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Characterizing Credibilistic Comonotonicity of Fuzzy Vector in Fuzzy Decision Systems

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Abstract

The concept of comonotonicity is a useful tool for solving various problems in insurance and financial economics. The credibilistic comonotonicity of fuzzy vector is defined via its comonotonic support. In general case, the properties of comonotonic fuzzy vector are discussed based on the relation between the joint monotone distribution of a fuzzy vector and its marginal monotone distributions. A useful property of mutually independent fuzzy variables is characterized by comonotonic fuzzy vector. In the case of two-dimensional fuzzy vector, several equivalent characterizations for comonotonic fuzzy vector are established. The theoretical results obtained in this paper have potential applications in practical risk management problems.

Keywords: fuzzy vector, support, comonotonicity, independence, equivalent characterization

1 Introduction

Since credibility measure was first introduced in [14], the theory of credibility measure has been well-studied and extended in both theory and applications. Among the recent theoretic developments, the interested reader may refer to [1, 2, 15] and the references therein.

Among the practical applications of credibility measure theory, Dolatabadi et al. [5] designed supply chain networks with multi-product and multi-customer by minimization per unit cost of supply and its variance in risky environments, where credibility measure was applied to the assurance that risk consequences are lower than a given risk level with maximum possible confidence level. Fan et al. [6] presented a fuzzy chance constrained programming approach to the day-ahead scheduling of virtual power plant, where the fuzzy chance constraint was converted into its crisp equivalent by utilising credibility theory. Garg [8] addressed the fuzzy system reliability analysis to construct the membership and non-membership functions by considering the different types of intuitionistic fuzzy failure rates, where functions of intuitionistic fuzzy numbers were calculated using credibility theory. Jiang et al. [9] proposed a credibility-based fuzzy chance constrained model to optimize the dynamic control bound, and fuzzy simulation technology was used to solve the model. Kalhori and Zarandi [10] presented a new approach to interval type-2 fuzzy clustering, where the credibility degrees are transformed to interval type-2 form to handle different sources of uncertainty. Based on probability and credibility measure, Liu [11] developed the approximation solution method for a class of two-stage fuzzy random minimum risk problem, and the convergence modes of the approximation method have been documented in [16, 17]. Liu and Bai [12] dealt with the interconnections between two types of risk aversion two-stage credibilistic optimization problems. Liu and Tian [18] addressed the approximation solution method for a class of two-stage fuzzy programming with minimum-risk criteria in the sense of credibilistic value-at-risk. Lu et al. [19] presented a credibility-based chance-constrained optimization model for integrated agricultural irrigation and water resources management. The model provided a credibility level to indicate the confidence level

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of the generated optimal management strategies. Mehlawat [20] dealt with fuzzy multi-objective multi-period portfolio selection problems, where the portfolio risk was quantified using credibilistic entropy of the fuzzy returns. Wang et al. [21] developed an environmental-friendly modeling system and applied it to an agriculture nonpoint source (AGNPS) management in Ulansuhai Nur watershed, where water environmental capacity, credibility-based chance-constrained programming, and AGNPS optimization models were integrated into a general modeling framework. Wu et al. [22] characterized incomplete information in the agent's ability by fuzzy variable and applied credibility measure to study optimal contracting problems. Yang et al. [23] proposed a new two-stage credibilistic optimization method for multi-objective supply chain network design problem with uncertain transportation costs and uncertain customer demands. Zhai et al. [24] developed a new two-stage uncapacitated hub location problem with recourse, in which uncertain parameters are characterized by both probability and possibility distributions. Zhang et al. [25] developed a full credibility-based chance-constrained programming method by introducing the new concept of credibility into the modeling framework.

It is known that the concept of comonotonicity is a useful tool for solving various problems in insurance and financial economics [3, 4]. The purpose of this paper is to study the credibilistic comonotonicity of fuzzy vector, which has potential applications in fuzzy decision systems. We first introduce the definition about the comonotonicity of a fuzzy vector. In general case, we study the properties of comonotonic fuzzy vector based on comonotonic set in Euclidean space, and characterize the mutually independent fuzzy variables by comonotonic fuzzy vector. Particularly, we establish several equivalent characterizations for two-dimensional comonotonic fuzzy vector.

The rest of paper is organized as follows. Section 2 reviews some fundamental concepts required in the rest of the paper. Section 3 first defines the concept of comonotonic fuzzy vector, then discusses its general properties based on the relation between the joint monotone distribution of fuzzy vector and its marginal monotone distributions. Section 4 establishes some equivalent characterizations for the comonotonicity of two-dimensional fuzzy vector. Section 5 draws the conclusions of this paper and suggests our future research.

2 Fundamental Concepts

Let Γ be an abstract space of generic element γ, $\mathcal{A}$ the ample field consisting of a collection of subsets of Γ, and Cr the credibility measure [14] defined on $\mathcal{A}$. The triplet $(\Gamma, \mathcal{A}, \mathrm{Cr})$ is referred to as a credibility measure space.

Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ be a fuzzy vector defined on a credibility measure space $(\Gamma, \mathcal{A}, \mathrm{Cr})$. The credibility distribution function of $\boldsymbol{\xi}$ is denoted as

$$\mu_{\boldsymbol{\xi}}(\boldsymbol{t}) = \mathrm{Cr}\{\gamma \in \Gamma \mid \boldsymbol{\xi}(\gamma) = \boldsymbol{t}\}, \quad \boldsymbol{t} \in \Re^n.$$

A fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is said to be continuous if its credibility distribution function $\mu_{\boldsymbol{\xi}}(\boldsymbol{t})$ is continuous with respect to $\boldsymbol{t}$. The one-to-one correspondence between credibility measures and credibility distribution functions has been documented in [7].

We next define the notion of support of an n-dimensional fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$. Any subset Ξ^n of $\Re^n$ is called a support of $\boldsymbol{\xi}$ if $\mathrm{Cr}\{\boldsymbol{\xi} \in \Xi^n\} = 1$ holds true. In general, we are interested in supports which are as small as possible. Using the credibility distribution function, the following set

$$\Xi^n = \mathrm{cl}\{\boldsymbol{t} \in \Re^n \mid \mu_{\boldsymbol{\xi}}(\boldsymbol{t}) > 0\}$$

is a support of $\boldsymbol{\xi}$, where $\mathrm{cl}(A)$ is the closure of set A. If Ξ^n is a bounded subset of $\Re^n$, then fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is said to be bounded.

In [13], the independence of fuzzy vectors can be characterized by credibility measure in the following sense:

If $\boldsymbol{\xi}_i$ $(i = 1, 2, \ldots, n)$ are n_i-ary fuzzy vectors, then they are mutually independent fuzzy vectors if and only if

$$\mathrm{Cr}\left\{\gamma \mid \boldsymbol{\xi}_1(\gamma) \in B_1, \ldots, \boldsymbol{\xi}_n(\gamma) \in B_n\right\} = \min_{1 \le i \le n} \mathrm{Cr}\left\{\gamma \mid \boldsymbol{\xi}_i(\gamma) \in B_i\right\}$$

for any $B_i \subseteq \Re^{n_i}$ $(i = 1, 2, \ldots, n)$.

In what follows, we will denote

$$F_{\boldsymbol{\xi}}(x_1, x_2, \cdots, x_n) = \mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \ldots, \xi_n \le x_n\}, \quad \boldsymbol{x} \in \Re^n,$$

as the joint monotone increasing distribution of $\boldsymbol{\xi}$, and write

$$F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}, \quad x_i \in \Re, i = 1, 2, \ldots, n$$

as the marginal monotone increasing distributions of $\boldsymbol{\xi}$.

3 Comonotonicity of Fuzzy Vector: General Case

We first review the comonotonicity of a set of n-ary vectors in Euclidean space $\Re^n$. An n-ary vector $(x_1, x_2, \cdots, x_n)$ will be denoted by $\boldsymbol{x}$. For two n-ary vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, the notation $\boldsymbol{x} \le \boldsymbol{y}$ will be used for the componentwise order which is defined by $x_i \le y_i$ for $i = 1, 2, \ldots, n$.

A subset A of $\Re^n$ is said to be comonotonic if for any $\boldsymbol{x}, \boldsymbol{y} \in A$, either $\boldsymbol{x} \le \boldsymbol{y}$ or $\boldsymbol{y} \le \boldsymbol{x}$ holds [3].

On the basis of the comonotonic set, it is easy to check that a subset A of $\Re^n$ is comonotonic if for any $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ in A,

$$(x_i - y_i)(x_j - y_j) \ge 0$$

for any $i, j \in \{1, 2, \ldots, n\}$ and $i \ne j$.

We will denote the (i, j)-projection of a subset A of $\Re^n$ by $A_{i,j}$. It is defined by

$$A_{i,j} = \{(x_i, x_j) \mid (x_1, x_2, \cdots, x_n) \in A\}.$$

A comonotonic subset A of $\Re^n$ has the following property [3]:

A subset A of $\Re^n$ is comonotonic if and only if $A_{i,j}$ is comonotonic for all $i, j \in \{1, 2, \ldots, n\}$ and $i \ne j$.

We next formally define the concept of comonotonicity for a general fuzzy vector:

Definition 1. *A fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is called credibilistic comonotonicity if it has a comonotonic support Ξ^n, i.e. a comonotonic subset Ξ^n of $\Re^n$ such that $\mathrm{Cr}\{\boldsymbol{\xi} \in \Xi^n\} = 1$ holds true.*

The following theorem deals with a property of comonotonic fuzzy vector:

Theorem 1. *If fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is comonotonic, then for all $\boldsymbol{x}$, one has*

$$\mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \ldots, \xi_n \le x_n\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \le x_i\},$$

i.e., $F_{\boldsymbol{\xi}}(x_1, x_2, \cdots, x_n) = \min_{1 \le i \le n} F_{\xi_i}(x_i)$.

Proof. Suppose Ξ^n is the comonotonic support of fuzzy vector $\boldsymbol{\xi}$. Let $\boldsymbol{x} \in \Re^n$ and A_j be defined as

$$A_j = \{\boldsymbol{y} \in \Xi^n \mid y_j \le x_j\}, \quad j = 1, 2, \ldots, n.$$

Since Ξ^n is a comonotonic set, there is an index i such that

$$A_i = \bigcap_{j=1}^{n} A_j.$$

As a consequence, one has

$$F_{\boldsymbol{\xi}}(x_1, x_2, \cdots, x_n) = \mathrm{Cr}\left\{\boldsymbol{\xi} \in \cap_{j=1}^{n} A_j\right\} = \mathrm{Cr}\{\boldsymbol{\xi} \in A_i\} = F_{\xi_i}(x_i).$$

Since $A_i \subseteq A_j$, it follows from the monotonicity of Cr that

$$F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \ldots, \xi_n \le x_n\} \le \mathrm{Cr}\{\xi_j \le x_j\} = F_{\xi_j}(x_j)$$

for all j. As a consequence, one has

$$F_{\xi_i}(x_i) = \min\left\{F_{\xi_1}(x_1), F_{\xi_2}(x_2), \ldots, F_{\xi_n}(x_n)\right\},$$

which completes the proof of proposition. □

Suppose $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are two n-ary fuzzy vectors defined on a credibility measure space $(\Gamma, \mathcal{A}, \mathrm{Cr})$. Then the following notation

$$\boldsymbol{\xi} \stackrel{\mathrm{d}}{=} \boldsymbol{\zeta}$$

stands for 'equality in distribution' in the sense that

$$\mathrm{Cr}\{\boldsymbol{\xi} \le x\} = \mathrm{Cr}\{\boldsymbol{\zeta} \le x\}$$

for all $x \in \Re^n$.

For a fuzzy variable, we have the following result about the equality in distribution:

Theorem 2. *Let ξ be a fuzzy variable and its monotone increasing distribution $F_\xi(x) = \mathrm{Cr}\{\xi \le x\}$ is right continuous. Then for triangular fuzzy variable $T \sim \mathrm{Tri}(0, 1/2, 1)$, one has*

$$\xi \stackrel{\mathrm{d}}{=} F_\xi^{-1}(T),$$

where $F_\xi^{-1}(c) = \inf\{t \in \Re \mid F_\xi(t) \ge c\}$, $c \in 0, 1]$.

Proof. Since T is the triangular fuzzy variable $(0, 1/2, 1)$, the credibility of event $\{T \le t\}$ is t whenever $t \in [0, 1]$, i.e.,

$$\mathrm{Cr}\{T \le t\} = t, \quad t \in [0,\, 1].$$

From the right continuity of the monotone increasing distribution $F_\xi(x) = \mathrm{Cr}\{\xi \le x\}$, we find

$$T \le F_\xi(x) \Longleftrightarrow F_\xi^{-1}(T) \le x.$$

As a result, one has

$$F_\xi(x) = \mathrm{Cr}\{T \le F_\xi(x)\} = \mathrm{Cr}\{F_\xi^{-1}(T) \le x\},$$

which completes the proof of theorem. □

The following theorem extends the result of Theorem 2 to the case of fuzzy vector:

Theorem 3. *If $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is a fuzzy vector and its marginal monotone increasing distributions $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$ are right continuous, and the following condition holds true*

$$\mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \ldots, \xi_n \le x_n\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \le x_i\}$$

for all $\boldsymbol{x}$, then for triangular fuzzy variable $T \sim \mathrm{Tri}(0, 1/2, 1)$, one has

$$\boldsymbol{\xi} \stackrel{\mathrm{d}}{=} \left(F_{\xi_1}^{-1}(T), F_{\xi_2}^{-1}(T), \cdots, F_{\xi_n}^{-1}(T)\right),$$

where $F_{\xi_i}^{-1}(c) = \inf\{t \in \Re \mid F_{\xi_i}(t) \ge c\}$, $c \in (0, 1]$, $i = 1, 2, \ldots, n$.

Proof. Suppose the following condition holds true

$$\mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \ldots, \xi_n \le x_n\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \le x_i\}, \quad x = (x_1, \cdots, x_n) \in \Re^n.$$

Since T is the triangular fuzzy variable $(0, 1/2, 1)$, one has $\mathrm{Cr}\{T \le t\} = t$ for $t \in [0, 1]$. From the right continuity of the monotone increasing distribution $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$, we find

$$F_{\xi_i}^{-1}(T) \le x_i \Longleftrightarrow T \le F_{\xi_i}(x_i), i = 1, 2, \ldots, n.$$

As a consequence, one has

$$\begin{aligned}
&\mathrm{Cr}\left\{F_{\xi_1}^{-1}(T) \le x_1, \ldots, F_{\xi_n}^{-1}(T) \le x_n\right\} \\
&= \mathrm{Cr}\left\{T \le F_{\xi_1}(x_1), \ldots, T \le F_{\xi_n}(x_n)\right\} \\
&= \mathrm{Cr}\left\{T \le \min\nolimits_{1 \le i \le n} F_{\xi_i}(x_i)\right\} \\
&= \min\nolimits_{1 \le i \le n} F_{\xi_i}(x_i) \\
&= F_{\boldsymbol{\xi}}(x_1, x_2, \cdots, x_n),
\end{aligned}$$

which completes the proof of theorem. □

As a corollary of Theorem 3, we have the following result:

Corollary 1. *If $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is a fuzzy vector and its marginal monotone increasing distributions $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$ are right continuous, and the following condition holds true*

$$\mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \dots, \xi_n \le x_n\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \le x_i\}$$

for all $\boldsymbol{x}$, then for triangular fuzzy variable $T \sim \mathrm{Tri}(0, 1/2, 1)$, one has

$$\boldsymbol{\xi} \stackrel{\mathrm{d}}{=} \left(F_{\xi_1}^{-1}(1-T), F_{\xi_2}^{-1}(1-T), \cdots, F_{\xi_n}^{-1}(1-T)\right),$$

where $F_{\xi_i}^{-1}(c) = \inf\{t \in \Re \mid F_{\xi_i}(t) \ge c\}$, $c \in (0,1]$, $i = 1, 2, \dots, n$.

Proof. Since T is triangular fuzzy variable $(0, 1/2, 1)$, the credibility distribution function of $1-T$ is

$$\mu_{1-T}(t) = \mathrm{Cr}\{1 - T = t\} = \mathrm{Cr}\{T = 1 - t\}$$

which implies

$$1 - T \sim \mathrm{Tri}\left(0, \frac{1}{2}, 1\right).$$

By Theorem 3, the assertion of corollary is proved. □

According to Definition 1, a fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is comonotonic if and only if there exists a set $N \in \mathcal{A}$ with $\mathrm{Cr}(N) = 0$ such that for any $\gamma, \gamma' \notin N$ and any $i \ne j$,

$$(\xi_i(\gamma) - \xi_i(\gamma'))(\xi_j(\gamma) - \xi_j(\gamma')) \ge 0.$$

That is, the credibilistic comonotonicity of a fuzzy vector is equivalent to its pairwise credibilistic comonotonicity, which is stated as the following proposition:

Proposition 1. *A fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ is comonotonic if and only if (ξ_i, ξ_j) is comonotonic for all $i, j \in \{1, 2, \dots, n\}$ and $i \ne j$.*

We next present a useful property of credibilistic independence:

Theorem 4. *Let $\xi_1, \xi_2, \dots, \xi_n$ be mutually independent fuzzy variables and each monotone increasing distribution $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$ is right continuous. Then there exists a fuzzy variable Z, and non-decreasing functions f_i on $\Re$ such that*

$$\boldsymbol{\xi} \stackrel{\mathrm{d}}{=} (f_1(Z), f_2(Z), \cdots, f_n(Z)).$$

Proof. Since fuzzy variables ξ_i, $i = 1, 2, \dots, n$, are mutually independent, one has

$$\mathrm{Cr}\{\xi_1 \in B_1, \xi_2 \in B_2, \dots, \xi_n \in B_n\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \in B_i\}$$

for any $B_i \subseteq \Re, i = 1, \dots, n$.

Particularly, letting $B_i = (-\infty, t_i]$, one has

$$\mathrm{Cr}\{\xi_1 \le x_1, \xi_2 \le x_2, \dots, \xi_n \le x_n\} = \min_{1 \le i \le n} \mathrm{Cr}\{\xi_i \le x_i\},$$

which is equivalent to

$$F_{\boldsymbol{\xi}}(x_1, x_2, \cdots, x_n) = \min_{1 \le i \le n} F_{\xi_i}(x_i)$$

for all $x = (x_1, x_2, \cdots, x_n) \in \Re^n$. Let Z be fuzzy variable $\mathrm{Tri}(0, 1/2, 1)$, and

$$f_i(c) = F_{\xi_i}^{-1}(c) = \inf\{t \in \Re \mid F_{\xi_i}(t) \ge c\}, c \in (0, 1]$$

for $i = 1, 2, \dots, n$. By Theorem 3, one has

$$\boldsymbol{\xi} \stackrel{\mathrm{d}}{=} (f_1(Z), f_2(Z), \cdots, f_n(Z)).$$

which completes the proof of theorem. □

As a corollary of Theorem 4, one has

Corollary 2. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent fuzzy variables and each monotone increasing distribution $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$ is right continuous. Then exists a fuzzy variable Z, and non-increasing functions g_i on $\Re$ such that*

$$\boldsymbol{\xi} \overset{\mathrm{d}}{=} (g_1(Z), g_2(Z), \cdots, g_n(Z)).$$

For a general fuzzy vector, we have the following result:

Theorem 5. *Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ be a fuzzy vector and its marginal monotone increasing distributions $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$ are right continuous. Then there exist a comonotonic fuzzy vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_n)$ such that for each i, one has $\xi_i \overset{\mathrm{d}}{=} \eta_i$.*

Proof. Let triangular fuzzy variable $T \sim \mathrm{Tri}(0, 1/2, 1)$, and

$$\boldsymbol{\eta} = \left(F_{\xi_1}^{-1}(T), F_{\xi_2}^{-1}(T), \cdots, F_{\xi_n}^{-1}(T)\right),$$

where $F_{\xi_i}^{-1}(c) = \inf\{t \in \Re \mid F_{\xi_i}(t) \ge c\}$, $c \in (0,1]$, $i = 1, 2, \ldots, n$. Then $\boldsymbol{\eta}$ is a comonotonic fuzzy vector with $\eta_i = F_{\xi_i}^{-1}(T), i = 1, 2, \ldots, n$.

By the premise of theorem, $F_{\xi_i}(x_i) = \mathrm{Cr}\{\xi_i \le x_i\}$ is right continuous for each i. It follows from Theorem 2 that $\xi_i \overset{\mathrm{d}}{=} \eta_i$ for each i. □

4 The Case of Two-Dimensional Fuzzy Vector

In this section, we study the credibilistic comonotonicity of two-dimensional fuzzy vector (ξ_1, ξ_2). Some equivalent characterizations are summarized in the following theorem for the credibilistic comonotonicity of (ξ_1, ξ_2).

Theorem 6. *Let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ be a fuzzy vector. Then the following conditions are equivalent:*

(i) ξ_1 *and* ξ_2 *are comonotonic in the sense of Definition 1.*

(ii) There exists a set $N \in \mathcal{A}$ with $\mathrm{Cr}(N) = 0$ such that there is no pair $\gamma_1, \gamma_2 \in \Gamma \backslash N$ with $\xi_1(\gamma_1) < \xi_2(\gamma_2)$ and $\xi_2(\gamma_1) > \xi_2(\gamma_2)$.

(iii) There exists a set $N \in \mathcal{A}$ with $\mathrm{Cr}(N) = 0$ such that the set $\{(\xi_1(\gamma), \xi_2(\gamma)) \mid \gamma \in \Gamma \backslash N\}$ is a chain with respect to componentwise order in $\Re^2$.

(iv) There exists a fuzzy variable Z and non-decreasing functions f_i, $i = 1, 2$, on $\Re$ such that

$$\xi_i = f_i(Z)$$

in the sense of almost sure for $i = 1, 2$.

(v) There exists continuous and non-decreasing functions f_i, $i = 1, 2$, on $\Re$ such that $f_1(z) + f_2(z) = z, z \in \Re$, and

$$\xi_i = f_i(\xi_1 + \xi_2)$$

in the sense of almost sure for $i = 1, 2$.

Proof. $(i) \Longrightarrow (ii)$: If ξ_1 and ξ_2 are comonotonic, by Definition 1, there a comonotonic subset Ξ^2 of $\Re^2$ such that

$$\mathrm{Cr}\left\{(\xi_1, \xi_2) \in \Xi^2\right\} = 1,$$

which implies that $\mathrm{Cr}\left\{(\xi_1, \xi_2) \in \Re^2 \backslash \Xi^2\right\} = 0$. As a consequence, taking $N = \boldsymbol{\xi}^{-1}(\Re^2 \backslash \Xi^2)$, one has $\mathrm{Cr}(N) = 0$, assertion (ii) is proved.

$(ii) \Longleftrightarrow (iii)$: Straightforward.

$(iii) \Longrightarrow (i)$: If assertion (iii) holds true, then there is a set $N \in \mathcal{A}$ with $\text{Cr}(N) = 0$ such that the set $\{(\xi_1(\gamma), \xi_2(\gamma)) \mid \gamma \in \Gamma\backslash N\}$ is a chain with respect to componentwise order in $\Re^2$.

As a result, if we take

$$\Xi^2 = \{(\xi_1(\gamma), \xi_2(\gamma)) \mid \gamma \in \Gamma\backslash N\},$$

then Ξ^2 is a comonotonic subset of $\Re^2$. It is evident that $\text{Cr}(\Xi^2) = 1$. Thus assertion (i) is proved.

$(vi) \Longrightarrow (ii)$: Suppose that there exists a fuzzy variable Z and non-decreasing functions f_i, $i = 1, 2$, on $\Re$ such that

$$\xi_i = f_i(Z)$$

in the sense of almost sure for $i = 1, 2$. Then there is a set $N \in \mathcal{A}$ with $\text{Cr}(N) = 0$ such that

$$\xi_i(\gamma) = f_i(Z(\gamma))$$

for all $\gamma \in \Gamma\backslash N$ and $i = 1, 2$. It is evident that there is no pair $\gamma_1, \gamma_2 \in \Gamma\backslash N$ with $\xi_1(\gamma_1) < \xi_2(\gamma_2)$ and $\xi_2(\gamma_1) > \xi_2(\gamma_2)$, assertion (ii) is proved.

$(v) \Longrightarrow (iv)$: suppose that there exists continuous and non-decreasing functions f_i, $i = 1, 2$, on $\Re$ such that $f_1(z) + f_2(z) = z, z \in \Re$, and

$$\xi_i = f_i(\xi_1 + \xi_2)$$

in the sense of almost sure for $i = 1, 2$. Then there is a set $N \in \mathcal{A}$ with $\text{Cr}(N) = 0$ such that

$$\xi_i(\gamma) = f_i(\xi_1(\gamma) + \xi_2(\gamma))$$

for all $\gamma \in \Gamma\backslash N$ and $i = 1, 2$. If we denote $Z = \xi + \xi_2$, then

$$\xi_i(\gamma) = f_i(Z(\gamma))$$

for all $\gamma \in \Gamma\backslash N$ and $i = 1, 2$, which implies assertion (iv) is valid.

$(ii) \Longrightarrow (v)$: If assertion (ii) holds true, then there is a set $N \in \mathcal{A}$ with $\text{Cr}(N) = 0$ such that there is no pair $\gamma_1, \gamma_2 \in \Gamma\backslash N$ with $\xi_1(\gamma_1) < \xi_2(\gamma_2)$ and $\xi_2(\gamma_1) > \xi_2(\gamma_2)$. Write $Z = \xi_1 + \xi_2$. We first define f_1, f_2 on $Z(\Gamma\backslash N)$, the image of fuzzy variable Z on $\Gamma\backslash N$. If $z \in Z(\Gamma\backslash N)$, then there exist some $\gamma \in \Gamma\backslash N$ such that

$$z = x_1 + x_2,$$

where $x_1 = \xi_1(\gamma)$ and $x_2 = \xi_2(\gamma)$. If we define

$$f_1(z) = x_1, f_2(z) = x_2,$$

then it is necessary to check the uniqueness of the decomposition. In fact, suppose there exist $\gamma_1, \gamma_2 \in \Gamma\backslash N$ such that

$$\xi_1(\gamma_1) + \xi_2(\gamma_1) = z = \xi_1(\gamma_2) + \xi_2(\gamma_2).$$

Then, one has

$$\xi_1(\gamma_1) - \xi_1(\gamma_2) = -(\xi_2(\gamma_1) - \xi_2(\gamma_2)).$$

Since there is no pair $\gamma_1, \gamma_2 \in \Gamma\backslash N$ with $\xi_1(\gamma_1) < \xi_2(\gamma_2)$ and $\xi_2(\gamma_1) > \xi_2(\gamma_2)$, one has $\xi_1(\gamma_1) = \xi_1(\gamma_2)$ and $\xi_2(\gamma_1) = \xi_2(\gamma_2)$. The uniqueness of the decomposition is proved.

We next show that f_1 and f_2 are non-decreasing functions. In fact, let $z_1, z_2 \in \Gamma\backslash N$ with $z_1 < z_2$. Then there exist $\gamma_1, \gamma_2 \in \Gamma\backslash N$ such that

$$\xi_1(\gamma_1) + \xi_2(\gamma_1) = z_1 < z_2 = \xi_1(\gamma_2) + \xi_2(\gamma_2),$$

which implies that

$$\xi_1(\gamma_1) - \xi_1(\gamma_2) < -(\xi_2(\gamma_1) - \xi_2(\gamma_2)).$$

The above inequality is compatible with assertion (ii) only if

$$\xi_1(\gamma_1) - \xi_1(\gamma_2) \leq 0, (\xi_2(\gamma_1) - \xi_2(\gamma_2)) \leq 0,$$

which implies

$$f_1(z_1) \le f_1(z_2), f_2(z_1) \le f_2(z_2).$$

We now show the continuity of f_1 and f_2. For any $z, z+h \in Z(\Gamma\backslash N)$ with $h > 0$, by the monotonicity of f_1, one has $f_1(z) \le f_1(z+h)$. In addition, by

$$z + h = f_1(z+h) + f_2(z+h) \ge f_1(z+h) + f_2(z) = f_1(z+h) + z - f_1(z),$$

which implies $f_1(z+h) \le f_1(z) + h$. As a consequence, we find

$$f_1(z) \le f_1(z+h) \le f_1(z) + h.$$

It is similar to prove that

$$f_1(z) - h \le f_1(z-h) \le f_1(z)$$

for any $z, z-h \in Z(\Gamma\backslash N)$ with $h > 0$. Combining the above inequalities gives the continuity of f_1 at z. The continuity of f_2 can be proved similarly.

Finally, it is easy to extend the continuity of f_1 and f_2 from $Z(\Gamma\backslash N)$ to $\Re$. The validness of assertion (v) is proved. □

5 Conclusions and Future Research

In this study, we discussed the credibilistic comonotonicity of fuzzy vector in credibility measure theory and obtained the following major new results.

Based on the comonotonic set in Euclidean space, we first introduced the new concept of credibilistic comonotonicity via its comonotonic support.

In general case, we discussed the properties of comonotonic fuzzy vector based on the relation between the joint monotone distribution of a fuzzy vector and its marginal monotone distributions. The equivalence between the credibilistic comonotonicity and its pairwise credibilistic comonotonicity was also discussed. In particular, we further established several equivalent characterizations in Theorem 6 for two-dimensional fuzzy vector.

As an open problem, the extensions of assertions (iv) and (v) in Theorem 6 to general cases are significant issues in our future research. Applying the comonotonicity of fuzzy vector to credibilistic orders is another interesting research issue.

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