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On Similarity Measures for Fuzzy Sets with Applications to Pattern Recognition, Decision Making, Clustering, and Approximate Reasoning

G. Hesamian[1], J. Chachi[2,*]
[1] *Department of Statistics, Payame Noor University, Tehran 19395-3697, Iran*
[2] *Department of Mathematics, Statistics and Computer Sciences*
Semnan University, Semnan 35195-363, Iran

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Abstract

The paper deals with the well-known notion of similarity measures between fuzzy sets and proposes a variety of similarity measures for fuzzy sets. Moreover, the paper proves that the proposed measures satisfy the properties of the axiomatic definition for similarity measures and introduces several theorems. Several illustrative and practical examples from the areas of pattern recognition, decision making, fuzzy clustering analysis, and approximate reasoning system will be used to present the calculation of these similarity measures between fuzzy sets. The paper concludes with suggestions for future research.

Keywords: approximate reasoning system, decision making, fuzzy clustering analysis, pattern recognition, similarity measure

1 Introduction

Zadeh [47] initiated fuzzy sets which describe everything as a matter of degree and can be used to capture the uncertainty in imprecision and vagueness in a mathematical way. Since then, fuzzy set theory has been successfully used in various fields as a powerful tool for processing imprecise or vague information, e.g., decision making, approximate reasoning, logic programming, fuzzy risk analysis, pattern recognition, and cluster analysis, fuzzy time series, medical diagnosis (see, for instance, [6, 10, 20, 30, 31, 34, 39, 45]). Lots of these studies have been done on the issue of similarity measures. A similarity between fuzzy sets is an important way to measure the degree of similarity between two fuzzy concepts. In this regard, many researchers have conducted extensive studies on similarity measures between fuzzy sets. Zwick et al. [54] reviewed and compared several similarity measures between fuzzy sets based on both geometric and set-theoretic ways. Pappis and Karacapilidis [35] introduced three similarity measures between fuzzy sets. After that, some researchers (see, for instance, [1, 2, 3, 5, 9, 17, 21, 25, 26, 33, 40, 44, 48]) gave more similarity measures in fuzzy environment. The literature about this kind of measure can be divided into two main blocks:

1. The papers based on introducing axiomatic definitions (see, for instance, [5, 21, 24, 38]);

2. The papers based on introducing or reviewing some measures or some parametric families of measures (see, for instance, [8, 12, 13, 14, 19, 20]). These kind of papers were mostly applied to specific subsequent problems.

In this paper, we will focus on the former (those ones devoted to axiomatic definitions) for introducing new definitions of similarity measures of fuzzy sets. Note that, most of the axiomatic definitions have been introduced independently from each other. They have been proposed in different contexts, with different purposes and using different nomenclatures. In this regard, Couso et al. [11] provided a formal study relating

*Corresponding author.
Emails: gh.hesamian@chb.pnu.ac.ir (G. Hesamian), jchachi@semnan.ac.ir (J. Chachi).

different axiomatic definitions of "similarity" between fuzzy sets (as well as some other dual definitions such as "dissimilarity", "equality", "inequality" or "difference" measures). They presented all these axioms in a uniform framework and showed the formal relationships among them.

The main idea of this paper proposes similarity measures between fuzzy sets. In this regard, based on the axioms provided by Couso et al. [11], we propose new classes of similarity measures between fuzzy sets in this paper. We also investigate the properties of these measures. For practical reasons, we would explain similarity measures between fuzzy sets by examples in pattern recognition and decision making. We then combine the proposed similarity measures with Yang and Shih's [45] algorithm for clustering fuzzy data. The clustering results are logically expressed in a hierarchical tree structure. We also validate the effectiveness of the proposed similarity measures in a bidirectional approximate reasoning system.

The remainder of this paper is organized as follows. In the next section, a brief review of fuzzy sets is given. In Section 3, new definitions and relevant properties with respect to similarity measures between fuzzy sets will be proposed and discussed. Several illustrative and practical examples from the areas of pattern recognition, decision making, fuzzy clustering analysis, and approximate reasoning system will be presented in Section 4. Finally, in Section 5, the paper concludes with suggestions for future research.

2 Preliminaries

Throughout this paper, the following definitions and notations are used for convenience of explaining general concepts concerned with fuzzy sets.

A fuzzy set $\widetilde{A}$ of the universal set $\mathbb{X}$ is defined by its membership function $\widetilde{A}(x) : \mathbb{X} \to [0,1]$ [47]. In this paper, we consider $\mathbb{R}$ (the real line) as the universal set. We denote by $\widetilde{A}_\alpha = \{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\}$ the α-level set (α-cut) of the fuzzy set $\widetilde{A}$, for every $\alpha \in (0,1]$, and for $\alpha = 0$, $\widetilde{A}_0$ is the closure of the set $\{x \in \mathbb{R} : \widetilde{A}(x) > 0\}$. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if for every $\alpha \in [0,1]$, the set $\widetilde{A}_\alpha$ is a non-empty compact interval. Such an interval will be denoted by $\widetilde{A}_\alpha = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}_\alpha\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}_\alpha\}$. We denote by $\mathcal{F}(\mathbb{R})$, the set of all fuzzy numbers of $\mathbb{R}$.

The imprecision or vagueness can be treated by means of a particular kind (family) of fuzzy numbers, the LR-fuzzy numbers. These are very useful in practice since they can be characterized by means of three real numbers: the center, the left spread, and the right spread. The term LR is due to the left (L) and the right (R) shape of the membership function referred to the fuzzy set [53]. Typically, the LR fuzzy number $\widetilde{N} = (n, l, r)_{LR}$ with central value $n \in \mathbb{R}$, left and right spreads $l \in \mathbb{R}^+$, $r \in \mathbb{R}^+$, decreasing left and right shape functions $L : \mathbb{R}^+ \to [0,1]$, $R : \mathbb{R}^+ \to [0,1]$, with $L(0) = R(0) = 1$, has the following membership function [53]

$$\widetilde{N}(x) = \begin{cases} L(\frac{n-x}{l}) & \text{if} \quad x \leq n, \\ R(\frac{x-n}{r}) & \text{if} \quad x \geq n. \end{cases}$$

We can easily obtain the α-cut of $\widetilde{N}$ as follows

$$\widetilde{N}_\alpha = [\widetilde{N}^L_\alpha, \widetilde{N}^U_\alpha] = [n - L^{-1}(\alpha)l, n + R^{-1}(\alpha)r], \quad \alpha \in [0,1].$$

An LR-fuzzy number $\widetilde{N} = (n, l, r)_{LR}$ with $L = R$ and $l = r = \lambda$ is called symmetric and is abbreviated by $\widetilde{N} = (n, \lambda)_L$.

In practice, it is usually preferred to use simple shapes for functions L and R such as triangular, i.e. $L(x) = R(x) = \max\{1 - x, 0\}$. The membership function and the α-cut of the triangular fuzzy number $\widetilde{A} = (a, l, r)_T$ are given by

$$\begin{aligned} \widetilde{A}(x) &= \begin{cases} \frac{x-(a-l)}{l} & x \in [a-l, a], \\ \frac{(a+r)-x}{r} & x \in [a, a+r], \\ 0 & x \notin [a-l, a+r]. \end{cases} \\ \widetilde{A}_\alpha &= [a - (1-\alpha)l, a + (1-\alpha)r], \qquad \alpha \in [0,1], \end{aligned}$$

The following operators on fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are used in this paper [15, 16].

1. $\widetilde{A}=\widetilde{B}$ if and only if $\widetilde{A}(x)=\widetilde{B}(x)$, for all $x\in\mathbb{R}$.
2. $\widetilde{A}\subseteq\widetilde{B}$ if and only if $\widetilde{A}(x)\leq\widetilde{B}(x)$, for all $x\in\mathbb{R}$, which is equivalent to $\widetilde{A}_\alpha\subseteq\widetilde{B}_\alpha$ for every $\alpha\in[0,1]$.
3. $\widetilde{A}^c$ is the complement of fuzzy number $\widetilde{A}$ with the membership function $\widetilde{A}^c(x)=1-\widetilde{A}(x)$, for all $x\in\mathbb{R}$.
4. $\widetilde{A}\cup\widetilde{B}$ is the union of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ with the membership function $(\widetilde{A}\cup\widetilde{B})(x)=\max\{\widetilde{A}(x),\widetilde{B}(x)\}$ for all $x\in\mathbb{R}$.
5. $\widetilde{A}\cap\widetilde{B}$ is the intersection of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ with the membership function $(\widetilde{A}\cap\widetilde{B})(x)=\min\{\widetilde{A}(x),\widetilde{B}(x)\}$, for all $x\in\mathbb{R}$.

3 Proposed Similarity Measures Between Fuzzy Sets

In this section, we will propose new similarity measures between fuzzy sets, and then discuss by several theorems that the proposed measures satisfy the properties of the axiomatic definition provided by Couso et al. [11] for similarity measures.

Definition 3.1. *The similarity measure $S_1:\mathcal{F}(\mathbb{R})\otimes\mathcal{F}(\mathbb{R})\to\mathbb{R}$ between two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ is defined as follows*

$$S_1(\widetilde{A},\widetilde{B})=f\left(\int_0^1|\widetilde{A}_\alpha\triangle\widetilde{B}_\alpha|d\alpha\right),$$

in which $\widetilde{A}_\alpha\triangle\widetilde{B}_\alpha$ denotes the symmetric difference of two ordinary sets $\widetilde{A}_\alpha$ and $\widetilde{B}_\alpha$ (we recall that the symmetric difference between two ordinary sets A and B is defined as

$$\begin{aligned}A\triangle B &= (A\cup B)-(A\cap B)\\ &= \{\theta-\lambda|\theta\in(A\cup B)\ and\ \lambda\in(A\cap B)\},\end{aligned}$$

i.e. the set-theoretic difference between their union and their intersection), $|A|$ denotes the length of closed interval $A\subseteq\mathbb{R}$ and $f:\mathbb{R}^+\to[0,1]$ is a strictly decreasing function with $f(0)=1$ (for instance, $f(x)=(1-x)/(1+x)$, $f(x)=1/(1+x^p)$ or $f(x)=e^{-x^p}$, $p>0$).

Theorem 3.1. *Let $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ be three fuzzy numbers in $\mathcal{F}(\mathbb{R})$, then the proposed similarity measure S_1 satisfies the following properties:*

1. $S_1(\widetilde{A},\widetilde{B})\in[0,1]$.
2. $S_1(\widetilde{A},\widetilde{B})=S_1(\widetilde{B},\widetilde{A})$.
3. $S_1(\widetilde{A},\widetilde{B})=1$ *if and only if* $\widetilde{A}=\widetilde{B}$.
4. *If* $\widetilde{A}\subseteq\widetilde{B}\subseteq\widetilde{C}$, *then* $S_1(\widetilde{A},\widetilde{C})\leq\min\{S_1(\widetilde{A},\widetilde{B}),S_1(\widetilde{B},\widetilde{C})\}$.
5. $S_1(\widetilde{A}\cap\widetilde{B},\widetilde{A}\cup\widetilde{B})=S_1(\widetilde{A},\widetilde{B})$.
6. $S_1(\widetilde{A}^c,\widetilde{B}^c)=S_1(\widetilde{A},\widetilde{B})$.
7. $S_1(\widetilde{A}\cap\widetilde{C},\widetilde{B}\cap\widetilde{C})\geq S_1(\widetilde{A},\widetilde{B})$.
8. $S_1(\widetilde{A}\cup\widetilde{C},\widetilde{B}\cup\widetilde{C})\geq S_1(\widetilde{A},\widetilde{B})$.

Proof. 1. It is immediately derived from the properties of the function f.

2. The result can easily be checked because $|\widetilde{A}_\alpha\triangle\widetilde{B}_\alpha|=|\widetilde{B}_\alpha\triangle\widetilde{A}_\alpha|$, for every $\alpha\in[0,1]$.

3. If $\widetilde{A} = \widetilde{B}$, then, $S(\widetilde{A}, \widetilde{B}) = f(\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha) = f(0) = 1$. Conversely, assume that $S(\widetilde{A}, \widetilde{B}) = 1$. It is clear that $\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha = f^{-1}(1) = 0$. Therefore, $|\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| = 0$, for each $\alpha \in [0, 1]$, which concludes that $\widetilde{A}_\alpha = \widetilde{B}_\alpha$ for each $\alpha \in [0, 1]$, or equivalently $\widetilde{A} = \widetilde{B}$.

4. Let $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$ be three arbitrary nested fuzzy numbers, then $\widetilde{A}_\alpha \subseteq \widetilde{B}_\alpha \subseteq \widetilde{C}_\alpha$, for every $\alpha \in [0, 1]$. We can prove that for every $\alpha \in [0, 1]$, $\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha \subseteq \widetilde{A}_\alpha \triangle \widetilde{C}_\alpha$, and $\widetilde{B}_\alpha \triangle \widetilde{C}_\alpha \subseteq \widetilde{A}_\alpha \triangle \widetilde{C}_\alpha$. Therefore, we obtain the inequalities $|\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| \leq |\widetilde{A}_\alpha \triangle \widetilde{C}_\alpha|$, and $|\widetilde{B}_\alpha \triangle \widetilde{C}_\alpha| \leq |\widetilde{A}_\alpha \triangle \widetilde{C}_\alpha|$. The proof can easily be checked because f is a strictly decreasing function.

5. For every $\alpha \in [0, 1]$, because $(\widetilde{A}_\alpha \cup \widetilde{B}_\alpha) \triangle (\widetilde{A}_\alpha \cap \widetilde{B}_\alpha) = \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha$, therefore, we can get the result.

6. It can be proved that $(\widetilde{A}_\alpha)^c \triangle (\widetilde{B}_\alpha)^c = \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha$, for every $\alpha \in [0, 1]$.

7. For given fuzzy numbers $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, note that

$$(\widetilde{A}_\alpha \cap \widetilde{C}_\alpha) \triangle (\widetilde{B}_\alpha \cap \widetilde{C}_\alpha) \subseteq \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha.$$

Therefore, we obtain the following inequality

$$\int_0^1 |(\widetilde{A}_\alpha \cap \widetilde{C}_\alpha) \triangle (\widetilde{B}_\alpha \cap \widetilde{C}_\alpha)| \, d\alpha \leq \int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| \, d\alpha.$$

The proof can easily be completed, because f is a strictly decreasing function.

8. For every $\alpha \in [0, 1]$, we can obtain that

$$(\widetilde{A}_\alpha \cup \widetilde{C}_\alpha) \triangle (\widetilde{B}_\alpha \cup \widetilde{C}_\alpha) \subseteq \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha.$$

Now, similar to item 7, the proof can be completed.

□

Definition 3.2. *The similarity measure $S_2 : \mathcal{F}(\mathbb{R}) \otimes \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ between two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ is defined as follows*

$$S_2(\widetilde{A}, \widetilde{B}) = \frac{\int_0^1 |\widetilde{A}_\alpha \cap \widetilde{B}_\alpha| d\alpha}{\int_0^1 |\widetilde{A}_\alpha \cup \widetilde{B}_\alpha| d\alpha}.$$

Note that, in the above definition, we assumed $\min\{|\widetilde{A}_1|, |\widetilde{B}_1|\} > 0$. Otherwise, the bounds of the integrals are restricted to $(0, 1)$.

Theorem 3.2. *The proposed similarity measure S_2 meats the following properties:*

1. $S_2(\widetilde{A}, \widetilde{B}) \in [0, 1]$.

2. $S_2(\widetilde{A}, \widetilde{B}) = S_2(\widetilde{B}, \widetilde{A})$.

3. $S_2(\widetilde{A}, \widetilde{B}) = 1$ *if and only if* $\widetilde{A} = \widetilde{B}$.

4. *If* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, *then* $S_2(\widetilde{A}, \widetilde{C}) \leq \min\{S_2(\widetilde{A}, \widetilde{B}), S_2(\widetilde{B}, \widetilde{C})\}$.

5. *If* $\widetilde{A} \cap \widetilde{B} = \emptyset$ *then* $S_2(\widetilde{A}, \widetilde{B}) = 0$, *otherwise* $S_2(\widetilde{A}, \widetilde{B}) > 0$.

6. $S_2(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = S_2(\widetilde{A}, \widetilde{B})$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Definition 3.3. *Let $\mathbb{X} = \{x_1, x_2, \ldots, x_k\}$ and $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy sets of $\mathbb{X}$. The similarity measure between two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ is defined as follows*

$$S_3(\widetilde{A}, \widetilde{B}) = f(\sup_{\alpha\in[0,1]} |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha|),$$

where f is defined as in Definition 3.1 and $|A|$ denotes the total number of elements in set $A \subseteq \mathbb{X}$.

Remark 3.1. *One can verify that S_3 satisfies the items 1-8 in Theorem 3.1, including the following property:*

9. *$S_3(A, A^c) = 0$, for a crisp set A.*

Definition 3.4. *Let $\mathbb{X} = \{x_1, x_2, \ldots, x_k\}$ and $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy sets of $\mathbb{X}$. The similarity measure between two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ is defined as follows*

$$S_4(\widetilde{A}, \widetilde{B}) = \inf_{\alpha\in[0,1]} \frac{|\widetilde{A}_\alpha \cap \widetilde{B}_\alpha|}{|\widetilde{A}_\alpha \cup \widetilde{B}_\alpha|},$$

where $|A|$ denotes the total number of elements in set $A \subseteq \mathbb{X}$.

Remark 3.2. *Similar to Theorem 3.2, it is easy to verify that S_4 satisfies the items 1-6.*

4 Numerical Examples

In this section, we present some practical examples to demonstrate the similarity measures between fuzzy sets. For practical reasons, we would explain similarity measures between fuzzy sets by examples in pattern recognition, decision making, fuzzy clustering, and approximate reasoning system.

4.1 Pattern Recognition Examples

Example 4.1. *Assume that there are two patterns denoted with fuzzy sets $\widetilde{A}_1$ and $\widetilde{A}_2$ in $\mathbb{X} = \{x_1, x_2, \ldots, x_5\}$. The two patterns are denoted as follows*

$$\begin{aligned} \widetilde{A}_1 &= \{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.8}{x_3}, \frac{1}{x_4}, \frac{0.7}{x_5}\}, \\ \widetilde{A}_2 &= \{\frac{0.3}{x_1}, \frac{1}{x_2}, \frac{0.8}{x_3}, \frac{0.6}{x_4}, \frac{0.4}{x_5}\}. \end{aligned}$$

Assume that a sample pattern $\widetilde{A}$ in $\mathbb{X}$ is given with

$$\widetilde{A} = \{\frac{0.3}{x_1}, \frac{0.6}{x_2}, \frac{1}{x_3}, \frac{0.9}{x_4}, \frac{0.5}{x_5}\}.$$

Now, by using Definition 3.2 with $f(x) = 1/(1+x)$ the following formula is obtained

$$S(\widetilde{A}, \widetilde{A}_i) = \frac{1}{1 + \sup_\alpha |A_\alpha \Delta A_{i\alpha}|}, \quad i = 1, 2,$$

to calculate the similarity degree between fuzzy sets $\widetilde{A}_i$, $i = 1, 2$, and $\widetilde{A}$ as follows

$$\begin{aligned} S(\widetilde{A}, \widetilde{A}_1) &= \frac{1}{3}, \\ S(\widetilde{A}, \widetilde{A}_2) &= \frac{1}{4}. \end{aligned}$$

So, based on the above calculation results, it is evident that the sample $\widetilde{A}$ belongs to the pattern $\widetilde{A}_1$ more than $\widetilde{A}_2$, according to the principle of the maximum degree of similarity between fuzzy sets.

Example 4.2. *Let four patterns be represented by*

$$\begin{aligned}\widetilde{A}_1 &= (0.41, 0.05, 0.06)_T,\\ \widetilde{A}_2 &= (0.39, 0.04, 0.08)_T,\\ \widetilde{A}_3 &= (0.45, 0.08, 0.05)_T,\\ \widetilde{A}_4 &= (0.46, 0.07, 0.06)_T.\end{aligned}$$

Consider a new sample pattern $\widetilde{A}$ which will be recognized as the following triangular fuzzy number

$$\widetilde{A} = (0.43, 0.06, 0.03)_T.$$

Using Definition 3.1 with $f(x) = 1/(1+x^2)$, the following formula is obtained

$$S(\widetilde{A}, \widetilde{A}_i) = \frac{1}{1 + \left(\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{A}_{i\alpha}| d\alpha\right)^2}, \qquad i = 1, \ldots, 4,$$

to calculate the similarity degree between fuzzy numbers $\widetilde{A}_i$, $i = 1, \ldots, 4$, and $\widetilde{A}$. The similarities between these patterns are shown in Table 1. It is easy to see that sample $\widetilde{A}$ should belong to pattern $\widetilde{A}_1$, according to principle of maximum degree of similarity between fuzzy sets.

Table 1: The similarity measures between the patterns in Example 4.2

$S(\widetilde{A}, \widetilde{A}_1)$	$S(\widetilde{A}, \widetilde{A}_2)$	$S(\widetilde{A}, \widetilde{A}_3)$	$S(\widetilde{A}, \widetilde{A}_4)$	$\arg\max_{i=1,\ldots,4} S(\widetilde{A}, \widetilde{A}_i)$
0.968	0.906	0.901	0.765	$\widetilde{A}_1$

4.2 Decision Making Example

Example 4.3. *The problem is involved in 3 applicants ($\widetilde{A}_1$, $\widetilde{A}_2$, $\widetilde{A}_3$) for a position, each is evaluated over four attributes including:*

1. *Experience in the specific job function, denoted as x_1,*
2. *Educational background, denoted as x_2,*
3. *Adaptability, denoted as x_3, and*
4. *Aptitude for teamwork, denoted as x_4.*

Suppose that we obtain the fuzzy decision matrix $\widetilde{\mathcal{D}}$ as follows:

$$\widetilde{\mathcal{D}} = \begin{array}{c} \\ \widetilde{A}_1 \\ \widetilde{A}_2 \\ \widetilde{A}_3 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 0.4 & 0.5 & 0.1 & 0.7 \\ 0.4 & 0.3 & 1.0 & 0.3 \\ 0.5 & 0.2 & 1.0 & 0.8 \end{pmatrix} \end{array},$$

where $[\widetilde{\mathcal{D}}]_{ij} = \widetilde{A}_i(x_j)$, $i = 1, 2, 3$ and $j = 1, \ldots, 4$. The following fuzzy reference sequence $\widetilde{\mathcal{O}}$ is composed of the optimal membership values of the indicator over all the alternatives

$$\widetilde{\mathcal{O}} = optimal \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 0.6 & 0.5 & 1.0 & 0.8 \end{pmatrix} \end{array}.$$

We need to choose the best candidate for the position by calculating the similarities between the reference sequence $\widetilde{\mathcal{O}}$ and the alternate sequences $\widetilde{A}_i$, i.e. $S_4(\widetilde{\mathcal{O}}, \widetilde{A}_i)$, $i = 1, 2, 3$ (where the similarity measure S_4 is introduced in Definition 3.4). A summary of the information on similarity measures is provided in Table 2.

However, it is seen that, by the similarity measure S_4 the optimal alternative is $\widetilde{A}_3$ according to principle of maximum degree of similarity between fuzzy sets.

We can also choose the best candidate for the position by calculating similarity measure

$$\begin{aligned} S_3(\widetilde{\mathcal{O}}, \widetilde{A}_i) &= f(\sup_{\alpha\in[0,1]} |\widetilde{\mathcal{O}}_\alpha \triangle \widetilde{A}_{i\alpha}|) \\ &= \frac{1}{1+\sup_{\alpha\in[0,1]} |\widetilde{\mathcal{O}}_\alpha \triangle \widetilde{A}_{i\alpha}|}, \qquad i = 1,2,3. \end{aligned}$$

A summary of the information on similarity measures is provided in Table 2. However, it is seen that, by the similarity measure S_3 the optimal alternative is $\widetilde{A}_3$ according to principle of maximum degree of similarity between fuzzy sets.

Table 2: The optimal alternate based on similarity measure S_4 introduced in Definition 3.4

$S_4(\widetilde{\mathcal{O}}, \widetilde{A}_1)$	$S_4(\widetilde{\mathcal{O}}, \widetilde{A}_2)$	$S_4(\widetilde{\mathcal{O}}, \widetilde{A}_3)$	$\arg\max_{i=1,2,3} S_4(\widetilde{\mathcal{O}}, \widetilde{A}_i)$
0.500	0.250	0.667	$\widetilde{A}_3$
$S_3(\widetilde{\mathcal{O}}, \widetilde{A}_1)$	$S_3(\widetilde{\mathcal{O}}, \widetilde{A}_2)$	$S_3(\widetilde{\mathcal{O}}, \widetilde{A}_3)$	$\arg\max_{i=1,2,3} S_3(\widetilde{\mathcal{O}}, \widetilde{A}_i)$
$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\widetilde{A}_3$

4.3 Clustering Example

Fuzzy clustering, which is one of the major techniques in pattern recognition, is a method for decomposing a given data set into groups or clusters of similar individuals with uncertain boundaries. Different algorithms have been developed in fuzzy cluster analysis, which can be roughly divided into two main categories: the first category is based on objective functions, while the other one is based on a relation matrix such as similarity relation [29, 32, 42, 45, 46], correlation relation, fuzzy equivalence relations [23, 28], and the like (for more on this topic, see e.g. [4, 36, 41, 43, 49, 52]). Although the second category of clustering methods is eventually the novel method of agglomerative hierarchical clustering, they are simple and useful in application systems [5, 22]. The investigation considered in the following example focuses on the second category of fuzzy clustering methods based on fuzzy relations that can be made in the beginning with a similarity matrix.

Example 4.4. *Assume that there are nine patterns denoted with fuzzy triangular numbers as follows:*

$$\begin{aligned} &\widetilde{A}_1 = (13.00, 0.27, 1.00)_T, \quad \widetilde{A}_2 = (14.00, 1.95, 0.93)_T, \quad \widetilde{A}_3 = (14.40, 0.56, 1.17)_T, \\ &\widetilde{A}_4 = (14.70, 0.89, 0.88)_T, \quad \widetilde{A}_5 = (14.90, 0.12, 1.21)_T, \quad \widetilde{A}_6 = (15.30, 1.19, 0.41)_T, \\ &\widetilde{A}_7 = (15.10, 1.82, 0.90)_T, \quad \widetilde{A}_8 = (15.60, 0.38, 1.38)_T, \quad \widetilde{A}_9 = (16.00, 1.97, 0.12)_T. \end{aligned}$$

For the sake of convenience, we consider the similarity measure S_2 defined in Definition 3.2. The similarity measures between these patterns are shown in Table 3. To obtain a fuzzy cluster for these patterns, we are using Yang and Shih's algorithm [45], which creates a clustering algorithm for the $\max - \triangle$ similarity relation matrix. Then, by $\max - \triangle$ composition, we also have $R^{(0)} = R^{(1)}$ which is the $\max - \triangle$ similarity relation matrix. Applying the clustering algorithm to these patterns, by beginning with Table 3 as the initial similarity matrix $R^{(0)}$ on $\{\widetilde{A}_1, \ldots, \widetilde{A}_9\}$, we obtain the hierarchical clustering results presented in Table 4.

The hierarchical clustering illustrates the partitions that have been made at levels of β. For example, when the level β is in $(0.189, 0.246]$, the following three distinct clusters are obtained

$$\begin{aligned} \mathcal{C}_1 &= \{\widetilde{A}_1, \widetilde{A}_2\}, \\ \mathcal{C}_2 &= \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \\ \mathcal{C}_3 &= \{\widetilde{A}_5, \widetilde{A}_8\}. \end{aligned}$$

Now, considering the above clusters, suppose that we want to determine to which cluster the new pattern $\widetilde{A} = (13.7, 1.1, 0.8)_T$ belongs. To do this, we use principle of largest similarity between the new pattern $\widetilde{A}$ and *clusters* $\mathcal{C}_j$, $j = 1,2,3$, *i.e.* $\arg\max_{j\in\{1,2,3\}}\{S_2(\widetilde{A},\mathcal{C}_j)\}$, *where*

$$\begin{aligned} S_2(\widetilde{A},\mathcal{C}_1) &= \max\{S_2(\widetilde{A},\widetilde{A}_i)|\widetilde{A}_i \in \mathcal{C}_1\} \\ &= \max\{S_2(\widetilde{A},\widetilde{A}_1), S_2(\widetilde{A},\widetilde{A}_2)\} \\ &= \max\{0.400, 0.460\} = 0.460. \end{aligned}$$

Similarly, the similarities between $\widetilde{A}$ *and* $\mathcal{C}_2$, $\mathcal{C}_3$ *are obtained as follows*

$$\begin{aligned} S_2(\widetilde{A},\mathcal{C}_2) &= \max\{S_2(\widetilde{A},\widetilde{A}_i)|\widetilde{A}_i \in \mathcal{C}_2\} \\ &= \max\{0.098, 0.084, 0.230, 0.142, 0.021\} = 0.230, \\ S_2(\widetilde{A},\mathcal{C}_3) &= \max\{S_2(\widetilde{A},\widetilde{A}_i)|\widetilde{A}_i \in \mathcal{C}_3\} \\ &= \max\{0, 0\} = 0. \end{aligned}$$

It is clear that $\arg\max_{j\in\{1,2,3\}}\{S_2(\widetilde{A},\mathcal{C}_j)\} = \mathcal{C}_1$. *Therefore, using principle of maximum degree of similarity, it indicates that the new pattern* $\widetilde{A}$ *belongs to the cluster* $\mathcal{C}_1$.

Table 3: Similarity measures between nine patterns $\widetilde{A}_1 - \widetilde{A}_9$, using S_2, in Example 4.4

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$	$\widetilde{A}_6$	$\widetilde{A}_7$	$\widetilde{A}_8$	$\widetilde{A}_9$
$\widetilde{A}_1$	1.000	0.246	0.006	0.006	0.000	0.000	0.000	0.000	0.000
$\widetilde{A}_2$	0.246	1.000	0.394	0.355	0.143	0.233	0.399	0.041	0.186
$\widetilde{A}_3$	0.006	0.394	1.000	0.753	0.189	0.367	0.512	0.240	0.250
$\widetilde{A}_4$	0.006	0.355	0.753	1.000	0.261	0.452	0.585	0.031	0.283
$\widetilde{A}_5$	0.000	0.143	0.189	0.261	1.000	0.551	0.462	0.494	0.449
$\widetilde{A}_6$	0.000	0.233	0.367	0.452	0.551	1.000	0.553	0.100	0.472
$\widetilde{A}_7$	0.000	0.399	0.512	0.585	0.462	0.553	1.000	0.119	0.394
$\widetilde{A}_8$	0.000	0.041	0.240	0.031	0.494	0.100	0.119	1.000	0.408
$\widetilde{A}_9$	0.000	0.186	0.250	0.283	0.449	0.472	0.394	0.408	1.000

Table 4: Clustering results according to different levels in Example 4.4

β	Clustering results
$0.000 < \beta \leq 0.031$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_8, \widetilde{A}_9\}$
$0.031 < \beta \leq 0.143$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \{\widetilde{A}_8\}$
$0.143 < \beta \leq 0.189$	$\{\widetilde{A}_1, \widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \{\widetilde{A}_8\}$
$0.189 < \beta \leq 0.246$	$\{\widetilde{A}_1, \widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \{\widetilde{A}_5, \widetilde{A}_8\}$
$0.246 < \beta \leq 0.250$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \{\widetilde{A}_5, \widetilde{A}_8\}$
$0.250 < \beta \leq 0.367$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_6, \widetilde{A}_7\}, \{\widetilde{A}_5, \widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.367 < \beta \leq 0.408$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_7\}, \{\widetilde{A}_5, \widetilde{A}_6\}, \{\widetilde{A}_8, \widetilde{A}_9\}$
$0.408 < \beta \leq 0.512$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_7\}, \{\widetilde{A}_5, \widetilde{A}_6\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.512 < \beta \leq 0.553$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4\}, \{\widetilde{A}_5\}, \{\widetilde{A}_6, \widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.553 < \beta \leq 0.753$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4\}, \{\widetilde{A}_5\}, \{\widetilde{A}_6\}, \{\widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.753 < \beta \leq 1.000$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3\}, \{\widetilde{A}_4\}, \{\widetilde{A}_5\}, \{\widetilde{A}_6\}, \{\widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$

4.4 Approximate Reasoning Examples

In the following two examples, we validate the effectiveness of the proposed similarity measures in a bidirectional approximate reasoning system. The basic structure of the fuzzy inference system consists of three components: a rule base, a database, and an inference procedure. The rule base contains the selection of fuzzy if-then rules activated by a certain value of interest, the database defines the membership functions adopted in the fuzzy if-then rules, then the inference procedure provides a fuzzy reasoning based on information aggregation from the activated fuzzy rules. The structure of a typical fuzzy if-then rule that uses the "and" fuzzy operator is demonstrated in the example statement below (see, for instance, [37, 53]):

$$R_i: \quad \text{if } \widetilde{X}_1 \text{ is } \widetilde{A}_{1i}, \; \textit{and } \widetilde{X}_2 \text{ is } \widetilde{A}_{2i}, \ldots, \; \textit{and } \widetilde{X}_p \text{ is } \widetilde{A}_{pi}, \text{ then } \widetilde{Y} \text{ is } \widetilde{B}_i.$$

In this scheme, R_i $(i = 1, \ldots, n)$ is the ith production rule, n is the number of rules, $\widetilde{X}_j$ $(j = 1, \ldots, p)$ is the fuzzy input (antecedent) variables, $\widetilde{Y}$ is the fuzzy output (consequent) variable, $\widetilde{A}_{ji}$'s are fuzzy sets of the universe of discourse $\mathbb{X}_j$ for the antecedent variables, and $\widetilde{B}_i$'s are fuzzy sets of the universe of discourse $\mathbb{Y}$ for the consequent variable.

Let us first consider the forward approximate reasoning scheme based on fuzzy sets. Suppose that the antecedent statement is demonstrated as follows

$$\textit{Antecedent}: \quad \widetilde{X}_1 \text{ is } \widetilde{A}^*_1, \; \textit{and } \widetilde{X}_2 \text{ is } \widetilde{A}^*_2, \ldots, \; \textit{and } \widetilde{X}_p \text{ is } \widetilde{A}^*_p,$$

where $\widetilde{A}^*_j$'s $(j = 1, \ldots, p)$ are fuzzy sets of the universe of discourse $\mathbb{X}_j$ for the antecedent variables. We need to determine the consequence of the approximate reasoning scheme, which is

$$\textit{Consequence}: \quad \widetilde{Y} \text{ is } \widetilde{B}^*,$$

where $\widetilde{B}^*$ is a fuzzy set of the universe of discourse $\mathbb{Y}$ for the consequent variable.

Applying a method similar to that described by Chen et al. [7] (see also [50]), we propose the following algorithm for ith rule:

1. Compute $s_{ji} = S(\widetilde{A}_{ji}, \widetilde{A}^*_j)$, the similarity measurement between fuzzy sets $\widetilde{A}_{ji}$, and $\widetilde{A}^*_j$,
2. Let $s_i = \max_{1 \leq j \leq p} s_{ji}$, and $\widetilde{B}^*_i = s_i \otimes \widetilde{B}_i$,
3. The deduced consequence of rule R_i is "$\widetilde{Y}$ is $\widetilde{B}^*_i$".

Thus, the deduced consequence of the approximate reasoning scheme is "$\widetilde{Y}$ is $\widetilde{B}^*$", where

$$\widetilde{B}^* = \widetilde{B}^*_1 \cup \widetilde{B}^*_2 \cup \cdots \cup \widetilde{B}^*_n.$$

Conversely, let us consider the backward approximate reasoning scheme based on fuzzy sets. Suppose that the consequence statement is demonstrated as follows

$$\textit{Consequence}: \quad \widetilde{Y} \text{ is } \widetilde{B}^*.$$

We need to determine the antecedent of the approximate reasoning scheme, which is

$$\textit{Antecedent}: \quad \widetilde{X}_1 \text{ is } \widetilde{A}^*_1, \; \textit{and } \widetilde{X}_2 \text{ is } \widetilde{A}^*_2, \ldots, \; \textit{and } \widetilde{X}_p \text{ is } \widetilde{A}^*_p.$$

Similarly, we can derive the following results for ith rule:

1. Compute $s_i = S(\widetilde{B}_i, \widetilde{B}^*)$, the similarity measurement between fuzzy sets $\widetilde{B}_i$, and $\widetilde{B}^*$,
2. Let $\widetilde{A}^*_{ji} = s_i \otimes \widetilde{A}_{ji}$,
3. The deduced antecedent of rule R_i is

$$\widetilde{X}_1 \text{ is } \widetilde{A}^*_{1i}, \; \textit{and } \widetilde{X}_2 \text{ is } \widetilde{A}^*_{2i}, \ldots, \; \textit{and } \widetilde{X}_p \text{ is } \widetilde{A}^*_{pi}.$$

By aggregating above results, the deduced antecedent of the approximate reasoning scheme is

$$\textit{Antecedent}: \quad \widetilde{X}_1 \text{ is } \widetilde{A}_1^*, \ \textit{and} \ \widetilde{X}_2 \text{ is } \widetilde{A}_2^*, \ldots, \ \textit{and} \ \widetilde{X}_p \text{ is } \widetilde{A}_p^*,$$

where

$$\widetilde{A}_j^* = \widetilde{A}_{j1}^* \cup \widetilde{A}_{j2}^* \cup \cdots \cup \widetilde{A}_{jn}^*, \quad j = 1, \ldots, p.$$

Remark 4.1. *Note that, while all fuzzy sets are LR-fuzzy numbers in the approximate reasoning scheme, the fuzzy set of the obtained $\widetilde{A}_j^*$ could be any irregular shapes. But, it is suitable that these fuzzy sets are LR-fuzzy numbers. To achieve this, the obtained $\widetilde{A}_j^*$ should be transformed into a LR-fuzzy number. So, we consider the transformed $\widetilde{A}_j^*$ as $\widetilde{A}_j^* = (a_j^*, l_j, r_j)_{LR}$, where a_j^* is the defuzzified value of the $\widetilde{A}_j^*$ calculated by center of gravity, l_j and r_j are set as the minimum and maximum of the possible values of $\widetilde{A}_{j1}^*, \widetilde{A}_{j2}^*, \ldots, \widetilde{A}_{jn}^*$, respectively.*

Table 5: The fuzzy sets in the forward approximate reasoning in Example 4.5

i	$\widetilde{A}_{1i}$	$\widetilde{A}_{2i}$	$\widetilde{A}_{3i}$	$\widetilde{B}_i$
1	$(0.40, 0.20, 0.15)_T$	$(0.80, 0.40, 0.50)_T$	$(25, 15, 15)_T$	$(26, 6, 6)_T$
2	$(0.40, 0.20, 0.15)_T$	$(0.80, 0.40, 0.50)_T$	$(45, 15, 15)_T$	$(26, 6, 6)_T$
3	$(0.60, 0.25, 0.10)_T$	$(1.20, 0.40, 0.40)_T$	$(45, 15, 15)_T$	$(18, 6, 4)_T$
4	$(0.60, 0.25, 0.10)_T$	$(0.80, 0.40, 0.50)_T$	$(25, 15, 15)_T$	$(18, 6, 4)_T$
5	$(0.55, 0.30, 0.05)_T$	$(1.20, 0.40, 0.40)_T$	$(25, 15, 15)_T$	$(10, 8, 5)_T$
6	$(0.55, 0.30, 0.05)_T$	$(1.20, 0.40, 0.40)_T$	$(45, 15, 15)_T$	$(10, 8, 5)_T$

Example 4.5. *Let us consider the following forward approximate reasoning scheme based on fuzzy sets*

$$\begin{aligned}
R_1 &: \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{11}, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{21}, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{31}, \textit{ then } \widetilde{Y} \textit{ is } \widetilde{B}_1, \\
R_2 &: \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{12}, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{22}, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{32}, \textit{ then } \widetilde{Y} \textit{ is } \widetilde{B}_2, \\
R_3 &: \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{13}, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{23}, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{33}, \textit{ then } \widetilde{Y} \textit{ is } \widetilde{B}_3, \\
R_4 &: \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{14}, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{24}, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{34}, \textit{ then } \widetilde{Y} \textit{ is } \widetilde{B}_4, \\
R_5 &: \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{15}, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{25}, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{35}, \textit{ then } \widetilde{Y} \textit{ is } \widetilde{B}_5, \\
R_6 &: \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{16}, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{26}, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{36}, \textit{ then } \widetilde{Y} \textit{ is } \widetilde{B}_6.
\end{aligned}$$

The fuzzy sets used in this scheme are given in Table 5. Now, suppose that the antecedent statement is demonstrated as follows

$$\textit{Antecedent}: \quad \widetilde{X}_1 \textit{ is } \widetilde{A}_1^*, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_2^*, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_3^*,$$

where

$$\widetilde{A}_1^* = (0.45, 0.45, 0.20)_T, \quad \widetilde{A}_2^* = (1, 0.80, 0.80)_T, \quad \widetilde{A}_3^* = (40, 15, 15)_T.$$

We need to determine the consequence of the approximate reasoning scheme, which is

$$\textit{Consequence}: \quad \widetilde{Y} \textit{ is } \widetilde{B}^*.$$

Applying the following similarity measure between fuzzy sets $\widetilde{A}$ and $\widetilde{B}$

$$S(\widetilde{A}, \widetilde{B}) = \frac{1}{1 + \int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha},$$

we obtain the similarities shown in Table 6. Thus,

$$\widetilde{B}^* = \cup_{i=1}^6 \widetilde{B}_i^*,$$

where the fuzzy sets $\widetilde{B}_i^*$, $i = 1, \ldots, 6$, are given in Table 6.

Table 6: Similarity measures $s_{ji} = S(\widetilde{A}_{ji}, \widetilde{A}_i^*)$ $(j = 1, 2, 3,\ i = 1, \ldots, 6)$ between fuzzy sets in the forward approximate reasoning in Example 4.5

i	s_{1i}	s_{2i}	s_{3i}	$s_i = \max_{j\in\{1,2,3\}} s_{ji}$	$\widetilde{B}_i^* = s_i \otimes \widetilde{B}_i$
1	0.480	0.290	0.004	0.480	$(12.48, 2.88, 2.88)_T$
2	0.480	0.290	0.011	0.480	$(12.48, 2.88, 2.88)_T$
3	0.420	0.210	0.011	0.420	$(7.56, 2.52, 1.68)_T$
4	0.420	0.290	0.004	0.420	$(7.56, 2.52, 1.68)_T$
5	0.630	0.210	0.004	0.630	$(6.30, 5.04, 3.15)_T$
6	0.630	0.210	0.011	0.630	$(6.30, 5.04, 3.15)_T$

Example 4.6. *Conversely, let us consider the forward approximate reasoning scheme in Example 4.5 as a backward approximate reasoning scheme. Now, suppose that the consequence statement of the approximate reasoning scheme is demonstrated as follows*

$$\textit{Consequence}: \qquad \widetilde{Y} \textit{ is } \widetilde{B}^* = (18, 6, 6)_T.$$

We need to determine the antecedent statement of the approximate reasoning scheme, which is

$$\textit{Antecedent}: \qquad \widetilde{X}_1 \textit{ is } \widetilde{A}_1^*, \textit{ and } \widetilde{X}_2 \textit{ is } \widetilde{A}_2^*, \textit{ and } \widetilde{X}_3 \textit{ is } \widetilde{A}_3^*.$$

Similarly to the previous example, we can derive the results summarized in Table 7. Thus, we obtain

$$\begin{aligned} \widetilde{A}_1^* &= \cup_{i=1}^6 \widetilde{A}_{1i}^*, \\ \widetilde{A}_2^* &= \cup_{i=1}^6 \widetilde{A}_{2i}^*, \\ \widetilde{A}_3^* &= \cup_{i=1}^6 \widetilde{A}_{3i}^*, \end{aligned}$$

where the fuzzy sets $\widetilde{A}_{ji}^*$, $j = 1, 2, 3$, and $i = 1, \ldots, 6$, are given in Table 7.

Table 7: Similarity measures in backward approximate reasoning in Example 4.6

i	$s_i = S(\widetilde{B}_i, \widetilde{B}^*)$	$\widetilde{A}_{1i}^* = s_i \otimes \widetilde{A}_{1i}$	$\widetilde{A}_{2i}^* = s_i \otimes \widetilde{A}_{2i}$	$\widetilde{A}_{3i}^* = s_i \otimes \widetilde{A}_{3i}$
1	0.08	$(0.032, 0.016, 0.012)_T$	$(0.064, 0.032, 0.040)_T$	$(2.0, 1.2, 1.2)_T$
2	0.08	$(0.032, 0.016, 0.012)_T$	$(0.064, 0.032, 0.040)_T$	$(3.6, 1.2, 1.2)_T$
3	0.50	$(0.300, 0.125, 0.050)_T$	$(0.600, 0.200, 0.200)_T$	$(22.5, 7.5, 7.5)_T$
4	0.50	$(0.300, 0.125, 0.050)_T$	$(0.400, 0.200, 0.250)_T$	$(12.5, 7.5, 7.5)_T$
5	0.08	$(0.044, 0.024, 0.004)_T$	$(0.096, 0.032, 0.032)_T$	$(2.0, 1.2, 1.2)_T$
6	0.08	$(0.044, 0.024, 0.004)_T$	$(0.096, 0.032, 0.032)_T$	$(3.6, 1.2, 1.2)_T$

5 Conclusion

In this paper, we proposed new definitions of similarity measures for measuring the degree of similarity between fuzzy sets based on an axiomatic approach. We also investigated the properties of the axiomatic definition for these measures. Our discussion of applicability, distinguishably, and effectiveness of proposed measures, carried out by results of examples related to pattern recognition, decision making, fuzzy clustering analysis, and approximate reasoning system, suggests useful references of measures that cover these areas adequately. The immediate concerns of the examples provided in this text are to indicate that these measures can provide a useful way for measuring the degree of similarity between fuzzy sets and that the proposed approach performs well in many practical situations and can be applied for future research in many other fields, such as image processing, fuzzy neural networks, fuzzy reasoning, fuzzy risk analysis, and fuzzy control.

On the other hand, over the past several years, many studies have introduced similarity measures, distance measures, inclusion measures, entropy and the fuzziness of fuzzy sets. They also discussed their properties and also considered the relationships between them and showed that these measures can be deduced by each other based on their axiomatic definitions [18, 26, 27, 48, 51]. In this regard, based on the similarity measures introduced in this paper, one can investigate the relations among similarity measures, inclusion measures, distance measures, entropy and the fuzziness of fuzzy sets, and prove some theorems that inclusion measures, similarity measures, distance measures, entropy and the fuzziness of fuzzy sets can be transformed by each other based on their axiomatic definitions. Therefore, some new formulas can also be put forward to calculate inclusion measures, similarity measures, distance measures, entropy and the fuzziness of fuzzy sets.

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