Journal of Uncertain Systems
Vol.11, No.1, pp.49-67, 2017
Online at: www.jus.org.uk

Novel Properties of Intuitionistic Fuzzy Competition Graphs

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Received 21 September 2016; Revised 1 November 2016

Abstract

We investigate some new properties of intuitionistic fuzzy competition graphs. We present the construction of intuitionistic fuzzy competition graph. We study new type of intuitionistic fuzzy graphs by considering intuitionistic fuzzy open neighbourhood and intuitionistic fuzzy closed neighbourhood of the vertices. We also present an application of intuitionistic fuzzy competition graphs.

Keywords: intuitionistic fuzzy competition graph, intuitionistic fuzzy closed neighbouhood graphs, underlying intuitionistic fuzzy graph

1 Introduction

The intimation of competition graphs was first popularized by Cohen [11], in 1968, in association with a problem in ecology. Suppose a digraph $\overrightarrow{\mathcal{G}} = (X, \overrightarrow{E})$, which harmonizes to a food cycle. A vertex $x \in X$ indicates a specie in the food cycle and an arc $\overrightarrow{(x, a)} \in \overrightarrow{E}$ shows that x feeds on the species a. If two species x and y have a common feed a, they will strive for the feed a. Depending on this homology, Cohen stated a graph which shows the relations of competition among the species in the food cycle. The competition graph $C(\overrightarrow{\mathcal{G}})$ of a digraph $\overrightarrow{\mathcal{G}} = (X, \overrightarrow{E})$ is an undirected graph which has the same vertex set X as in $\overrightarrow{\mathcal{G}}$ and has an edge between two distinct vertices $x, y \in X$ if there exists a vertex $a \in X$ and arcs $\overrightarrow{(x, a)}, \overrightarrow{(y, a)} \in \overrightarrow{E}$, that is, if x and y have a common neighbour, then there is an edge between x and y.

Nowadays, science and technology are featured with complex processes and phenomena for which complete information is not always reachable. For such cases, to handle types of systems containing elements of uncertainty, mathematical models are developed. A variety of these models is based on fuzzy sets, which is an extension of the ordinary set theory. A fuzzy set gives the degree of membership of an object in a given set. Atanassov [7] generalized this idea and popularized the notion of intuitionistic fuzzy sets. He introduced a new component, degree of non-membership, in the definition of a fuzzy set with the condition that sum of two degrees must be less than or equal to one. In modeling real time systems where the level of information inherent in the system varies with different levels of precision, fuzzy models are finding an increasing number of applications. Fuzzy models are becoming fruitful because of their aim in reducing the distinctions between the conventional numerical models used in sciences and engineering and the symbolic models used in expert systems. Initial definition of a fuzzy graph [14], given by Kaufmann, was based on Zadeh's fuzzy relations [26]. The fuzzy relations between fuzzy sets were also investigated by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of certain graph theoretical concepts. Later on, Koczy [15] introduced the concepts of fuzzy edge graphs, fuzzy vertex graphs and fuzzy graphs to present some networks models. Mordeson and Nair [16] studied several concepts of fuzzy graphs. The concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs were introduced by Shannon and Atanassov [24]. Operations on intuitionistic fuzzy graphs were defined by Parvathi et al. [18]. Karunambigai et al. used intuitionistic fuzzy graphs to find shortest paths in networks [13] and discussed self-centered intuitionistic fuzzy graphs [12]. Akram et al. [1-4] studied many new concepts, including strong intuitionistic fuzzy graphs, intuitionistic fuzzy hypergraphs, intuitionistic soft graphs and intuitionistic fuzzy digraphs in decision support systems. Fuzzy k-competition and p-competition graphs were introduced by Samanta and Pal [22]. Samanta et al. [21] introduced m-step fuzzy competition graphs. On the other hand, the concepts of bipolar fuzzy competition graphs and

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intuitionistic fuzzy competition graphs are discussed in [23, 20]. In this research article, we present some new properties of intuitionistic fuzzy competition graphs. We investigate the construction of intuitionistic fuzzy competition graph. We also discuss new type of intuitionistic fuzzy graphs by considering intuitionistic fuzzy open and closed neighbourhood of the vertices.

2 Intuitionistic Fuzzy Competition Graphs

Definition 2.1. [20] An intuitionistic fuzzy out neighbourhood(IFON) of a vertex x in an intuitionistic fuzzy digraph(IFD) $\overrightarrow{\mathcal{G}} = (\mathcal{A}, \overrightarrow{\mathcal{B}})$ is an intuitionistic fuzzy set(IFS) $\mathfrak{N}^+(x) = (X_x^+, \phi_{\mathcal{A}_x}, \psi_{\mathcal{A}_x})$, where $X_x^+ = \{y \in X | \phi_{\overrightarrow{\mathcal{B}}}(xy) > 0, \psi_{\overrightarrow{\mathcal{B}}}(xy) > 0\}$ and $\phi_{\mathcal{A}_x} : X_x^+ \to [0, 1]$ and $\psi_{\mathcal{A}_x} : X_x^+ \to [0, 1]$ are defined by $\phi_{\mathcal{A}_x}(y) = \phi_{\overrightarrow{\mathcal{B}}}(xy)$ and $\psi_{\mathcal{A}_x}(y) = \psi_{\overrightarrow{\mathcal{B}}}(xy)$.

An intuitionistic fuzzy in neighbourhood(IFIN) of a vertex x in an IFD $\overrightarrow{\mathcal{G}} = (\mathcal{A}, \overrightarrow{\mathcal{B}})$ is an IFS $\mathfrak{N}^-(x) = (X_x^-, \phi_{\mathcal{A}_x}, \psi_{\mathcal{A}_x})$, where $X_x^- = \{y \in X | \phi_{\overrightarrow{\mathcal{B}}}(xy) > 0, \psi_{\overrightarrow{\mathcal{B}}}(xy) > 0\}$ and $\phi_{\mathcal{A}_x} : X_x^- \to [0, 1]$ and $\psi_{\mathcal{A}_x} : X_x^- \to [0, 1]$ are defined by $\phi_{\mathcal{A}_x}(y) = \phi_{\overrightarrow{\mathcal{B}}}(xy)$ and $\psi_{\mathcal{A}_x}(y) = \psi_{\overrightarrow{\mathcal{B}}}(xy)$.

Definition 2.2. [20] An intuitionistic fuzzy competition graph(IFCG) $\mathfrak{C}(\overrightarrow{\mathcal{G}})$ of an IFD $\overrightarrow{\mathcal{G}} = (X\ \phi,\ \psi)$ is an undirected IFG $\mathcal{G} = (X, \phi, \psi)$ which has the same intuitionistic fuzzy vertex set as in $\overrightarrow{\mathcal{G}}$ and has an intuitionistic fuzzy edge between two vertices $x, y \in X$ in $\mathfrak{C}(\overrightarrow{\mathcal{G}})$ if and only if $\mathfrak{N}^+(x) \cap \mathfrak{N}^+(y)$ is a non-empty IFS in $\overrightarrow{\mathcal{G}}$. The membership and non-membership values of the edge (x, y) in $\mathfrak{C}(\overrightarrow{\mathcal{G}})$ are $\psi_1(x, y) = (\phi_1(x) \wedge \phi_1(y))h_1(\mathfrak{N}^+(x) \cap \mathfrak{N}^+(y))$ and $\psi_2(x, y) = (\phi_2(x) \vee \phi_2(y))h_2(\mathfrak{N}^+(x) \cap \mathfrak{N}^+(y))$, respectively.

Example 2.1. Consider $\overrightarrow{\mathcal{G}} = (X, \phi, \psi)$ be an IFD. Let $\{a, b, c, d, e\}$ be the vertex set with $\phi(a) = (0.5, 0.4)$, $\phi(b) = (0.4, 0.4)$, $\phi(c) = (0.4, 0.5)$, $\phi(d) = (0.5, 0.3)$, $\phi(e) = (0.5, 0.3)$, and $\psi\overrightarrow{(a, b)} = (0.3, 0.4)$, $\psi\overrightarrow{(a, b)} = (0.3, 0.4)$, $\psi\overrightarrow{(d, e)} = (0.5, 0.1)$, $\psi\overrightarrow{(a, d)} = (0.4, 0.2)$, $\psi\overrightarrow{(d, c)} = (0.4, 0.3)$, $\psi\overrightarrow{(c, e)} = (0.3, 0.5)$, and $\psi\overrightarrow{(b, c)} = (0.4, 0.4)$ as shown in Fig. 1.

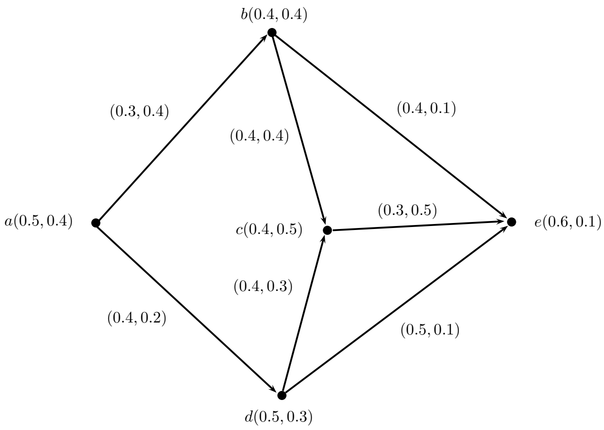

Figure 1: IFD

By direct calculations, we have

Table 1: Intuitionistic fuzzy out and in neighborhoods

x	$\mathfrak{N}^+(x)$	x	$\mathfrak{N}^+(x)$
a	b(0.3, 0.4), d(0.4, 0.2)	d	c(0.4, 0.3), e(0.5, 0.1)
b	c(0.4, 0.4), e(0.4, 0.1)	e	$\emptyset$
c	e(0.3, 0.5)		

The IFCG of Fig. 1 is shown in Fig. 2.

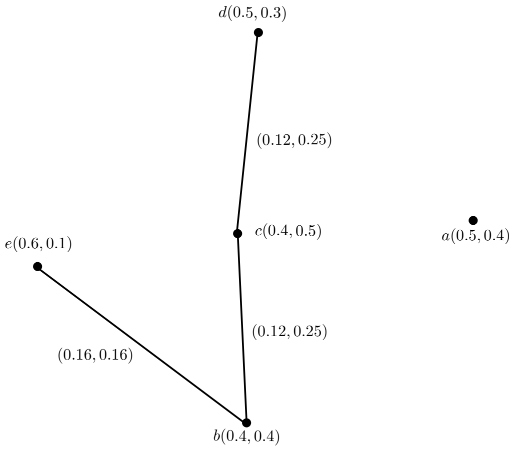

Figure 2: IFCG

We now discuss the method of construction of IFCG of the Cartesian product of IFD in following Theorem.

Theorem 2.1. *Let* $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) = (\mathcal{A}_1,\, \mathcal{S}_1)$ *and* $\mathfrak{C}(\overrightarrow{\mathcal{G}_2}) = (\mathcal{A}_2,\, \mathcal{S}_2)$ *be two IFCGs of IFDs* $\overrightarrow{\mathcal{G}_1} = (\mathcal{A}_1,\, \overrightarrow{\mathcal{B}_1})$ *and* $\overrightarrow{\mathcal{G}_2} = (\mathcal{A}_2,\, \overrightarrow{\mathcal{B}_2})$*, respectively. Then* $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_2}) = \mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\Box}$*, where* $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}$ *is an IFG on the crisp graph* $(X_1 \times X_2,\, E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*})$, $\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*$ *and* $\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*$ *are the crisp competition graphs of* $\overrightarrow{\mathcal{G}_1}$ *and* $\overrightarrow{\mathcal{G}_2}$*, respectively.* $\mathcal{G}^{\Box}$ *is an IFG on* $(X_1 \times X_2,\, E^{\Box})$ *such that:*

- $E^{\Box} = \{(a_1,\, a_2)(b_1,\, b_2) : b_1 \in \mathfrak{N}^-(a_1)^*, b_2 \in \mathfrak{N}^+(a_2)^*\}$,
  $E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} = \{(a_1,\, a_2)(a_1,\, b_2) : a_1 \in X_1, a_2b_2 \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}\}$
  $\cup \{(a_1,\, a_2)(b_1,\, a_2) : a_2 \in X_2, a_1b_1 \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\}$.

- $\phi_{\mathcal{A}_1\Box\mathcal{A}_2} = \phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2), \qquad \psi_{\mathcal{A}_1\Box\mathcal{A}_2} = \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2)$.

- $\phi_{\mathcal{S}}((a_1,\, a_2)(a_1,\, b_2)) = [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)] \times \vee_{x_2}\{\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2x_2) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(b_2x_2)\}$,
  $(a_1,\, a_2)(a_1,\, b_2) \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}, \quad x_2 \in (\mathfrak{N}^+(a_2) \cap \mathfrak{N}^+(b_2))^*$.

- $\psi_{\mathcal{S}}((a_1,\, a_2)(a_1,\, b_2)) = [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)] \times \vee_{x_2}\{\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2x_2) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(b_2x_2)\}$,
  $(a_1, a_2)(a_1, b_2) \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}, \quad x_2 \in (\mathfrak{N}^+(a_2) \cap \mathfrak{N}^+(b_2))^*$.

- $\phi_{\mathcal{S}}((a_1,\, a_2)(b_1,\, a_2)) = [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2)] \times \vee_{x_1}\{\phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(a_1x_1) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1x_1)\}$,
  $(a_1,\, a_2)(b_1,\, a_2) \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}, \quad x_1 \in (\mathfrak{N}^+(a_1) \cap \mathfrak{N}^+(b_1))^*$.

- $\psi_{\mathcal{S}}((a_1,\, a_2)(b_1,\, a_2)) = [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(b_2) \vee \psi_{\mathcal{A}_2}(a_2)] \times \vee_{x_1}\{\psi_{\mathcal{A}_2}(a_2) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(a_1x_1) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1x_1)\}$,
  $(a_1,\, a_2)(b_1,\, a_2) \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}, \quad x_1 \in (\mathfrak{N}^+(a_1) \cap \mathfrak{N}^+(b_1))^*$.

- $\phi_{\mathcal{S}}((a_1,\, a_2)(b_1,\, b_2)) = [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)] \times [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1a_1) \wedge \phi_{\mathcal{A}_2}(b_2) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2b_2)]$,
  $(a_1,\, a_2)(b_1,\, b_2) \in E^{\Box}$.

- $\psi_{\mathcal{S}}((a_1,\, a_2)(b_1,\, b_2)) = [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)]$
  $\times [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1a_1) \vee \psi_{\mathcal{A}_2}(b_2) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2b_2)]$,
  $(a_1,\, a_2)(b_1,\, b_2) \in E^{\Box}$.

Proof. Consider an edge $(a_1, a_2)(b_1, b_2)$ of $G_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\Box}$. Then, we consider three cases:
(1) If $a_1 = b_1$, $a_2 \neq b_2$, then $(a_1, a_2)(a_1, b_2) \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}$. By conditions (3) and (4),

$$\begin{aligned}
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{x_2}\{\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2x_2) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(b_2x_2)\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(b_2)] \\
&\times \vee x_2[\{\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2x_2)\} \wedge \{\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(b_2x_2)\}], \\
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \\
&\times \vee_{x_2}[\phi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(a_1,x_2)) \wedge \phi_{\overrightarrow{\mathcal{B}}}((a_1,b_2)(a_1,x_2))], \\
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \times \vee_{x_2}[\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)],
\end{aligned}$$

$$\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) = [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \times h[\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)]. \tag{1}$$

$$\begin{aligned}
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{x_2}\{\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2x_2) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(b_2x_2)\}, \\
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{x_2}[\{\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2x_2)\} \vee \{\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(b_2x_2)\}, \\
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \\
&\times \vee_{x_2}[\psi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(a_1,x_2)) \vee \psi_{\overrightarrow{\mathcal{B}}}((a_1,b_2)(a_1,x_2))], \\
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =&[\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \times \vee_{x_2}[\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)],
\end{aligned}$$

$$\psi((a_1,a_2)(a_1,b_2)) = [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \times h[\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)]. \tag{2}$$

From equations (1) and (2), $(a_1, a_2)(a_1, b_2)$ is an edge of $\mathfrak{C}(\mathcal{G}_1\Box\mathcal{G}_2)$.

(2) If $a_1 \neq b_1$, $a_2 = b_2$, then $(a_1, a_2)(b_1, a_2) \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\Box E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}$. Using conditions (5) and (6), we have

$$\begin{aligned}
\phi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2)] \\
&\times \vee_{x_1}\{\phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(a_1x_1) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1x_1)\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\{\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2)\} \wedge \{\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2)\}] \\
&\times \vee_{x_1}[\{\phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(a_1x_1)\} \wedge \{\phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1x_1)\}], \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(b_1,a_2)] \\
&\times \vee_{x_1}[\phi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(x_1,a_2)) \wedge \phi_{\overrightarrow{\mathcal{B}}}((b_1,a_2)(x_1,a_2))], \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(b_1,a_2)] \times \vee[\phi_{\mathfrak{N}^+(a_1,a_2)\bigcap\mathfrak{N}^+(b_1,a_2)}],
\end{aligned}$$

$$\phi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) = [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(b_1,a_2)] \times h[\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(b_1,a_2)]. \tag{3}$$

$$\begin{aligned}
\psi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2)] \\
&\times \vee_{x_1}\{\psi_{\mathcal{A}_2}(a_2) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(a_1x_1) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1x_1)\}, \\
\psi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\{\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2)\} \vee \{\psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2)\}] \\
&\times \vee_{x_1}[\{\psi_{\mathcal{A}_2}(a_2) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(a_1x_1)\} \vee \{\psi_{\mathcal{A}_2}(a_2) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1x_1)\}], \\
\psi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(b_1,a_2)] \\
&\times \vee_{x_1}[\psi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(x_1,a_2)) \vee \psi_{\overrightarrow{\mathcal{B}}}((b_1,a_2)(x_1,a_2))], \\
\psi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) =&[\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(b_1,a_2)] \times \vee[\psi_{\mathfrak{N}^+(a_1,a_2)\cap\mathfrak{N}^+(b_1,a_2)}],
\end{aligned}$$

$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,a_2)) = [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(b_1,a_2)] \times h[\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(b_1,a_2)]. \tag{4}$$

Equations (3) and (4) show that $(a_1,\, a_2)(b_1,\, a_2)$ is an edge of $\mathfrak{C}(\mathcal{G}_1\Box\mathcal{G}_2)$.

(3) If $a_1 \neq b_1$, $a_2 \neq b_2$, then $(a_1,\, a_2)(b_1,\, b_2) \in E^{\Box}$. Using conditions (7) and (8), we have

$$\begin{aligned}\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}_1}(a_1)\wedge\phi_{\mathcal{A}_1}(b_1)\phi_{\mathcal{A}_2}(a_2)\wedge\phi_{\mathcal{A}_2}(b_2)]\\ &\times[\phi_{\mathcal{A}_1}(a_1)\wedge\phi_{\overrightarrow{\mathcal{B}_2}}(a_2b_2)\wedge\phi_{\mathcal{A}_2}(b_2)\wedge\phi_{\overrightarrow{\mathcal{B}_1}}(b_1a_1)],\\ \phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}}(a_1,a_2)\wedge\phi_{\mathcal{A}}(b_1,b_2)]\\ &\times[\psi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(a_1,b_2))\wedge\psi_{\overrightarrow{\mathcal{B}}}((b_1,b_2)(a_1,b_2))],\\ \phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}}(a_1,a_2)\wedge\phi_{\mathcal{A}}(b_1,b_2)]\times\phi_{\mathfrak{N}^+(a_1,a_2)\cap\mathfrak{N}^+(b_1,b_2)},\end{aligned}$$

$$\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\phi_{\mathcal{A}}(a_1,a_2)\wedge\phi_{\mathcal{A}}(b_1,b_2)]\times h(\mathfrak{N}^+(a_1,a_2)\cap\mathfrak{N}^+(b_1,b_2)). \tag{5}$$

$$\begin{aligned}\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\psi_{\mathcal{A}_1}(a_1)\vee\psi_{\mathcal{A}_1}(b_1)\psi_{\mathcal{A}_2}(a_2)\vee\psi_{\mathcal{A}_2}(b_2)]\\ &\times[\psi_{\mathcal{A}_1}(a_1)\vee\psi_{\overrightarrow{\mathcal{B}_2}(a_2b_2)}\vee\psi_{\mathcal{A}_2}(b_2)\vee\psi_{\overrightarrow{\mathcal{B}_1}(b_1a_1)}],\\ \psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\psi_{\mathcal{A}}(a_1,a_2)\vee\psi_{\mathcal{A}}(b_1,b_2)]\\ &\times[\psi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(a_1,b_2))\vee\psi_{\overrightarrow{\mathcal{B}}}((b_1,b_2)(a_1,b_2))],\\ \psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\psi_{\mathcal{A}}(a_1,a_2)\vee\psi_{\mathcal{A}}(b_1,b_2)]\times\psi_{\mathfrak{N}^+(a_1,a_2)\cap\mathfrak{N}^+(b_1,b_2)},\end{aligned}$$

$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\psi_{\mathcal{A}}(a_1,a_2)\vee\psi_{\mathcal{A}}(b_1,b_2)]\times h(\mathfrak{N}^+(a_1,a_2)\cap\mathfrak{N}^+(b_1,b_2)). \tag{6}$$

Equations (5) and (6) imply that $(a_1,\, a_2)(b_1,\, b_2)$ is an edge of $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_1})$.

Hence, $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_1}) \subseteq \mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\Box}$. Conversely, using the same techniques we can show that $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\Box} \subseteq \mathfrak{C}(\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_1})$. The converse part is obvious, so we omit it. This completes the proof. □

Example 2.2. Consider $\overrightarrow{\mathcal{G}_1} = (X_1,\, \mathcal{A}_1,\, \mathcal{B}_1)$ and $\overrightarrow{\mathcal{G}_2} = (X_2,\, \mathcal{A}_2,\, \mathcal{B}_2)$ be two IFDs of the crisp digraphs $\overrightarrow{\mathcal{G}_1^*} = (X_1,\, \overrightarrow{E_1})$ and $\overrightarrow{\mathcal{G}_2^*} = (X_2,\, \overrightarrow{E_2})$, respectively, as shown in Fig. 3. The IFONs and IFINs of $\overrightarrow{\mathcal{G}_1}$ and $\overrightarrow{\mathcal{G}_2}$ are given in Tables 2 and 3.

The IFCGs $\mathfrak{C}(\overrightarrow{\mathcal{G}_1})$ and $\mathfrak{C}(\overrightarrow{\mathcal{G}_2})$ are given in Fig. 4.

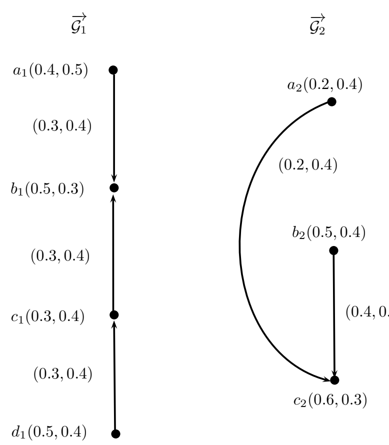

Figure 3: IFDs

Table 2: IFONs and IFINs of $\overrightarrow{\mathcal{G}_1}$

$x \in X_1$	$\mathfrak{N}^+(x)$	$\mathfrak{N}^-(x)$
a_1	$\{b_1(0.3, 0.4)\}$	$\emptyset$
b_1	$\emptyset$	$\{a_1(0.3, 0.4), c_1(0.3, 0.4)\}$
c_1	$\{b_1(0.3, 0.4)\}$	$\{d_1(0.3, 0.4)\}$
d_1	$\{c_1(0.3, 0.4)\}$	$\emptyset$

Table 3: IFONs and IFINs of $\overrightarrow{\mathcal{G}_2}$

$x \in X_2$	$\mathfrak{N}^+(x)$	$\mathfrak{N}^-(x)$
a_2	$\{c_2(0.2, 0.4)\}$	$\emptyset$
b_2	$\{c_2(0.4, 0.4)\}$	$\emptyset$
c_2	$\emptyset$	$\{a_2(0.2, 0.4), b_2(0.4, 0.4)\}$

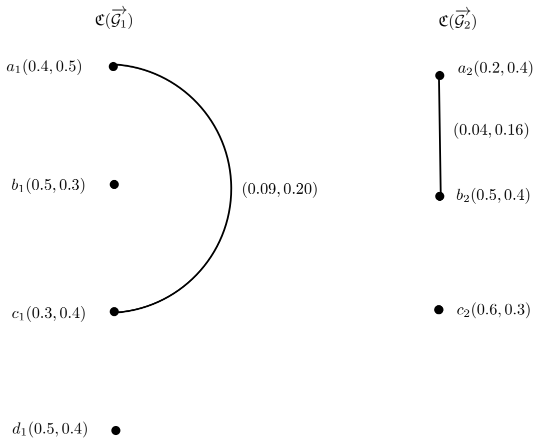

Figure 4: IFCGs of $\overrightarrow{\mathcal{G}_1}$ and $\overrightarrow{\mathcal{G}_2}$

We now construct the IFCG $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^* \Box \mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\Box} = (\mathcal{C}, \mathcal{S})$ where $\mathcal{C} = (\phi_{\mathcal{C}}, \psi_{\mathcal{C}})$ and $\mathcal{S} = (\phi_{\mathcal{S}}, \psi_{\mathcal{S}})$, from $\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*$ and $\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*$ using Theorem 2.1. According to condition (1), the two sets of edges are

$$\begin{aligned}
E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^* \Box \mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} =& \{(a_1, a_2)(a_1, b_2), (b_1, a_2)(b_1, b_2), (c_1, a_2)(c_1, b_2), \\
& (d_1, a_2)(d_1, b_2), (a_1, a_2)(c_1, a_2), (a_1, b_2)(c_1, b_2), (a_1, c_2)(c_1, c_2)\}, \\
E^{\Box} =& \{(b_1, a_2)(a_1, c_2), (b_1, a_2)(c_1, c_2), (b_1, b_2)(a_1, c_2), \\
& (b_1, b_2)(c_1, c_2), (c_1, a_2)(d_1, c_2), (c_1, b_2)(d_1, c_2)\}.
\end{aligned}$$

According to conditions (3) to (8), the degrees of membership and non-membership of the edges can be calculated as

$$\begin{aligned}
\mathcal{S}((a_1, a_2)(a_1, b_2)) =& (\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2), \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)) \\
& \times (\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{B}_2}(b_2c_2), \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{B}_2}(b_2c_2) \\
=& (0.2, 0.5) \times (0.2, 0.5) \\
=& (0.04, 0.25),
\end{aligned}$$

$$\begin{aligned}\mathcal{S}((b_1,a_2)(a_1,c_2)) =&(\phi_{\mathcal{A}_1}(b_1)\wedge\phi_{\mathcal{A}_1}(a_1)\wedge\phi_{\mathcal{A}_2}(a_2)\wedge\phi_{\mathcal{A}_2}(c_2),\psi_{\mathcal{A}_1}(b_1)\vee\psi_{\mathcal{A}_1}(a_1)\vee\psi_{\mathcal{A}_2}(a_2)\vee\psi_{\mathcal{A}_2}(c_2))\\ &\times(\phi_{\mathcal{A}_1}(b_1)\wedge\phi_{\mathcal{B}_1}(a_1b_1)\wedge\phi_{\mathcal{B}_2}(c_2)\wedge\phi_{\mathcal{B}_2}(a_2c_2),\psi_{\mathcal{A}_1}(b_1)\vee\psi_{\mathcal{B}_1}(a_1b_1)\vee\psi_{\mathcal{A}_2}(c_2)\vee\psi_{\mathcal{B}_2}(a_2c_2))\\ =&(0.2,0.5)\times(0.2,0.4)\\ =&(0.04,0.20).\end{aligned}$$

All the membership and non-membership degrees of adjacent edges of $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}$ and $\mathcal{G}^{\Box}$ are given in Table 4.

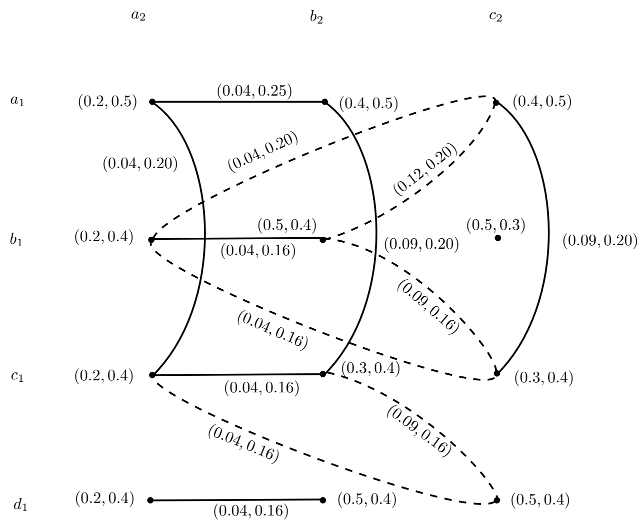

Figure 5: $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}\cup\mathcal{G}^{\Box}$

The IFCG obtained using this method is given in Fig. 5 where the solid lines indicate the part of IFCG obtained from $G_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}$, the dotted lines represent the part $\mathcal{G}^{\Box}$.

The Cartesian product $\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_2}$ of IFDs $\overrightarrow{\mathcal{G}_1}$ and $\overrightarrow{\mathcal{G}_1}$ is shown in Fig. 6. The IFONs of $\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_2}$ are calculated in Table 5. The IFCG of $\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_2}$ is shown in Fig. 7. It is clear from Figs. 5 and 7 that $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*\Box\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}\cup\mathcal{G}^{\Box}\cong\mathfrak{C}(\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_2})$.

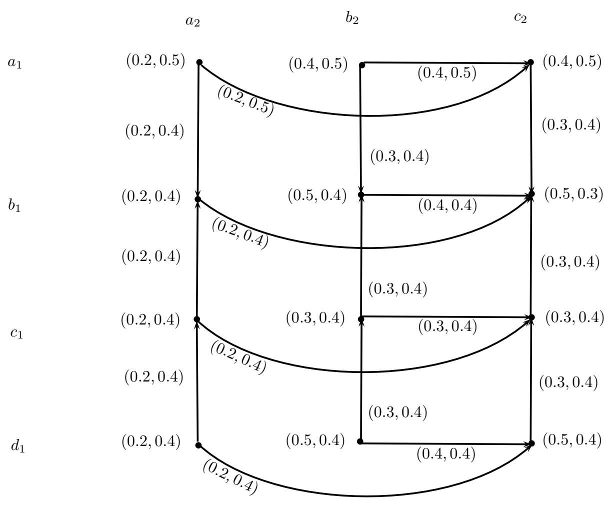

Figure 6: $\overrightarrow{\mathcal{G}_1}\Box\overrightarrow{\mathcal{G}_2}$

Table 4: Adjacent edges of $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^* \square \mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\square}$

$(x_1, x_2)(y_1, y_2)$	$\mathcal{S}(x_1, x_2)(y_1, y_2)$
$(a_1, a_2)(a_1, b_2)$	$(\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2),\ \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2))$ $\times(\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{B}_2}(b_2c_2),\ \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{B}_2}(b_2c_2))$ $= (0.04, 0.25)$
$(b_1, a_2)(b_1, b_2)$	$(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2))$ $\times(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{B}_2}(b_2c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{B}_2}(b_2c_2))$ $= (0.04, 0.16)$
$(c_1, a_2)(c_1, b_2)$	$(\phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2),\ \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2))$ $\times(\phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{B}_2}(b_2c_2),\ \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{B}_2}(b_2c_2))$ $= (0.04, 0.16)$
$(d_1, a_2)(d_1, b_2)$	$(\phi_{\mathcal{A}_1}(d_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2),\ \psi_{\mathcal{A}_1}(d_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2))$ $\times(\phi_{\mathcal{A}_1}(d_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{B}_2}(b_2c_2),\ \psi_{\mathcal{A}_1}(d_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{B}_2}(b_2c_2))$ $= (0.04, 0.16)$
$(a_1, a_2)(c_1, a_2)$	$(\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_2}(a_2),\ \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_2}(a_2))$ $\times(\phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{B}_1}(a_1b_1) \wedge \phi_{\mathcal{B}_1}(c_1b_1),\ \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{B}_1}(a_1b_1) \vee \psi_{\mathcal{B}_1}(c_1b_1))$ $= (0.04, 0.20)$
$(a_1, b_2)(c_1, b_2)$	$(\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_2}(b_2),\ \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_2}(b_2))$ $\times(\phi_{\mathcal{A}_2}(b_2) \wedge \phi_{\mathcal{B}_1}(a_1b_1) \wedge \phi_{\mathcal{B}_1}(c_1b_1),\ \psi_{\mathcal{A}_2}(b_2) \vee \psi_{\mathcal{B}_1}(a_1b_1) \vee \psi_{\mathcal{B}_1}(c_1b_1))$ $= (0.09, 0.20)$
$(a_1, c_2)(c_1, c_2)$	$(\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_2}(c_2) \wedge \phi_{\mathcal{B}_1}(a_1b_1) \wedge \phi_{\mathcal{B}_1}(c_1b_1),\ \psi_{\mathcal{A}_2}(c_2) \vee \psi_{\mathcal{B}_1}(a_1b_1) \vee \psi_{\mathcal{B}_1}(c_1b_1))$ $= (0.09, 0.20)$
$(b_1, a_2)(a_1, c_2)$	$(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{B}_1}(a_1b_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{B}_1}(a_1b_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $= (0.04, 0.20)$
$(b_1, a_2)(c_1, c_2)$	$(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{B}_1}(c_1b_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{B}_1}(c_1b_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $= (0.04, 0.16)$
$(b_1, b_2)(a_1, c_2)$	$(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(b_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(b_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_1}(a_1b_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{B}_1}(a_1b_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $= (0.12, 0.20)$
$(b_1, b_2)(c_1, c_2)$	$(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_2}(b_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_2}(b_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{B}_1}(c_1b_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{B}_1}(c_1b_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $= (0.09, 0.16)$
$(c_1, a_2)(d_1, c_2)$	$(\phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_1}(d_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_1}(d_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{B}_1}(d_1c_1) \wedge \phi_{\mathcal{B}_2}(a_2c_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{B}_1}(d_1c_1) \vee \psi_{\mathcal{B}_2}(a_2c_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $= (0.04, 0.16)$
$(c_1, b_2)(d_1, c_2)$	$(\phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{A}_1}(d_1) \wedge \phi_{\mathcal{A}_2}(b_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{A}_1}(d_1) \vee \psi_{\mathcal{A}_2}(b_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $\times(\phi_{\mathcal{A}_1}(c_1) \wedge \phi_{\mathcal{B}_1}(d_1c_1) \wedge \phi_{\mathcal{B}_2}(b_2c_2) \wedge \phi_{\mathcal{A}_2}(c_2),\ \psi_{\mathcal{A}_1}(c_1) \vee \psi_{\mathcal{B}_1}(d_1c_1) \vee \psi_{\mathcal{B}_2}(b_2c_2) \vee \psi_{\mathcal{A}_2}(c_2))$ $= (0.09, 0.16)$

Table 5: IFONs of $\overrightarrow{\mathcal{G}_1}\square\overrightarrow{\mathcal{G}_2}$

$(x,\, y)$	$\mathfrak{N}^+(x,\, y)$	$(x,\, y)$	$\mathfrak{N}^+(x,\, y)$
$(a_1,\, a_2)$	$\{((a_1,\, c_2),\, 0.2,\, 0.5),\, ((b_1,\, a_2),\, 0.2,\, 0.4)\}$	$(a_1,\, b_2)$	$\{((a_1,\, c_2),\, 0.4,\, 0.5),\, ((b_1,\, b_2),\, 0.3,\, 0.4)\}$
$(a_1,\, c_2)$	$\{((b_1,\, c_2),\, 0.3,\, 0.4)\}$	$(b_1,\, a_2)$	$\{((b_1,\, c_2),\, 0.2,\, 0.4)\}$
$(b_1,\, b_2)$	$\{((b_1,\, c_2),\, 0.4,\, 0.4)\}$	$(b_1,\, c_2)$	$\emptyset$
$(c_1,\, a_2)$	$\{((c_1,\, c_2),\, 0.2,\, 0.4),\, ((b_1,\, a_2),\, 0.2,\, 0.4)\}$	$(c_1,\, b_2)$	$\{((b_1,\, b_2),\, 0.3,\, 0.4),\, ((c_1,\, c_2),\, 0.3,\, 0.4)\}$
$(c_1,\, c_2)$	$\{((b_1,\, c_2),\, 0.3,\, 0.4)\}$	$(d_1,\, a_2)$	$\{((d_1,\, c_2),\, 0.2,\, 0.4),\, ((c_1,\, a_2),\, 0.2,\, 0.4)\}$
$(d_1,\, b_2)$	$\{((d_1,\, c_2),\, 0.4,\, 0.4),\, (c_1,\, b_2),\, 0.3,\, 0.4)\}$	$(d_1,\, c_2)$	$\{((c_1,\, c_2),\, 0.3,\, 0.4)\}$

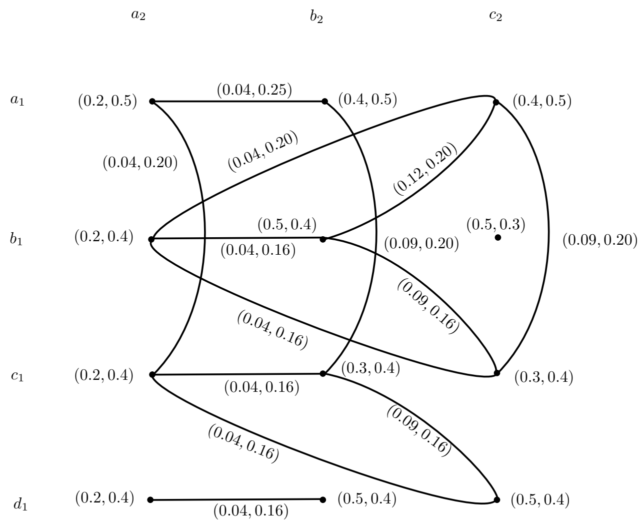

Figure 7: $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}\square\overrightarrow{\mathcal{G}_2})$

Definition 2.3. The direct product of two intuitionistic fuzzy graphs $\mathcal{G}_1 = (\mathcal{A}_1,\, \mathcal{B}_1)$ and $\mathcal{G}_2 = (\mathcal{A}_2,\, \mathcal{B}_2)$ is denoted by $\mathcal{G}_1 \times \mathcal{G}_2$ and defined as a pair $(\mathcal{A}_1 \times \mathcal{A}_2,\, \mathcal{B}_1 \times \mathcal{B}_2)$, such that for each $1 \leq j \leq m$,

1. $\phi_{\mathcal{A}_1\times\mathcal{A}_2}(x_1,\, x_2) = \phi_{\mathcal{A}_1}(x_1) \wedge \phi_{\mathcal{A}_2}(x_2)$,
   $\psi_{\mathcal{A}_1\times\mathcal{A}_2}(x_1,\, x_2) = \psi_{\mathcal{A}_1}(x_1) \vee \psi_{\mathcal{A}_2}(x_2)$, for all $(x_1,\, x_2) \in X_1 \times X_2$.

2. $\phi_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,\, x_2)(y_1,\, y_2)) = \phi_{\mathcal{B}_1}(x_1y_1) \wedge \phi_{\mathcal{B}_2}(x_2y_2)$,
   $\psi_{\mathcal{B}_1\times\mathcal{B}_2}((x_1,\, x_2)(y_1,\, y_2)) = \psi_{\mathcal{B}_1}(x_1y_1) \vee \psi_{\mathcal{B}_2}(x_2y_2)$, for all $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$.

We now discuss the construction of IFCG of the direct product of IFDs from respective IFCGs of the IFDs.

Theorem 2.2. *Let $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) = (\mathcal{A}_1,\, \mathcal{S}_1)$ and $\mathfrak{C}(\overrightarrow{\mathcal{G}_2}) = (\mathcal{A}_2,\, \mathcal{S}_2)$ be two IFCGs of IFDs $\overrightarrow{\mathcal{G}_1} = (\mathcal{A}_1,\, \overrightarrow{\mathcal{B}_1})$ and $\overrightarrow{\mathcal{G}_2} = (\mathcal{A}_2,\, \overrightarrow{\mathcal{B}_2})$, respectively, without isolated vertices such that neither is an intuitionistic fuzzy empty graph. Then $\mathfrak{C}(\overrightarrow{\mathcal{G}_1} \times \overrightarrow{\mathcal{G}_2}) = [\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) \times \mathfrak{C}(\overrightarrow{\mathcal{G}_2})] \cup \mathcal{G}^\times$, where $\mathcal{G}^\times = (\mathcal{A},\, \mathcal{S})$ is a IFG on the crisp graph $(X_1 \times X_2,\, E^\times)$ defined as*

1. $\mathcal{C}((a_1,\, a_2)) = (\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2),\, \psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2)),\, a_1 \in X_1,\, a_2 \in X_2.$

2. $E^\times = \{(a_1,\, a_2)(a_1,\, b_2) | a_1 \in X_1, a_2, b_2 \in X_2, \mathfrak{N}^+(a_1) \neq \emptyset, a_2b_2 \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}\}$
   $\cup \{(a_1,\, a_2)(b_1,\, a_2) | a_1, b_1 \in X_1, a_2, \in X_2, \mathfrak{N}^+(a_2) \neq \emptyset, a_1b_1 \in E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\}.$

3. $\phi_{\mathcal{S}}((a_1,\, a_2)(a_1,\, b_2)) = [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)]$
   $\times \vee_{c_1\in X_1, c_2\in X_2}\{\phi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(b_2c_2) | c_1 \in \mathfrak{N}^+(a_1)^*, c_2 \in \mathfrak{N}^+(a_2)^* \cap \mathfrak{N}^+(b_2)^*\},$
   $(a_1, a_2)(a_1, b_2) \in E^\times.$

4. $\psi_{\mathcal{S}}((a_1,\ a_2)(a_1,\ b_2)) = [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)]$
$\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\psi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(b_2c_2) | c_1 \in \mathfrak{N}^+(a_1)^*, c_2 \in \mathfrak{N}^+(a_2)^* \cap \mathfrak{N}^+(b_2)^*\}$,
$(a_1, a_2)(a_1, b_2) \in E^\times$.

5. $\phi_{\mathcal{S}}((a_1,\ a_2)(b_1,\ a_2)) = [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2)]$
$\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) | c_2 \in \mathfrak{N}^+(a_2)^*, c_1 \in \mathfrak{N}^+(a_1)^* \cap \mathfrak{N}^+(b_1)^*\}$,
$(a_1, a_2)(a_1, b_2) \in E^\times$.

6. $\psi_{\mathcal{S}}((a_1,\ a_2)(b_1,\ a_2)) = [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2)]$,
$\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\psi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) | c_2 \in \mathfrak{N}^+(a_2)^*, c_1 \in \mathfrak{N}^+(a_1)^* \cap \mathfrak{N}^+(b_1)^*\}$,
$(a_1, a_2)(a_1, b_2) \in E^\times$.

Proof. Let $(a_1, a_2)(b_1, b_2)$ be an edge of $[\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) \times \mathfrak{C}(\overrightarrow{\mathcal{G}_2})] \cup \mathcal{G}^\times$. Then there are three cases:
(1) If $a_1 = b_1, a_2 \neq b_2$, then $(a_1, a_2)(a_1, b_2) \in E^\times$. Using conditions (3) and (4),

$$\begin{aligned}
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =& [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(b_2c_2)\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =& [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(c_1,c_2)) \wedge \phi_{\overrightarrow{\mathcal{B}}}((a_1,b_2)(c_1,c_2))\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =& [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)}\},
\end{aligned}$$

$$\phi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) = [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \times h(\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)). \tag{7}$$

$$\begin{aligned}
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =& [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\psi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(b_2c_2)\}, \\
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =& [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\psi_{\overrightarrow{\mathcal{B}}}((a_1,a_2)(c_1,c_2)) \vee \psi_{\overrightarrow{\mathcal{B}}}((a_1,b_2)(c_1,c_2))\}, \\
\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) =& [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\psi_{\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)}\},
\end{aligned}$$

$$\psi_{\mathcal{S}}((a_1,a_2)(a_1,b_2)) = [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \times h(\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(a_1,b_2)). \tag{8}$$

Equations (7) and (8) show that $(a_1, a_2)(a_1, b_2)$ is an edge of $\mathfrak{C}(\overrightarrow{\mathcal{G}_1} \times \overrightarrow{\mathcal{G}_2})$.
(2) If $a_1 \neq b_1, a_2 = b_2$, then $(a_1, a_2)(b_1, a_2) \in E^\times$. It can be verified on the same lines as Case 1.

(3) If $a_1 \neq b_1, a_2 \neq b_2$, then $(a_1, a_2)(b_1, b_2)$ is an edge of $\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) \times \mathfrak{C}(\overrightarrow{\mathcal{G}_2})$. From the definition of direct product of two IFGs, we have

$$\begin{aligned}
\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& \phi_{\mathcal{S}_1}(a_1b_1) \wedge \phi_{\mathcal{S}_2}(a_2b_2), \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2)\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}_1}(a_1) \wedge \phi_{\mathcal{A}_1}(b_1) \wedge \phi_{\mathcal{A}_2}(a_2) \wedge \phi_{\mathcal{A}_2}(b_2)] \\
&\times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \wedge \phi_{\overrightarrow{\mathcal{B}_1}}(b_1c_1) \wedge \phi_{\overrightarrow{\mathcal{B}_2}}(b_2c_2)\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(b_1,b_2)] \times \vee_{c_1 \in X_1, c_2 \in X_2}\{\phi_{\overrightarrow{\mathcal{B}}}(a_1,a_2)(c_1,c_2) \wedge \phi_{\overrightarrow{\mathcal{B}}}(b_1,b_2)(c_1,c_2)\}, \\
\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) =& [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(a_1,b_2)] \times \vee_{c_1 \in X_1, c_2 \in V_2}\{\phi_{\mathfrak{N}^+((a_1,a_2) \cap \mathfrak{N}^+(b_1,b_2))}(c_1,c_2)\},
\end{aligned}$$

$$\phi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\phi_{\mathcal{A}}(a_1,a_2) \wedge \phi_{\mathcal{A}}(b_1,b_2)] \times h(\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(b_1,b_2)). \tag{9}$$

$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = \psi_{\mathcal{S}_1}(a_1b_1) \vee \psi_{\mathcal{S}_2}(a_2b_2),$$
$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\phi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)] \times \vee_{c_1\in X_1, c_2\in X_2}\{\psi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2)\},$$
$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\psi_{\mathcal{A}_1}(a_1) \vee \psi_{\mathcal{A}_1}(b_1) \vee \psi_{\mathcal{A}_2}(a_2) \vee \psi_{\mathcal{A}_2}(b_2)] \times \vee_{c_1\in X_1, c_2\in X_2}\{\psi_{\overrightarrow{\mathcal{B}_1}}(a_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(a_2c_2) \vee \psi_{\overrightarrow{\mathcal{B}_1}}(b_1c_1) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(b_2c_2)\},$$
$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(b_1,b_2)] \times \vee_{c_1\in X_1, c_2\in X_2}\{\psi_{\overrightarrow{\mathcal{B}}}(a_1,a_2)(c_1,c_2) \vee \psi_{\overrightarrow{\mathcal{B}}}(b_1,b_2)(c_1,c_2)\},$$
$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(a_1,b_2)] \times \vee_{c_1\in X_1, c_2\in V_2}\{\psi_{\mathfrak{N}^+((a_1,a_2)\cap\mathfrak{N}^+(b_1,b_2))}(c_1,c_2)\},$$

$$\psi_{\mathcal{S}}((a_1,a_2)(b_1,b_2)) = [\psi_{\mathcal{A}}(a_1,a_2) \vee \psi_{\mathcal{A}}(b_1,b_2)] \times h(\mathfrak{N}^+(a_1,a_2) \cap \mathfrak{N}^+(b_1,b_2)). \tag{10}$$

Equations (9) and (10) imply that $(a_1, a_2)(b_1, b_2)$ is an edge of $\mathfrak{C}(\overrightarrow{\mathcal{G}_1} \times \overrightarrow{\mathcal{G}_2})$. Hence, $[\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) \times \mathfrak{C}(\overrightarrow{\mathcal{G}_2})] \cup G^{\times} \subseteq \mathfrak{C}(\overrightarrow{\mathcal{G}_1} \times \overrightarrow{\mathcal{G}_2})$.

The converse part can be proved using the same lines. So, we omit it.

This completes the proof. □

Example 2.3. Consider IFDs $\overrightarrow{\mathcal{G}_1} = (X_1, \phi_1, \psi_1)$ and $\overrightarrow{\mathcal{G}_2} = (X_2, \phi_2, \psi_2)$ on the crisp digraphs $\mathcal{G}_1^* = (X_1, \overrightarrow{E}_1)$ and $\mathcal{G}_2^* = (X_2, \overrightarrow{E}_2)$ shown in Fig. 8 and their IFCGs are given in Fig. 9. We will show that $\mathfrak{C}(\overrightarrow{\mathcal{G}_1} \times \overrightarrow{\mathcal{G}_2}) = [\mathfrak{C}(\overrightarrow{\mathcal{G}_1}) \times \mathfrak{C}(\overrightarrow{\mathcal{G}_2})] \cup \mathcal{G}^{\times}$, where $\mathcal{G}^{\times}$ is an IFD on $(X_1 \times X_2, E^{\times})$. The edge set $E^{\times}$ using condition (2) is constructed as

$$E^{\times} = \{(a_1,b_2)(a_1,d_2), (c_1,b_2)(c_1,d_2), (a_1,a_2)(c_1,a_2), (a_1,b_2)(c_1,b_2), (a_1,d_2)(c_1,d_2)\}.$$

Using conditions (3) and (4), the degrees of membership of all the edges from $E^{\times}$ are calculated as under:

Table 6: IFONs of $\overrightarrow{\mathcal{G}_1}$

$x \in X_1$	$\mathfrak{N}^+(x)$
a_1	$\{b_1(0.1,\ 0.3)\}$
b_1	$\emptyset$
c_1	$\{b_1(0.1,\ 0.3)\}$

Table 7: IFONs of $\overrightarrow{\mathcal{G}_2}$

$x \in X_2$	$\mathfrak{N}^+(x)$
a_2	$\{c_2(0.1,\ 0.2)\}$
b_2	$\{a_2(0.1,\ 0.2)\}$
c_2	$\emptyset$
d_2	$\{a_2(0.1,\ 0.2)\}$

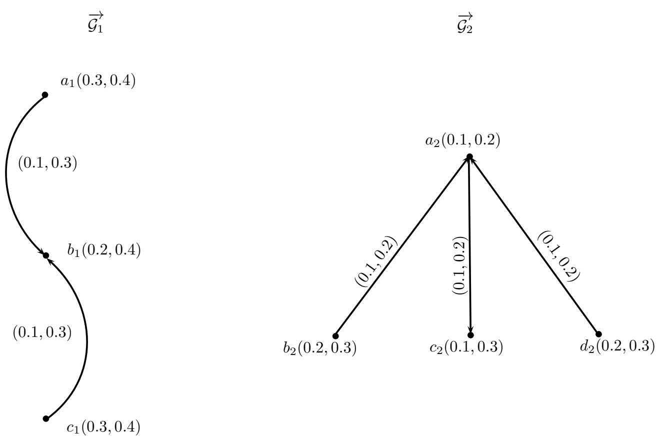

Figure 8: IFDs $\overrightarrow{\mathcal{G}_1}$ and $\overrightarrow{\mathcal{G}_2}$

$$\begin{aligned}\mathcal{S}((a_1,b_2)(a_1,d_2)) &= [(\phi_1(a_1)\wedge\phi_2(b_2)\wedge\phi_2(d_2),\psi_1(a_1)\vee\psi_2(b_2)\vee\psi_2(d_2))\\ &\quad\times(\phi_1(a_1b_1)\wedge\phi_2(b_2a_2)\wedge\phi_2(d_2a_2),\psi_1(a_1b_1)\vee\psi_2(b_2a_2)\vee\psi_2(d_2a_2))]\\ &= (0.2,0.4)(0.1,0.3).\end{aligned}$$

The IFG obtained using Theorem 2.2 is shown in Fig. 10. The solid lines represent the part of IFG obtained from $\mathfrak{C}(\overrightarrow{\mathcal{G}_1})\times\mathfrak{C}(\overrightarrow{\mathcal{G}_2})$ and dashed lines indicate the part of IFG obtained from $\mathcal{G}^{\times}$. The direct product of $\overrightarrow{\mathcal{G}_1}$ and $\overrightarrow{\mathcal{G}_2}$ is presented in Fig. 11 and its intuitionistic fuzzy competition graph in Fig. 12 which is similar as Fig. 10. Clearly, $[\mathfrak{C}(\overrightarrow{\mathcal{G}_1})\times\mathfrak{C}(\overrightarrow{\mathcal{G}_2})\cup\mathcal{G}^{\times}\cong\mathfrak{C}(\overrightarrow{\mathcal{G}_1}\times\overrightarrow{\mathcal{G}_2})$.

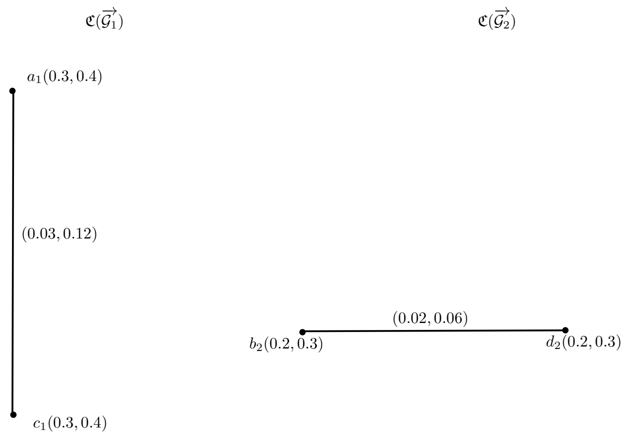

Figure 9: IFCGs

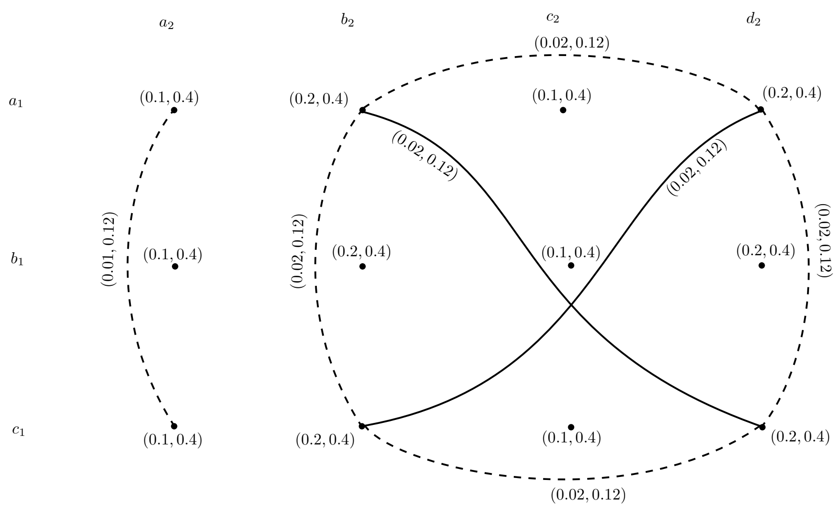

Figure 10: $[\mathfrak{C}(\overrightarrow{\mathcal{G}_1})\times\mathfrak{C}(\overrightarrow{\mathcal{G}_2})]\cup\mathcal{G}^{\times}$

Table 8: Adjacent edges of $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})\times\mathfrak{C}(\overrightarrow{\mathcal{G}_2})}\cup\mathcal{G}^{\times}$

$(x_1, x_2)(y_1, y_2)$	$\mathcal{S}(x_1, x_2)(y_1, y_2)$
$(a_1, b_2)(a_1, d_2)$	$(\phi_1(a_1)\wedge\phi_2(b_2)\wedge\phi_2(d_2),\ \psi_1(a_1)\vee\psi_2(b_2)\vee\psi_2(d_2))$ $\times(\phi_1(a_1b_1)\wedge\phi_2(b_2a_2)\wedge\phi_2(d_2a_2),\ \psi_1(a_1b_1)\vee\psi_2(b_2a_2)\vee\psi_2(d_2a_2))$ $=(0.02,\ 0.12)$
$(c_1, b_2)(c_1, d_2)$	$(\phi_1(c_1)\wedge\phi_2(b_2)\wedge\phi_2(d_2),\psi_1(c_1)\vee\psi_2(b_2)\vee\psi_2(d_2))$ $\times(\phi_1(c_1b_1)\wedge\phi_2(b_2a_2)\wedge\phi_2(d_2a_2),\ \psi_1(c_1b_1)\vee\psi_2(b_2a_2)\vee\psi_2(d_2a_2))$ $=(0.02,\ 0.12)$
$(a_1, a_2)(c_1, a_2)$	$(\phi_1(a_1)\wedge\phi_1(c_1)\wedge\phi_2(a_2),\ \psi_1(a_1)\vee\psi_1(c_1)\vee\psi_2(a_2))$ $\times(\phi_1(a_1b_1)\wedge\phi_1(c_1b_1)\wedge\phi_2(a_2c_2),\psi_1(a_1b_1)\vee\psi_1(c_1b_1)\vee\psi_2(a_2c_2))$ $=(0.01,\ 0.12)$
$(a_1, b_2)(c_1, b_2)$	$(\phi_1(a_1)\wedge\phi_1(c_1)\wedge\phi_2(b_2),\ \psi_1(a_1)\vee\psi_1(c_1)\vee\psi_2(b_2))$ $\times(\phi_1(a_1b_1)\wedge\phi_1(c_1b_1)\wedge\phi_2(b_2a_2),\ \psi_1(a_1b_1)\vee\psi_1(c_1b_1)\vee\psi_2(b_2a_2))$ $=(0.02,\ 0.12)$
$(a_1, d_2)(c_1, d_2)$	$(\phi_1(a_1)\wedge\phi_1(c_1)\wedge\phi_2(d_2),\ \psi_1(a_1)\vee\psi_1(c_1)\vee\psi_2(d_2))$ $\times(\phi_1(a_1b_1)\wedge\phi_1(c_1b_1)\wedge\phi_2(d_2a_2),\ \psi_1(a_1b_1)\vee\psi_1(c_1b_1)\vee\psi_2(d_2a_2))$ $=(0.02,\ 0.12)$

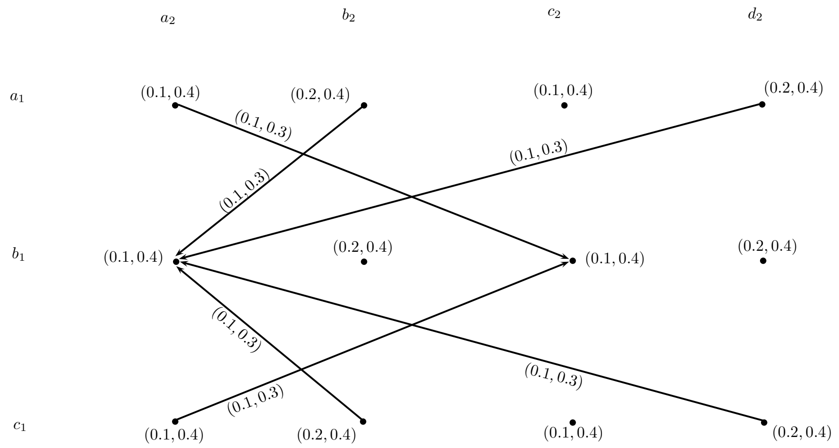

Figure 11: $\overrightarrow{\mathcal{G}_1}\times\overrightarrow{\mathcal{G}_2}$

Proposition 2.3. *Let* $\mathfrak{C}(\overrightarrow{\mathcal{G}_1})=(\mathcal{A}_1,\ \mathcal{S}_1)$ *and* $\mathfrak{C}(\overrightarrow{\mathcal{G}_2})=(\mathcal{A}_2,\ \mathcal{S}_2)$ *be two IFCGs of IFDs* $\overrightarrow{\mathcal{G}_1}=(\mathcal{A}_1,\ \overrightarrow{\mathcal{B}}_1)$ *and* $\overrightarrow{\mathcal{G}_2}=(\mathcal{A}_2,\ \overrightarrow{\mathcal{B}}_2)$ *of crisp diagraphs* $\overrightarrow{\mathcal{G}_1}^*=(X_1,\ \overrightarrow{E}_1)$ *and* $\overrightarrow{\mathcal{G}_2}^*=(X_2,\ \overrightarrow{E}_2)$*, respectively. Then,* $\mathfrak{C}(\overrightarrow{\mathcal{G}}_1\cup\overrightarrow{\mathcal{G}}_2)=(\mathcal{A}_1\cup\mathcal{A}_2,\ \mathcal{S})$ *can be constructed from and* $\mathfrak{C}(\overrightarrow{\mathcal{G}}_1)=(\mathcal{A}_1,\ \mathcal{S}_1)$ *and* $\mathfrak{C}(\overrightarrow{\mathcal{G}}_2)=(\mathcal{A}_2,\ \mathcal{S}_2)$ *as*

1. *If* $X_1\cap X_2=\emptyset$*, then* $\mathfrak{C}(\overrightarrow{\mathcal{G}}_1\cup\overrightarrow{\mathcal{G}}_2)=\mathfrak{C}(\overrightarrow{\mathcal{G}}_1)\cup\mathfrak{C}(\overrightarrow{\mathcal{G}}_2)$.

2. *If* $X_1\cap X_2\neq\emptyset$*, then* $\mathcal{S}:E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^*}\cup E_{\mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*}\cup\breve{E}\to[0,\ 1]\times[0,\ 1]$ *is an IFS, where*

$$\breve{E}=\{a_1a_2|e\in X_1\cap X_2\quad and\quad \overrightarrow{a_1e}\in\overrightarrow{E},\overrightarrow{a_2e}\notin\overrightarrow{E},\overrightarrow{a_2e}\in\overrightarrow{E},\overrightarrow{a_2e}\notin\overrightarrow{E}\}.$$

3. $\phi_{\mathcal{S}}(ab)=[(\phi_{\mathcal{A}_1}(a)\vee\phi_{\mathcal{A}_2}(a))\wedge(\phi_{\mathcal{A}_1}(b)\vee\phi_{\mathcal{A}_2}(b))]$ $\times\max_c\{(\phi_{\overrightarrow{\mathcal{B}_1}}(ac)\vee\phi_{\overrightarrow{\mathcal{B}_2}}(ac))\wedge(\phi_{\overrightarrow{\mathcal{B}_1}}(bc)\vee\phi_{\overrightarrow{\mathcal{B}_2}}(bc))|c\in\mathfrak{N}^+(a)\cap\mathfrak{N}^+(b)\}$,
   $ab\in E_{\mathfrak{C}(\mathcal{G}_1)^*}\cup E_{\mathfrak{C}(\mathcal{G}_2)^*}$.

4. $\psi_{\mathcal{S}}(ab) = [(\psi_{\mathcal{A}_1}(a) \wedge \psi_{\mathcal{A}_2}(a)) \vee (\psi_{\mathcal{A}_1}(b) \wedge \psi_{\mathcal{A}_2}(b))] \times \max_c\{(\psi_{\overrightarrow{\mathcal{B}_1}}(ac) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(ac)) \wedge (\psi_{\overrightarrow{\mathcal{B}_1}}(bc) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(bc)) | c \in \mathfrak{N}^+(a) \cap \mathfrak{N}^+(b)\}$, $ab \in E_{\mathfrak{C}(\mathcal{G}_1)^*} \cup E_{\mathfrak{C}(\mathcal{G}_2)^*}$.

5. $\phi_{\mathcal{S}}(ab) = [(\phi_{\mathcal{A}_1}(a) \vee \phi_{\mathcal{A}_2}(a)) \wedge (\phi_{\mathcal{A}_1}(b) \vee \phi_{\mathcal{A}_2}(b))] \times \max_c\{(\phi_{\overrightarrow{\mathcal{B}_1}}(ae) \vee \phi_{\overrightarrow{\mathcal{B}_2}}(be)) | e \in \mathfrak{N}^+(a) \cap \mathfrak{N}^+(b)\}$, $ab \in \breve{E}$.

6. $\psi_{\mathcal{S}}(ab) = [(\psi_{\mathcal{A}_1}(a) \wedge \psi_{\mathcal{A}_2}(a)) \vee (\psi_{\mathcal{A}_1}(b) \wedge \psi_{\mathcal{A}_2}(b))] \times \max_c\{(\psi_{\overrightarrow{\mathcal{B}_1}}(ae) \vee \psi_{\overrightarrow{\mathcal{B}_2}}(be)) | e \in \mathfrak{N}^+(a) \cap \mathfrak{N}^+(b)\}$, $ab \in \breve{E}$.

Table 9: Adjacent edges of $\mathcal{G}_{\mathfrak{C}(\overrightarrow{\mathcal{G}_1})^* \Box \mathfrak{C}(\overrightarrow{\mathcal{G}_2})^*} \cup \mathcal{G}^{\Box}$

$(x_1, x_2)(y_1, y_2)$	$\mathcal{S}(x_1, x_2)(y_1, y_2)$
$(a_1, a_2)(a_1, b_2)$	$(\phi_1(a_1) \wedge \phi_2(a_2) \wedge \phi_2(b_2), \psi_1(a_1) \vee \psi_2(a_2) \vee \psi_2(b_2))$ $\times(\phi_1(a_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(b_2c_2), \psi_1(a_1) \vee \psi_2(a_2c_2) \vee \psi_2(b_2c_2))$ $= (0.04, 0.25)$
$(b_1, a_2)(b_1, b_2)$	$(\phi_1(b_1) \wedge \phi_2(a_2) \wedge \phi_2(b_2), \psi_1(b_1) \vee \psi_2(a_2) \vee \psi_2(b_2))$ $\times(\phi_1(b_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(b_2c_2), \psi_1(b_1) \vee \psi_2(a_2c_2) \vee \psi_2(b_2c_2))$ $= (0.04, 0.16)$
$(c_1, a_2)(c_1, b_2)$	$(\phi_1(c_1) \wedge \phi_2(a_2) \wedge \phi_2(b_2), \psi_1(c_1) \vee \psi_2(a_2) \vee \psi_2(b_2))$ $\times(\phi_1(c_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(b_2c_2), \psi_1(c_1) \vee \psi_2(a_2c_2) \vee \psi_2(b_2c_2))$ $= (0.04, 0.16)$
$(d_1, a_2)(d_1, b_2)$	$(\phi_1(d_1) \wedge \phi_2(a_2) \wedge \phi_2(b_2), \psi_1(d_1) \vee \psi_2(a_2) \vee \psi_2(b_2))$ $\times(\phi_1(d_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(b_2c_2), \psi_1(d_1) \vee \psi_2(a_2c_2) \vee \psi_2(b_2c_2))$ $= (0.04, 0.16)$
$(a_1, a_2)(c_1, a_2)$	$(\phi_1(a_1) \wedge \phi_1(c_1) \wedge \phi_2(a_2), \psi_1(a_1) \vee \psi_1(c_1) \vee \psi_2(a_2))$ $\times(\phi_2(a_2) \wedge \phi_1(a_1b_1) \wedge \phi_1(c_1b_1), \psi_2(a_2) \vee \psi_1(a_1b_1) \vee \psi_1(c_1b_1))$ $= (0.04, 0.20)$
$(a_1, b_2)(c_1, b_2)$	$(\phi_1(a_1) \wedge \phi_1(c_1) \wedge \phi_2(b_2), \psi_1(a_1) \vee \psi_1(c_1) \vee \psi_2(b_2))$ $\times(\phi_2(b_2) \wedge \phi_1(a_1b_1) \wedge \phi_1(c_1b_1), \psi_2(b_2) \vee \psi_1(a_1b_1) \vee \psi_1(c_1b_1))$ $= (0.09, 0.20)$
$(a_1, c_2)(c_1, c_2)$	$(\phi_1(a_1) \wedge \phi_1(c_1) \wedge \phi_2(c_2), \psi_1(a_1) \vee \psi_1(c_1) \vee \psi_2(c_2))$ $\times(\phi_2(c_2) \wedge \phi_1(a_1b_1) \wedge \phi_1(c_1b_1), \psi_2(c_2) \vee \psi_1(a_1b_1) \vee \psi_1(c_1b_1))$ $= (0.09, 0.20)$
$(b_1, a_2)(a_1, c_2)$	$(\phi_1(b_1) \wedge \phi_1(a_1) \wedge \phi_2(a_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(a_1) \vee \psi_2(a_2) \vee \psi_2(c_2))$ $\times(\phi_1(b_1) \wedge \phi_1(a_1b_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(a_1b_1) \vee \psi_2(a_2c_2) \vee \psi_2(c_2))$ $= (0.04, 0.20)$
$(b_1, a_2)(c_1, c_2)$	$(\phi_1(b_1) \wedge \phi_1(c_1) \wedge \phi_2(a_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(c_1) \vee \psi_2(a_2) \vee \psi_2(c_2))$ $\times(\phi_1(b_1) \wedge \phi_1(c_1b_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(c_1b_1) \vee \psi_2(a_2c_2) \vee \psi_2(c_2))$ $= (0.04, 0.16)$
$(b_1, b_2)(a_1, c_2)$	$(\phi_1(b_1) \wedge \phi_1(a_1) \wedge \phi_2(b_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(a_1) \vee \psi_2(b_2) \vee \psi_2(c_2))$ $\times(\phi_1(b_1) \wedge \phi_1(a_1b_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(a_1b_1) \vee \psi_2(a_2c_2) \vee \psi_2(c_2))$ $= (0.12, 0.20)$
$(b_1, b_2)(c_1, c_2)$	$(\phi_1(b_1) \wedge \phi_1(c_1) \wedge \phi_2(b_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(c_1) \vee \psi_2(b_2) \vee \psi_2(c_2))$ $\times(\phi_1(b_1) \wedge \phi_1(c_1b_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(c_2), \psi_1(b_1) \vee \psi_1(c_1b_1) \vee \psi_2(a_2c_2) \vee \psi_2(c_2))$ $= (0.09, 0.16)$
$(c_1, a_2)(d_1, c_2)$	$(\phi_1(c_1) \wedge \phi_1(d_1) \wedge \phi_2(a_2) \wedge \phi_2(c_2), \psi_1(c_1) \vee \psi_1(d_1) \vee \psi_2(a_2) \vee \psi_2(c_2))$ $\times(\phi_1(c_1) \wedge \phi_1(d_1c_1) \wedge \phi_2(a_2c_2) \wedge \phi_2(c_2), \psi_1(c_1) \vee \psi_1(d_1c_1) \vee \psi_2(a_2c_2) \vee \psi_2(c_2))$ $= (0.04, 0.16)$
$(c_1, b_2)(d_1, c_2)$	$(\phi_1(c_1) \wedge \phi_1(d_1) \wedge \phi_2(b_2) \wedge \phi_2(c_2), \psi_1(c_1) \vee \psi_1(d_1) \vee \psi_2(b_2) \vee \psi_2(c_2))$ $\times(\phi_1(c_1) \wedge \phi_1(d_1c_1) \wedge \phi_2(b_2c_2) \wedge \phi_2(c_2), \psi_1(c_1) \vee \psi_1(d_1c_1) \vee \psi_2(b_2c_2) \vee \psi_2(c_2))$ $= (0.09, 0.16)$

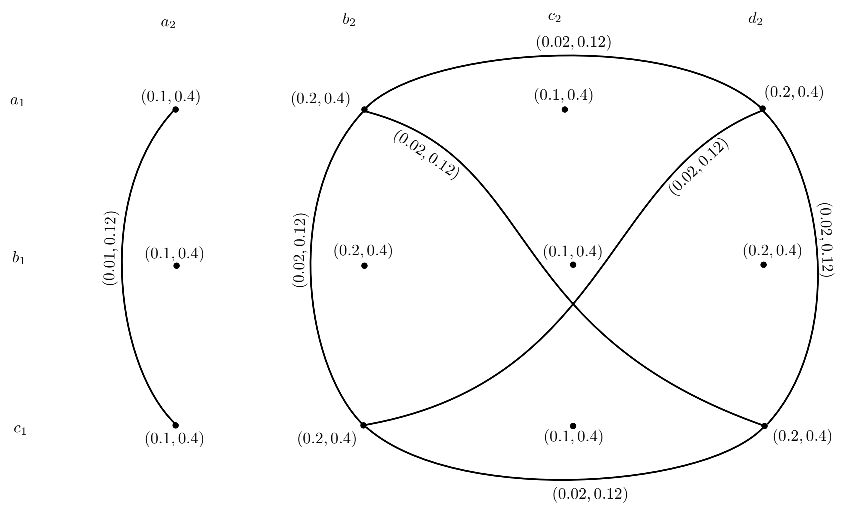

Figure 12: $\mathfrak{C}(\overrightarrow{\mathcal{G}_1} \times \overrightarrow{\mathcal{G}_2})$

Now we introduce the concepts of intuitionistic fuzzy open neighbourhood and intuitionistic fuzzy closed neighbourhood graphs.

Definition 2.4. Let $\mathcal{G} = (X, \mathcal{A}, \mathcal{B})$ be an IFG. An intuitionistic fuzzy open neighbourhood $\mathfrak{N}(a)$ of a vertex a in $\mathcal{G}$ is an IFS $\mathfrak{N}(a) = (X_a, \phi_{\mathcal{A}_a}, \psi_{\mathcal{A}_a})$, where $X_a = \{b|\phi_{\mathcal{B}}(ab) > 0, \psi_{\mathcal{B}}(ab) > 0\}$ and $\phi_{\mathcal{A}_a} : X_a \to [0, 1]$ and $\psi_{\mathcal{B}_a} : X_a \to [0, 1]$ are the membership functions stated as $\phi_{\mathcal{A}_a}(a) = \phi_{\mathcal{B}}(ab)$ and $\psi_{\mathcal{A}_a}(a) = \psi_{\mathcal{B}}(ab)$. The intuitionistic fuzzy closed neighbourhood $\mathfrak{N}[a]$ is defined as $\mathfrak{N}[a] = \mathfrak{N}(a) \cup \{(a, \phi(a), \psi(a))\}$.

Definition 2.5. Let $\mathcal{G} = (X, \mathcal{A}, \mathcal{B})$ be an IFG. An intuitionistic fuzzy open neighbourhood graph of $\mathcal{G}$ is an IFG $\mathfrak{N}(\mathcal{G}) = (X, \mathcal{A}, \mathcal{B}')$ whose set of vertices is same as the set of vertices of $\mathcal{G}$ and there is an edge between two vertices a and b if $\mathfrak{N}(a) \cap \mathfrak{N}(a) \neq \emptyset$ and $\mathcal{B}' = (X, \phi_{\mathcal{B}'}, \psi_{\mathcal{B}'})$ is an IFS, where $\phi_{\mathcal{B}'} : X \times X \to [0, 1]$ and $\psi_{\mathcal{B}'} : X \times X \to [0, 1]$ are the membership functions stated as

$$\begin{aligned}\phi_{\mathcal{B}'}(ab) &= (\phi_{\mathcal{A}}(a) \wedge \phi_{\mathcal{A}}(b)) \times h(\mathfrak{N}(a) \cap \mathfrak{N}(b)),\\ \psi_{\mathcal{B}'}(ab) &= (\psi_{\mathcal{A}}(a) \vee \psi_{\mathcal{A}}(b)) \times h(\mathfrak{N}(a) \cap \mathfrak{N}(b)), \quad \text{for all } a, b \in X.\end{aligned}$$

Similarly, we can define an intuitionistic fuzzy closed neighbourhood graph as:

Definition 2.6. Let $\mathcal{G} = (X, \mathcal{A}, \mathcal{B})$ be an IFG. An intuitionistic fuzzy closed neighbourhood graph of $\mathcal{G}$ is an IFG $\mathfrak{N}[\mathcal{G}] = (X, \mathcal{A}, \mathcal{B}')$ whose set of vertices is same as the set of vertices of $\mathcal{G}$ and there is an edge between two vertices a and b if $\mathfrak{N}[a] \cap \mathfrak{N}[b] \neq \emptyset$ and $\mathcal{B}' = (\phi_{\mathcal{B}'}, \psi_{\mathcal{B}'})$ is an IFS, where $\phi_{\mathcal{B}'} : X \times X \to [0, 1]$ and $\psi_{\mathcal{B}'} : X \times X \to [0, 1]$ are the membership functions stated as

$$\begin{aligned}\phi_{\mathcal{B}'}(ab) &= (\phi_{\mathcal{A}}(a) \wedge \phi_{\mathcal{A}}(b)) \times h(\mathfrak{N}[a] \cap \mathfrak{N}(b)),\\ \psi_{\mathcal{B}'}(ab) &= (\psi_{\mathcal{A}}(a) \vee \psi_{\mathcal{A}}(b)) \times h(\mathfrak{N}[a] \cap \mathfrak{N}[b]), \quad \text{for all } a, b \in X.\end{aligned}$$

Taking into account the intuitionistic fuzzy open and closed neighbourhood of the vertices, another type of IFGs are defined as under:

Definition 2.7. Let k be a non-negative number then an intuitionistic fuzzy (k)-competition graph of an IFG $\mathcal{G} = (X, \mathcal{A}, \mathcal{S})$ is an IFG $\mathfrak{N}_k(\mathcal{G}) = (X, \mathcal{A}, \mathcal{S}')$ which has the same vertex set as the set of vertices of $\mathcal{G}$ and there is an edge between two vertices a and b if $|\mathfrak{N}(a) \cap \mathfrak{N}(b)| > k$. The membership value of the edge ab is stated as

$$\begin{aligned}\phi_{\mathcal{S}'}(ab) &= \frac{l-k}{l}(\phi_{\mathcal{A}}(a) \wedge \phi_{\mathcal{A}}(b)) \times h(\mathfrak{N}(a) \cap \mathfrak{N}(b)),\\ \psi_{\mathcal{S}'}(ab) &= \frac{l-k}{l}(\psi_{\mathcal{A}}(a) \vee \psi_{\mathcal{A}}(b)) \times h(\mathfrak{N}(a) \cap \mathfrak{N}(b)), \quad \text{for all } a, b \in X,\end{aligned}$$

where $|\mathfrak{N}(a) \cap \mathfrak{N}(b)| = l$.

Definition 2.8. Let k be a non-negative number then an intuitionistic fuzzy $[k]$-competition graph of an IFG $\mathcal{G} = (X,\, \mathcal{A},\, \mathcal{S})$ is an IFG $\mathfrak{N}_k[\mathcal{G}] = (X,\, \mathcal{A},\, \mathcal{S}')$ which has the same vertex set as in $\mathcal{G}$ and there is an edge between two vertices a and b if $|\mathfrak{N}[a] \cap \mathfrak{N}[b]| > k$. The membership value of the edge ab is stated as

$$\begin{aligned}
\phi_{\mathcal{S}'}(ab) &= \frac{l-k}{l}(\phi_{\mathcal{A}}(a) \wedge \phi_{\mathcal{A}}(b)) \times h(\mathfrak{N}[a] \cap \mathfrak{N}[b]),\\
\psi_{\mathcal{S}'}(ab) &= \frac{l-k}{l}(\psi_{\mathcal{A}}(a) \vee \psi_{\mathcal{A}}(b)) \times h(\mathfrak{N}[a] \cap \mathfrak{N}[b]), \quad \text{for all } a, b \in X,
\end{aligned}$$

where $|\mathfrak{N}[a] \cap \mathfrak{N}[b]| = l$.

Theorem 2.4. *For every edge of an IFG $\mathcal{G}$, there exists one edge in $\mathfrak{N}[\mathcal{G}]$.*

Proof. Let $\mathfrak{N}[\mathcal{G}] = (X,\, \mathcal{A},\, \mathcal{S}')$ be an intuitionistic fuzzy closed neighbourhood graph corresponding to the IFG $\mathcal{G} = (\mathcal{A},\, \mathcal{B})$. Let ab be an edge in $\mathcal{G}$ then, $a,\, b \in \mathfrak{N}[a]$ and $a,\, b \in \mathfrak{N}[b]$. So, $a,\, b \in \mathfrak{N}[a] \cap \mathfrak{N}[b]$ and therefore, $h(\mathfrak{N}[a] \cap \mathfrak{N}[b]) \notin \emptyset$. Hence,

$$\begin{aligned}
\phi_{\mathcal{S}'}(ab) &= \frac{l-k}{l}(\phi_{\mathcal{A}}(a) \wedge \phi_{\mathcal{A}}(b)) \times h(\mathfrak{N}[a] \cap \mathfrak{N}[b]),\\
\psi_{\mathcal{S}'}(ab) &= \frac{l-k}{l}(\psi_{\mathcal{A}}(a) \vee \psi_{\mathcal{A}}(b)) \times h(\mathfrak{N}[a] \cap \mathfrak{N}[b]),
\end{aligned}$$

that is, ab is an edge of $\mathfrak{N}[\mathcal{G}]$.

□

Definition 2.9. Let $\overrightarrow{\mathcal{G}} = (X,\, \mathcal{A},\, \overrightarrow{\mathcal{B}})$ be an IFD of the crisp fuzzy digraph $\overrightarrow{\mathcal{G}}^* = (X,\, \mathcal{A},\, \overrightarrow{E})$. The underlying IFG of $\overrightarrow{\mathcal{G}}$ is an IFG, $\mathfrak{U}(\overrightarrow{\mathcal{G}}) = (X,\, \mathcal{A},\, \mathcal{B})$ where $\mathcal{D} = (\phi_{\mathcal{B}},\, \psi_{\mathcal{B}})$ is defined as

$$\mathcal{B}(ab) = (\phi_{\mathcal{B}}(ab), \psi_{\mathcal{B}}(ab)) = \begin{cases}
(\phi_{\overrightarrow{\mathcal{B}}}(ab), \psi_{\overrightarrow{\mathcal{B}}}(ab), & if \quad \overrightarrow{ab} \in \overrightarrow{E}, \overrightarrow{ba} \notin \overrightarrow{E}\\
(\phi_{\overrightarrow{\mathcal{B}}}(ba), \psi_{\overrightarrow{\mathcal{B}}}(ba), & if \quad \overrightarrow{ba} \in \overrightarrow{E}, \overrightarrow{ab} \notin \overrightarrow{E}\\
(\phi_{\overrightarrow{\mathcal{B}}}(ab), \phi_{\overrightarrow{\mathcal{B}}}(ba), (\psi_{\overrightarrow{\mathcal{B}}}(ab), \psi_{\overrightarrow{\mathcal{B}}}(ba), & if \quad \overrightarrow{ab}, \overrightarrow{ba} \in \overrightarrow{E}.
\end{cases}$$

We now illustrate the relations between intuitionistic fuzzy neighbourhood graphs(IFNGs) and IFCGs.

Theorem 2.5. *Let $\overrightarrow{\mathcal{G}} = (\mathcal{A}, \overrightarrow{\mathcal{B}})$ be a symmetric IFD without any loops then, $\mathfrak{C}_k(\overrightarrow{\mathcal{G}}) = \mathfrak{N}_k(\mathfrak{U}(\overrightarrow{\mathcal{G}}))$, where $\mathfrak{U}(\overrightarrow{\mathcal{G}})$ is an underlying IFG of $\overrightarrow{\mathcal{G}}$.*

Theorem 2.6. *Let $\overrightarrow{\mathcal{G}} = (\mathcal{A}, \overrightarrow{\mathcal{B}})$ be a symmetric IFD having loops at every vertex then, $\mathfrak{C}_k(\overrightarrow{\mathcal{G}}) = \mathfrak{N}_k[\mathfrak{U}(\overrightarrow{\mathcal{G}})]$, where $\mathfrak{U}(\overrightarrow{\mathcal{G}})$ is an underlying IFG of $\overrightarrow{\mathcal{G}}$.*

3 Application

Competition graphs are starting point of interesting graph-theoretical concepts to represent the competition between objects. To cover all the competitions in real world, these graphical representations are insufficient. Therefore, we apply intuitionistic fuzzy competition graphs to study the strength of competition between objects.

Local governments are the public administration in districts, towns and cities to govern a particular area. In the cities of various countries local governments are very common. Since, the people of specific region can easily contact to the local government than the federal government, therefore, local governments have their own importance in that region. The criterion for the selection of local government officials in different countries is not same. There are direct pubic elections in some countries and some countries declare the local government officials by the judgement of a governing council. Local government has five common seats such as city manager, city council member, county commissioner, city attorney and mayor.

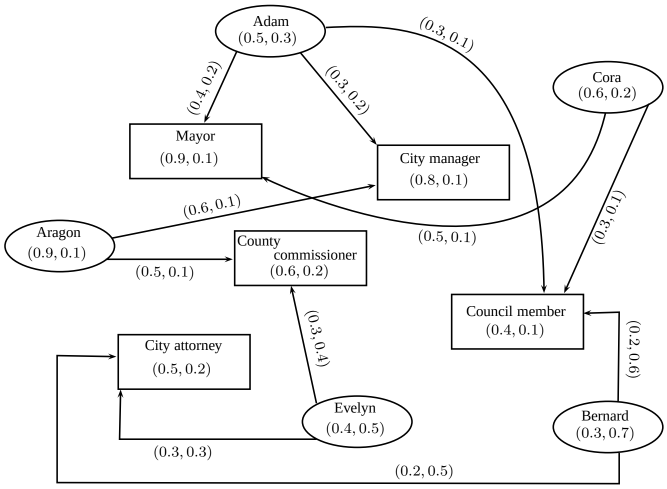

Figure 13: Intuitionistic fuzzy political digraph

Let us take the example of five competitors in a country named as Adam, Bernard, Aragon, Cora and Evelyn for different political seats. The corresponding IFD is shown in Fig. 13. The membership degree of each competitor represents the degree of leadership quality and non-membership degree indicates the non-leadership quality to manage social issues. The competitor characteristics can be written as:

$$\{\text{leadership, non-leadership}\}.$$

The membership degree of each directed edge between a competitor and a position indicates the degree of eligibility for the particular seat. The non-membership degree expresses the non-eligibilty for that seat. The membership properties can be represented as:

$$\{\text{eligibility, non-eligibilty}\}.$$

The membership and non-membership degrees of each political seat indicate the average past leadership and non-leadership quality of competitors on these seats. We construct an IFCG to interpret the competition between competitors for political seat. The IFONs are given in Table 10.

Table 10: IFONs of competitors

x	$\mathfrak{N}^+(x)$
Adam	$\{Mayor(0.4, 0.2), Citymanager(0.3, 0.2), Councilmember(0.3, 0.1)\}$
Cora	$\{Mayor(0.5, 0.1), Councilmember(0.3, 0.1)\}$
Bernard	$\{Cityattorney(0.2, 0.5), Councilmember(0.2, 0.6)\}$
Evelyn	$\{Cityattorney(0.3, 0.3), Countycommissioner(0.3, 0.4)\}$
Aragon	$\{Citymanager(0.6, 0.1), Countycommissioner(0.5, 0.1)\}$

The intuitionistic fuzzy competition graph is presented in Fig. 14. Here, dashed lines and solid lines represent the competition of competitors for particular seats and the strength of competition between two competitors, respectively. For instance, Adam and Cora are in competition for the seat of mayor. The strength of competition between them is (0.20, 0.03). $R(x, q)$, in Table 11, expresses the competition of competitor x for seat q with respect to leadership and non-leadership quality to handle the social issues. Table 11 represents the strength of competition between competitors for particular seats. From Table 10, Adam and Aragon have same strength of competition for the seat of city manager, Bernard for council member, Evelyn and Bernard have same strength of competition for the seat of city attorney, Aragon and Evelyn have same strength of competition for the seat of county commissioner.

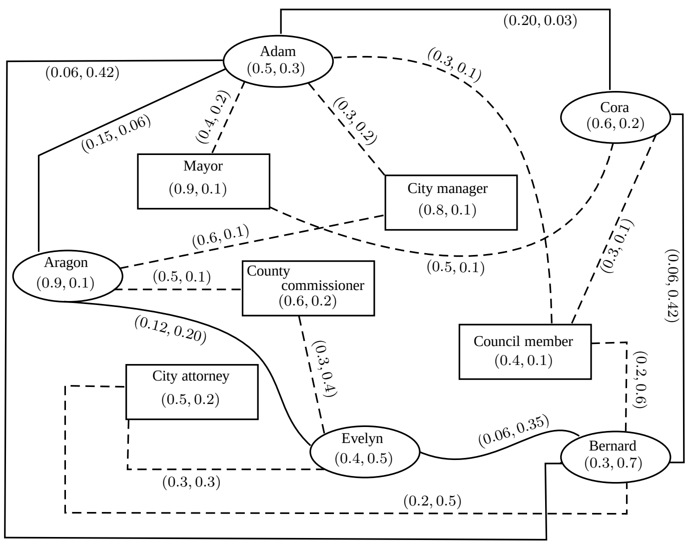

Figure 14: IFCG

Table 11: Strength of competition between competitors for political seats

$(competitor, seat)$	in competition	R$(competitor, seat)$	S$(competitor, seat)$
$(Adam, Mayor)$	$Cora$	(0.20, 0.03)	0.2231
$(Cora, Mayor)$	$Adam$	(0.20, 0.03)	0.2231
$(Adam, Citymanager)$	$Aragon$	(0.15, 0.06)	0.1974
$(Aragon, Citymanager)$	$Adam$	(0.15, 0.06)	0.1974
$(Adam, Councilmember)$	$Cora, Bernard$	(0.13, 0.225)	0.2751
$(Cora, Councilmember)$	$Adam, Bernard$	(0.13, 0.225)	0.2751
$(Bernard, Councilmember)$	$Adam, Cora$	(0.06, 0.42)	0.2784
$(Evelyn, Cityattorney)$	$Bernard$	(0.06, 0.35)	0.2665
$(Bernard, Cityattorney)$	$Evelyn$	(0.06, 0.35)	0.2665
$(Aragon, Countycommissioner)$	$Evelyn$	(0.12, 0.20)	0.3360
$(Evelyn, Countycommissioner)$	$Aragon$	(0.12, 0.20)	0.3360

Acknowledgements

The authors are thankful to the Editor-in-Chief and the referee for their valuable comments and suggestions.

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