuzzy variables to model fixed charge transportation problem and multi-item solid transportation problem. In order to deal with the Gaussian T2 fuzziness, Das et al. [6] developed two chance-constrained programming models based on generalized credibility measures for the objective function as well as the constraints sets with the help of the critical value reductions method [24]. Das et al. [7] derived reduction process for a trapezoidal T2 fuzzy number. In Pramanik et al. [23], the T2 fuzziness has been removed by using generalized credibility measure developed with the help of critical reduction method [24] and hence the models were reduced to chance constrained programming problems with different credibility labels. Zhou et al. [28] developed a multi-objective DEA model in a setting of T2 fuzzy modeling to evaluate and select the most appropriate sustainable suppliers. Mahapatra et al. [19] introduced a concept on solution technique for fuzzy variable based non-linear programming problem with both decision variables and restriction being fuzzy in nature, and applied the proposed procedure to complex system reliability model to evaluate the system reliability. Ma et al. [18] developed an integrated type 1 and T2 fuzzy sets chance-constrained programming model for tackling regional municipal solid waste management problem. Yang et al. [26] proposed a bi-objective hub-and-spoke network design problem with T2 fuzzy transportation cost and travel time described by parametric secondary possibility distributions, which are obtained using three types of mean value reduction methods [25].

The IV fuzzy variable is special case of general T2 fuzzy variable [16]. Liu and Liu [17] first studied a class of Normalized Parametric Interval-Valued (NPIV) fuzzy variables, and discussed the numerical

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characteristics for selection variables of NPIV fuzzy variables and their linear combinations. Guo et al. [10] extended the work of [17], and studied a class of GPIV fuzzy variables. In this paper, we further address this issue, and discussed the linear combinations of GPIV fuzzy variables.

The rest of paper is organized as follows. In Section 3, several important GPIV fuzzy variables are defined. In Section 4, the arithmetic about the linear combinations of GPIV fuzzy variables are discussed. Section 5 gives the conclusions of this paper and suggests future research areas.

2 Preliminaries

2.1 The IV Fuzzy Set

We next review some basic concepts in fuzzy set theory, including T2 fuzzy set, interval T2 (IT2) fuzzy set and interval-valued (IV) fuzzy set.

The concept of T2 fuzzy set was given by [27], and the following representation for a T2 fuzzy set is given by Mendel and John [21]:

Definition 1 ([21])**.** A T2 fuzzy set, denoted A, is characterized by a T2 membership function $\mu_A(x, u)$, for $x \in X$ and $u \in J_x \subseteq [0, 1]$, i.e.

$$A = \{((x, u), \mu_A(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\},$$

where $0 \leq \mu_A(x, u) \leq 1$.

The IT2 fuzzy set is a special T2 fuzzy set, it was introduced by Karnik et al. [11], and Mendel et al. [22] described the IT2 fuzzy set as follows:

Definition 2 ([22])**.** A T2 fuzzy set, denoted A, is characterized by a T2 membership function $\mu_A(x, u)$. If for $\forall x \in X, \forall u \in J_x \subseteq [0, 1]$, $\mu_A(x, u) = 1$, then A is an IT2 fuzzy set.

The IV fuzzy set is a particular case of IT2 fuzzy sets, it was introduced by Zadeh [27]. Let

$$L([0, 1]) = \{[\underline{x}, \overline{x}] \mid (\underline{x}, \overline{x}) \in [0, 1]^2 \text{ and } \underline{x} \leq \overline{x}\}.$$

Then Bustince et al. [4] used the following definition of an IV fuzzy set:

Definition 3 ([4])**.** An IV fuzzy set A on the universe $X \neq \emptyset$ is a mapping $A : X \to L([0, 1])$ such that the membership degree of $x \in X$ is given by $A(x) = [\underline{\mu}_A(x), \overline{\mu}_A(x)] \in L([0, 1])$, where $\underline{\mu}_A : X \to [0, 1]$ and $\overline{\mu}_A : X \to [0, 1]$ are mappings defining the lower and upper bounds of the membership interval $A(x)$, respectively.

2.2 The PIV Fuzzy Variable

We now review some basic concepts in fuzzy possibility theory [16], including fuzzy possibility measure, T2 fuzzy variable, secondary possibility distribution, IT2 fuzzy variable, and parametric interval-valued (PIV) fuzzy variable.

Let $\mathcal{P}(\Gamma)$ be the power set on the universe Γ, and $\tilde{\text{P}}\text{os} : \mathcal{P}(\Gamma) \mapsto \mathcal{R}([0, 1])$ a set function on $\mathcal{P}(\Gamma)$ such that $\{\tilde{\text{P}}\text{os}(A) \mid \mathcal{P}(\Gamma) \ni A \text{ atom}\}$ is a family of mutually independent regular fuzzy variables. We call $\tilde{\text{P}}\text{os}$ a fuzzy possibility measure if it satisfies the following conditions:
(i) $\tilde{\text{P}}\text{os}(\emptyset) = \tilde{0}$;
(ii) For any subclass $\{A_i \mid i \in I\}$ of $\mathcal{P}(\Gamma)$ (finite, countable or uncountable),

$$\tilde{\text{P}}\text{os}\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \tilde{\text{P}}\text{os}(A_i).$$

Moreover, if $\mu_{\tilde{\text{P}}\text{os}(\Gamma)}(1) = 1$, then we call $\tilde{\text{P}}\text{os}$ a regular fuzzy possibility measure. The triplet $(\Gamma, \mathcal{P}(\Gamma), \tilde{\text{P}}\text{os})$ is referred to as a fuzzy possibility space.

Definition 4 ([16])**.** Let $(\Gamma, \mathcal{P}(\Gamma), \tilde{\text{P}}\text{os})$ be a fuzzy possibility space. A map $\xi = (\xi_1, \xi_2, \cdots, \xi_n) : \Gamma \mapsto \Re^n$ is called a T2 fuzzy vector. As $n = 1$, the map $\xi : \Gamma \mapsto \Re$ is usually called a T2 fuzzy variable.

The secondary possibility distribution function $\tilde{\mu}_\xi(x)$ of the T2 fuzzy vector ξ is defined as

$$\tilde{\mu}_\xi(x) = \tilde{\text{P}}\text{os}\{\gamma \in \Gamma \mid \xi(\gamma) = x\}, x \in \Re^n, \tag{1}$$

and the T2 possibility distribution function $\mu_\xi(x, u)$ of ξ is defined as

$$\mu_\xi(x, u) = \text{Pos}\{\tilde{\mu}_\xi(x) = u\}, (x, u) \in \Re^n \times J_x, \tag{2}$$

where $J_x \subset [0, 1]$ is the support of $\tilde{\mu}_\xi(x)$.

An IT2 fuzzy variable is a special case of T2 fuzzy variables, it is defined by Liu and Liu [17] as follows:

Definition 5 ([17])**.** Assume that ξ is a T2 fuzzy variable with T2 possibility distribution function $\mu_\xi(x, u)$. If for any $x \in \Re, u \in J_x \subseteq [0, 1]$, $\mu_\xi(x, u) = 1$, then ξ is called an IT2 fuzzy variable.

If the secondary possibility distribution function $\tilde{\mu}_\xi(x)$ is a subinterval of $[0, 1]$, then Liu and Liu [17] defined a PIV fuzzy variable as follows.

Definition 6 ([17])**.** Assume that ξ is an IT2 fuzzy variable with the secondary possibility distribution function $\tilde{\mu}_\xi(x)$. If for any $x \in \Re$, $\tilde{\mu}_\xi(x)$ is a subinterval $[\mu_{\xi^L}(x; \theta_l), \mu_{\xi^U}(x; \theta_r)]$ of $[0, 1]$ with parameters $\theta_l, \theta_r \in [0, 1]$, then ξ is called a PIV fuzzy variable.

3 Common GPIV Fuzzy Variables

The GPIV fuzzy variable is first proposed by Guo et al. [10], and is employed to characterize uncertain demand in a three level supply chain problem. However, the work [10] just considers generalized PIV trapezoidal fuzzy variable. After recalling this concept in the following, we introduce several other important GPIV fuzzy variables. Some concepts used but not provided in this paper, the interested reader may refer to [14, 16] and the references therein.

Let $r_1 < r_2 \le r_3 < r_4$ be real numbers. An IT2 fuzzy variable $\tilde{\xi}$ is called a GPIV trapezoidal fuzzy variable, denoted as $\tilde{\xi} \sim \tilde{\text{T}}\text{ra}(r_1, r_2, r_3, \tilde{r}_4; \theta_l, \theta_r)$, if its secondary possibility distribution is the following subinterval of $[0, 1]$,

$$\left[\frac{r - r_1}{r_2 - r_1} - \theta_l \frac{r - r_1}{r_2 - r_1}, \frac{r - r_1}{r_2 - r_1} + \theta_r \frac{r_2 - r}{r_2 - r_1}\right], r \in [r_1, r_2],$$

the subinterval $[1 - \theta_l, 1]$ of $[0, 1]$ for $r \in [r_2, r_3]$, and the following subinterval of $[0, 1]$,

$$\left[\frac{r_4 - r}{r_4 - r_3} - \theta_l \frac{r_4 - r}{r_4 - r_3}, \frac{r_4 - r}{r_4 - r_3} + \theta_r \frac{r - r_3}{r_4 - r_3}\right], r \in [r_3, r_4],$$

where $\theta_l, \theta_r \in [0, 1]$ are two distribution parameters characterizing the degree of uncertainty that ξ takes on the value r.

It is evident that the possibility of event $\{\tilde{\xi} = r\}$ is an interval with variable boundaries characterized by parameters θ_l and θ_r. When $\theta_l = \theta_r = 0$, the corresponding secondary possibility distribution is called the nominal possibility distribution of $\tilde{\xi}$.

Particularly, when $r_2 = r_3$, we call an IT2 fuzzy variable ξ as a GPIV triangular fuzzy variable, and denoted as $\tilde{\xi} \sim \tilde{\text{T}}\text{ri}(r_1, r_2, r_3; \theta_l, \theta_r)$.

Definition 7. An IT2 fuzzy variable $\tilde{\eta}$ is called a GPIV normal fuzzy variable, denoted as $\tilde{\eta} \sim \tilde{\text{N}}\text{or}(a, \sigma^2)$, if its secondary possibility distribution is the following subinterval of $[0, 1]$,

$$[\mu(t) - \theta_l \mu(t), \mu(t) + \theta_r(1 - \mu(t))],$$

where $\theta_l, \theta_r \in [0, 1]$ and $\mu(t) = \exp\{-\frac{1}{2}(\frac{t-a}{\sigma})^2\}, t \in \Re$, where the parameters $a \in \Re$ and $\sigma > 0$. When $\theta_l = \theta_r = 0$, the corresponding secondary possibility distribution is called the nominal possibility distribution of $\tilde{\eta}$.

Definition 8. An IT2 fuzzy variable $\tilde{\zeta}$ is called a GPIV Gamma fuzzy variable, denoted as $\tilde{\zeta} \sim \widetilde{\mathrm{Gam}}(r, \lambda; \theta_l, \theta_r)$, if its secondary possibility distribution is the following subinterval of $[0, 1]$,

$$[\mu(t) - \theta_l \mu(t), \mu(t) + \theta_r(1 - \mu(t))],$$

where $\theta_l, \theta_r \in [0, 1]$, and $\mu(t) = (\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\}, t \geq 0$, where the parameters $r > 0$ and $\lambda > 0$. When $\theta_l = \theta_r = 0$, the corresponding secondary possibility distribution is called the nominal possibility distribution of $\tilde{\zeta}$.

Definition 9. An IT2 fuzzy variable $\tilde{\zeta}$ is said to be a GPIV Erlang fuzzy variable, denoted as $\tilde{\zeta} \sim \widetilde{\mathrm{Erl}}(\rho, \kappa; \theta_l, \theta_r)$, if its secondary possibility distribution is the following subinterval of $[0, 1]$,

$$[\mu(t) - \theta_l \mu(t), \mu(t) + \theta_r(1 - \mu(t))],$$

where $\theta_l, \theta_r \in [0, 1]$, and $\mu(t) = (\frac{t}{\kappa\rho})^\kappa \exp\{\kappa - \frac{t}{\rho}\}$, where $\rho > 0$ and κ is a positive integer. When $\theta_l = \theta_r = 0$, the corresponding secondary possibility distribution is called the nominal possibility distribution of $\tilde{\zeta}$.

Particularly, when $\kappa = 1$, $\tilde{\zeta}$ is called a GPIV exponential fuzzy variable, and denoted as $\tilde{\zeta} \sim \widetilde{\mathrm{Exp}}(\rho; \theta_l, \theta_r)$.

4 Linear Combinations of GPIV Fuzzy Variables

First, the following theorem deals with the linear combination of GPIV trapezoidal fuzzy variables:

Theorem 1. *Let $\tilde{\xi}_i \sim \widetilde{\mathrm{Tra}}(r_{i1}, r_{i2}, r_{i3}, r_{i4}; \theta_{il}, \theta_{ir})$ be GPIV trapezoidal fuzzy variables for $i \leq n$. Suppose the nominal possibility distributions $\mathrm{Tra}(r_{i1}, r_{i2}, r_{i3}, r_{i4})$'s are mutually independent, and x_i's are real numbers. Then one has*

$$\tilde{\xi} = \sum_{i=1}^n x_i \tilde{\xi}_i \sim \widetilde{\mathrm{Tra}}(r_1(x), r_2(x), r_3(x), r_4(x); \theta_l, \theta_r),$$

where the parameters $\theta_l = \max_{1 \leq i \leq n} \theta_{il}$, $\theta_r = \min_{1 \leq i \leq n} \theta_{ir}$, and

$$r_1(x) = \sum_{i=1}^n \left(x_i^+ r_{i1} - x_i^- r_{i4}\right), \quad r_2(x) = \sum_{i=1}^n \left(x_i^+ r_{i2} - x_i^- r_{i3}\right),$$

$$r_3(x) = \sum_{i=1}^n \left(x_i^+ r_{i3} - x_i^- r_{i2}\right), \quad r_4(x) = \sum_{i=1}^n \left(x_i^+ r_{i4} - x_i^- r_{i1}\right)$$

with $x_i^+ = \max\{x_i, 0\}$, and $x_i^- = \max\{-x_i, 0\}$.

Proof. Since the nominal possibility distributions $\mathrm{Tra}(r_{i1}, r_{i2}, r_{i3}, r_{i4})$'s are mutually independent (see [15]), for any nonzero real numbers x_i's, the linear combination $\sum_{i=1}^n x_i \tilde{\xi}_i$ has the following nominal possibility distribution $\mathrm{Tra}(r_1(x), r_2(x), r_3(x), r_4(x))$.

Furthermore, for any $z \in [r_1(x), r_2(x)]$, there exist real numbers z_i's such that $z = \sum_{i=1}^n x_i z_i$ and

$$\widetilde{\mathrm{Pos}}\{\tilde{\xi} = z\} = \min_{i=1}^n \widetilde{\mathrm{Pos}}_i\{\tilde{\xi}_i = z_i\}.$$

Since the secondary distribution function of $\tilde{\xi}_i$ is

$$\widetilde{\mathrm{Pos}}_i\{\tilde{\xi}_i = z_i\} = \left[\frac{z - r_1(x)}{r_2(x) - r_1(x)} - \theta_{il}\frac{z - r_1(x)}{r_2(x) - r_1(x)}, \frac{z - r_1(x)}{r_2(x) - r_1(x)} + \theta_{ir}\frac{r_2(x) - z}{r_2(x) - r_1(x)}\right]$$

for $i = 1, 2, \ldots, n$, by the logic arithmetic of interval numbers on unit interval [0,1], one has

$$\widetilde{\mathrm{Pos}}\{\tilde{\xi} = z\} = \left[\frac{z - r_1(x)}{r_2(x) - r_1(x)} - \theta_l\frac{z - r_1(x)}{r_2(x) - r_1(x)}, \frac{z - r_1(x)}{r_2(x) - r_1(x)} + \theta_r\frac{r_2(x) - z}{r_2(x) - r_1(x)}\right],$$

where $\theta_l = \max_{1 \leq i \leq n} \theta_{il}$ and $\theta_r = \min_{1 \leq i \leq n} \theta_{ir}$. The cases $z \in [r_2(x), r_3(x)]$ and $z \in [r_3(x), r_4(x)]$ can be proved similarly, which completes the proof of theorem. $\square$

As a consequence of Theorem 1, one has the following result:

Corollary 1. *Let $\tilde{\xi}_i \sim \tilde{\mathrm{Tri}}(r_{i1}, r_{i2}, r_{i3}; \theta_{il}, \theta_{ir})$ be GPIV trapezoidal fuzzy variables for $i \leq n$. Suppose the nominal possibility distributions $\mathrm{Tri}(r_{i1}, r_{i2}, r_{i3})$'s are mutually independent, and x_i's are real numbers. Then one has $\tilde{\xi} = \sum_{i=1}^n x_i \tilde{\xi}_i \sim \tilde{\mathrm{Tri}}(r_1(x), r_2(x), r_3(x); \theta_l, \theta_r)$, where the parameters $\theta_l = \max_{1\leq i\leq n} \theta_{il}$, $\theta_r = \min_{1\leq i\leq n} \theta_{ir}$, and*

$$r_1(x) = \sum_{i=1}^n \left(x_i^+ r_{i1} - x_i^- r_{i3}\right), \quad r_2(x) = \sum_{i=1}^n x_i r_{i2}, \quad r_3(x) = \sum_{i=1}^n \left(x_i^+ r_{i3} - x_i^- r_{i1}\right)$$

with $x_i^+ = \max\{x_i, 0\}$, and $x_i^- = \max\{-x_i, 0\}$.

The following theorem discusses the linear combination of GPIV normal fuzzy variables:

Theorem 2. *Let $\tilde{\eta}_i \sim \tilde{\mathrm{Nor}}(\mu_i, \sigma_i^2; \theta_{il}, \theta_{ir})$ be GPIV normal fuzzy variables for $i \leq n$. Suppose the nominal possibility distributions $\mathrm{Nor}(\mu_i, \sigma_i^2)$'s are mutually independent, and x_i's are real numbers. Then one has $\tilde{\eta} = \sum_{i=1}^n x_i \tilde{\eta}_i \sim \tilde{\mathrm{Nor}}(a(x), \sigma^2(x); \theta_l, \theta_r)$ with the parameters $a(x) = \sum_{i=1}^n x_i \mu_i$, $\sigma(x) = \sum_{i=1}^n x_i \sigma_i$, $\theta_l = \max_{1\leq i\leq n} \theta_{il}$ and $\theta_r = \min_{1\leq i\leq n} \theta_{ir}$.*

Proof. Since the nominal possibility distributions $\mathrm{Nor}(\mu_i, \sigma_i^2)$'s are mutually independent (see [15]), for any real numbers x_i's, the linear combination $\tilde{\eta} = \sum_{i=1}^n x_i \tilde{\eta}_i$ has norminal possibility distribution $\mathrm{Nor}(a(x), \sigma^2(x))$ with parameters $a(x) = \sum_{i=1}^n x_i \mu_i$, and $\sigma(x) = \sum_{i=1}^n x_i \sigma_i$.

Furthermore, for any $z \in \Re$, there exist real numbers z_i's such that $z = \sum_{i=1}^n x_i z_i$ and

$$\tilde{\mathrm{Pos}}\{\tilde{\eta} = z\} = \min_{i=1}^n \tilde{\mathrm{Pos}}_i\{\tilde{\eta}_i = z_i\}.$$

Since the secondary possibility distribution $\tilde{\mathrm{Pos}}_i\{\tilde{\eta}_i = z_i\}$ is the following subinterval of $[0, 1]$,

$$\left[\exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\} - \theta_{il} \exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\}, \exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\} + \theta_{ir}(1 - \exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\})\right]$$

for $i = 1, 2, \ldots, n$, by the logic arithmetic of interval numbers on unit interval [0,1], the secondary possibility distribution $\tilde{\mathrm{Pos}}\{\tilde{\eta} = z\}$ is the following subinterval of $[0, 1]$,

$$\left[\exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\} - \theta_l \exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\}, \exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\} + \theta_r(1 - \exp\{-\frac{1}{2}(\frac{z-a(x)}{\sigma(x)})^2\})\right]$$

where $\theta_l = \max_{1\leq i\leq n} \theta_{il}$ and $\theta_r = \min_{1\leq i\leq n} \theta_{ir}$. The proof of theorem is complete. □

Finally, the following theorem deals with the positive linear combination of generalized PIV Gamma fuzzy variables:

Theorem 3. *Let $\tilde{\zeta}_i \sim \tilde{\mathrm{Gam}}(r, \lambda_i; \theta_{il}, \theta_{ir})$ be GPIV Gamma fuzzy variables for $i \leq n$. Suppose the nominal possibility distributions $\mathrm{Gam}(r, \lambda_i)$'s are mutually independent, and x_i's are positive real numbers. Then $\tilde{\zeta} = \sum_{i=1}^n x_i \tilde{\zeta}_i \sim \tilde{\mathrm{Gam}}(r, \lambda(x); \theta_l, \theta_r)$ with the parameters $\lambda(x) = \sum_{i=1}^n x_i \lambda_i$, $\theta_l = \max_{1\leq i\leq n} \theta_{il}$ and $\theta_r = \min_{1\leq i\leq n} \theta_{ir}$.*

Proof. Since the nominal possibility distributions $\mathrm{Gam}(r, \lambda_i)$'s are mutually independent in the sense of [15], for any nonzero real numbers x_i's, the linear combination $\sum_{i=1}^n x_i \tilde{\zeta}_i$ has nominal possibility distribution $\mathrm{Gam}(r, \lambda(x))$ with parameter $\lambda(x) = \sum_{i=1}^n x_i \lambda_i$. Furthermore, for any $z \in \Re$, there exist real numbers z_i's such that $z = \sum_{i=1}^n z_i$ and $\tilde{\mathrm{Pos}}\{\tilde{\zeta} = z\} = \min_{i=1}^n \tilde{\mathrm{Pos}}_i\{\tilde{\zeta}_i = z_i\}$.

Since the secondary possibility distribution $\tilde{\mathrm{Pos}}_i\{\tilde{\zeta}_i = z_i\}$ is the following subinterval of $[0, 1]$,

$$\left[(\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\} - \theta_{il}(\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\}, (\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\} + \theta_{ir}(1 - (\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\})\right]$$

for $i = 1, 2, \ldots, n$, by the logic arithmetic of interval numbers on unit interval [0,1], the secondary possibility distribution $\tilde{\mathrm{Pos}}\{\tilde{\zeta} = z\}$ is the following subinterval of $[0, 1]$,

$$\left[(\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\} - \theta_l(\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\}, (\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\} + \theta_r(1 - (\frac{t}{\lambda r})^r \exp\{r - \frac{t}{\lambda}\})\right],$$

where $\theta_l = \max_{1\leq i\leq n} \theta_{il}$ and $\theta_r = \min_{1\leq i\leq n} \theta_{ir}$. The proof of theorem is complete. □

As immediate consequence of Theorem 3, we have the following results about GPIV Erlang and exponential fuzzy variables:

Corollary 2. *Let* $\tilde{\zeta}_i \sim \tilde{\mathrm{E}}\mathrm{rl}(\rho_i, \kappa; \theta_{il}, \theta_{ir})$ *be GPIV Erlang fuzzy variables for* $i \leq n$*. Suppose the Erlang possibility distributions* $\mathrm{Erl}(\rho_i, \kappa)$*'s are mutually independent, and* x_i*'s are posotive real numbers. Then* $\tilde{\zeta} = \sum_{i=1}^{n} x_i\tilde{\zeta}_i \sim \tilde{\mathrm{E}}\mathrm{rl}(\rho(x), \kappa; \theta_l, \theta_r)$ *with the parameters* $\rho(x) = \sum_{i=1}^{n} x_i\rho_i$*,* $\theta_l = \max_{1\leq i\leq n}\theta_{il}$ *and* $\theta_r = \min_{1\leq i\leq n}\theta_{ir}$*.*

Corollary 3. *Let* $\tilde{\zeta}_i \sim \tilde{\mathrm{E}}\mathrm{xp}(\rho_i; \theta_{il}, \theta_{ir})$ *be GPIV expotential fuzzy variables for* $i \leq n$*. Suppose the expotential possibility distributions* $\mathrm{Exp}(\rho_i)$*'s are mutually independent, and* x_i*'s are positive real numbers. Then* $\tilde{\zeta} = \sum_{i=1}^{n} x_i\tilde{\zeta}_i \sim \tilde{\mathrm{E}}\mathrm{xp}(\rho(x); \theta_l, \theta_r)$ *with the parameters* $\rho(x) = \sum_{i=1}^{n} x_i\rho_i$*,* $\theta_l = \max_{1\leq i\leq n}\theta_{il}$ *and* $\theta_r = \min_{1\leq i\leq n}\theta_{ir}$*.*

5 Conclusions and Future Research

This paper addressed the arithmetic of GPIV fuzzy variables in fuzzy possibility theory, and obtained the following major results:

Firstly, several new GPIV fuzzy variables were defined, including GPIV normal, Gamma, Erland and exponential fuzzy variables.

Secondly, the arithmetic about the linear combinations of common GPIV fuzzy variables were studied and several useful theoretical results were obtained (see Theorems 1–3 and Corollaries 1–3).

Along this direction, the credibility distribution functions for selections of GPIV fuzzy variables [8], and the credibilistic comonotonicity for GPIV fuzzy vector [9] are important issues in our future research.

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