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Partial Quadratic Entropy of Uncertain Random Variables

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Abstract

Partial entropy is a measure to characterize how much of entropy of an uncertain random variable belongs to uncertain variables. In this paper, a definition of partial quadratic entropy of uncertain random variables is proposed. Furthermore, some properties of partial quadratic entropy are derived such as positive linearity.

Keywords: chance theory, uncertain random variable, quadratic entropy, partial quadratic entropy

1 Introduction

In real world, when mathematical models are built, we always have to face two types of indeterminacy. One term of indeterminacy is the phenomena whose outcomes cannot be exactly predicted such as rolling a die, roulette wheel, lifetime, stock price, coal reserve, strength of bridge, bank deposit. For modelling these types of indeterminacy, probability theory is one of the most important tools. As is known to us all, probability theory is valid when the estimated probability is close enough to the real frequency according to the law of large numbers. However, it is usually difficult to obtain the observed data because of economic reasons or technical difficulties such as coal reserve, strength of bridge. So, we have to consult with some domain experts to estimate their degrees of belief on whether that each event will occur. In this case, information and knowledge cannot be described well by random variables. Although fuzzy set theory founded by Zadeh [30] is used to model fuzziness by some researchers, a series of paradoxes presented by Liu [20] show that fuzzy set is not suitable for modelling this type of uncertain phenomena. In order to model this type of human uncertainty, Liu [20] suggests to deal with it by uncertainty theory. Uncertainty theory founded by Liu [16] is a branch of mathematics based on the normality, duality, subadditivity, and product axioms. Nowadays, uncertainty theory is well developed in both theoretical and practical aspect, for more details, see [6, 18, 19]. It is mentioned that, the world is neither random nor uncertain, but sometimes it can be analyzed by probability theory, and sometimes by uncertainty theory.

Entropy is used to characterize the degree of uncertainty in information sciences. It was first proposed by Shannon [26] for random variables in 1948. In many real cases, only little information about random variables is available, but there is an infinite number of probability distributions satisfying the given information. In this case, Jaynes [10] presented maximum entropy principle, that is of all the probability distributions with common expected value and variance, to choose the one with maximum entropy. Inspired by the Shannon entropy of random variables, fuzzy entropy was first introduced by Zadeh [30] to quantify the fuzziness in 1968. After that, it has been studied by many researchers such as De Luca and Termini [5], Kaufmann [11], Yager [28], Kosko [12], Bhandari and Pal [1].

In uncertainty theory, Liu [17] provided a definition of entropy for uncertain variables. The properties of entropy for uncertain variables were investigated by Dai and Chen [4], and the maximum entropy principle for uncertain variable was proposed by Chen and Dai [2]. In addition, the concepts of sine entropy for uncertain variable was proposed by Yao and Dai [29]. Furthermore, Dai [3] introduced a definition of quadratic entropy for uncertain variables.

However, in many cases, randomness and uncertainty exist simultaneously in a complex system. Inspired by Kwakernaak [14, 15], Puri and Ralescu [25], Kruse and Meyer [13], and Liu and Liu [23, 24], uncertain random variable was first defined by Liu [21] to describe complex systems in which uncertainty and randomness always appear together.

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Thus, in order to describe such a system, Liu [21] first proposed chance theory, which is a mathematical methodology for modelling complex systems with both uncertainty and randomness, including chance measure, uncertain random variable, chance distribution, operational law, expected value, variance. Following that, Liu [22] presented the operational law of uncertain random variable, the formula of expected value and proposed uncertain random programming as a branch of mathematical programming involving uncertain random variables. After chance theory was introduced, some experts done some useful works. For example, Gao and Sheng [7] studied law of large numbers of uncertain random variables with different chance distributions. Then Gao et al. [8] discussed order statistic of uncertain random variables and its applications. Additionally, Sheng et al. [27] introduced several types of entropy for uncertain random variables such as quadratic entropy, sine entropy.

The rest of this paper is organized as follows. In Section 2, some concepts of uncertainty theory and chance theory are recalled as they are needed. In Section 3, a definition of partial quadratic entropy of uncertain random variables is proposed and some properties are presented. Finally, some conclusions are given in Section 4.

2 Preliminaries

In this section, we will review some basic concepts and properties of uncertain variables and uncertain random variables.

2.1 Uncertain Variables

In this subsection, we provide several definitions and elementary concepts of uncertainty theory that will be used in the next sections. For more details, the reader can refer to [16, 17].

Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ is called an uncertain measure if it satisfies the following axioms:
(i) (Normality Axiom) $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ.
(ii) (Duality Axiom) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ for any event Λ.
(iii) (Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\Lambda_2,\ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\left\{\Lambda_i\right\}.$$

(iv) (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\ldots$, the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\{\prod_{k=1}^{\infty}\Lambda_k\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$ where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\ldots$, respectively.

Definition 1. *An uncertain variable ξ is a function from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers such that $\{\xi\in B\}$ is an event for any Borel set B.*

Definition 2. *The uncertain variables $\xi_1,\xi_2,\ldots,\xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i\in B_i\}\right\}=\bigwedge_{i=1}^{n}\mathcal{M}\left\{\xi_i\in B_i\right\}$$

for any Borel sets $B_1,B_2,\ldots,B_n$ of real numbers.

Theorem 1. *Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables, and $f_1,f_2,\ldots,f_n$ be measurable functions. Then $f_1(\xi_1),f_2(\xi_2),\ldots,f_n(\xi_n)$ are independent uncertain variables.*

Definition 3. (Liu [17]) *Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

Theorem 2. (Liu [17]) *Let $\xi_1,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions Φ_1, $\Phi_2,\ldots,\Phi_n$, respectively. If f is a strictly increasing function, then $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ is an uncertain variable with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha)).$$

Definition 4. (Dai [3]) *Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its quadratic entropy is defined by*

$$Q[\xi] = \int_{-\infty}^{\infty} S(\Phi(x))dx,$$

where $S(t) = t(1-t)$.

Theorem 3. (Dai [3]) *Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$Q[\xi] = \int_0^1 \Phi^{-1}(\alpha)(2\alpha - 1)d\alpha.$$

Theorem 4. (Dai [3]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an entropy

$$Q[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \ln \frac{\alpha}{1-\alpha}(1-2\alpha)d\alpha.$$

2.2 Uncertain Random Variable

The chance space refers to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, in which $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \Pr)$ is a probability space.

Definition 5. (Liu [21]) *Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of Θ is defined by*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\} \mathrm{d}r.$$

Liu [21] proved a chance measure satisfies normality, duality, and monotonicity properties, that is (i) $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$; (ii) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ for any event Θ; (iii) $\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$. Besides, Hou [9] proved the subadditivity of chance measure, that is, $\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\}$ for a countable sequence of events $\Theta_1, \Theta_2, \ldots$.

Definition 6. (Liu [21]) *An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e., $\{\xi \in B\}$ is an event for any Borel set B.*

To calculate the chance measure, Liu [22] presented a definition of chance distribution.

Definition 7. (Liu [22]) *Let ξ be an uncertain random variable. Then its chance distribution is defined by*

$$\Phi(x) = \mathrm{Ch}\{\xi \leq x\}$$

for any $x \in \mathcal{R}$.

Theorem 5. (Liu [22]) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots$, Ψ_m, respectively, and let $\tau_1, \tau_2, \ldots, \tau_n$ be uncertain variables. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x, y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)$$

where $F(x, y_1, \cdots, y_m)$ is the uncertainty distribution of ξ for any real numbers $y_1, y_2, \ldots, y_m$.

The chance distribution is consistent with probability distribution and uncertainty distribution. That is, if an uncertain random variable degenerate to a random variable, then the chance distribution becomes the probability distribution. And if an uncertain random variable degenerate to a random variable, then the chance distribution becomes the uncertainty distribution.

Definition 8. (Liu [22]) *Let ξ be an uncertain random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Ch}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Let Φ denote the chance distribution of ξ. Liu [22] proved a formula to calculate the expected value of an uncertain random variable by using chance distribution, that is,

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x.$$

Theorem 6. (Liu [21]) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables (not necessarily independent), then the uncertain random variable*

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$$

has an expected value

$$E[\xi] = \int_{\mathbb{R}^m} E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)]\mathrm{d}\Psi_1 \cdots \mathrm{d}\Psi_m$$

where $E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)]$ is the expected value of ξ for any real numbers $y_1, \ldots, y_m$.

Theorem 7. (Liu [21], Linearity of Expected Value Operator) *Assume η_1 and η_2 are random variables (not necessarily independent), τ_1 and τ_2 are independent uncertain variables, and f_1 and f_2 are measurable functions. Then*

$$E[f_1(\eta_1, \tau_1) + f_2(\eta_2, \tau_2)] = E[f_1(\eta_1, \tau_1)] + E[f_2(\eta_2, \tau_2)].$$

3 Partial Quadratic Entropy of Uncertain Random Variables

Sheng et al. [27] introduced the concept of quadratic entropy for uncertain random variables. First, we review their definition for uncertain random variables.

Definition 9. [27] *Let ξ be an uncertain random variable with chance distribution $\Phi(x)$. Then its entropy is defined by*

$$Q[\xi] = \int_{-\infty}^{\infty} S(\Phi(x))dx,$$

where $S(t) = t(1 - t)$.

However, one question may arise. How much of entropy of an uncertain random variable associated to uncertain variable? For this purpose, we introduce the concept of partial quadratic entropy.

Definition 10. *Suppose that $\eta_1, \eta_2, \ldots, \eta_m$ are independent random variables and $\tau_1, \tau_2, \ldots, \tau_m$ are uncertain variables. Then partial quadratic entropy of uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_m)$ is defined as following*

$$PQ[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} S(F(x, y_1, \cdots, y_m))dxd\Psi_1(y_1) \cdots d\Psi_m(y_m),$$

where $S(t) = t(1-t)$ and $F(x, y_1, \cdots, y_m)$ is the uncertainty distribution of uncertain variable $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_m)$ for any real numbers $y_1, \ldots, y_m$.

Remark 1: Partial entropy measures how much entropy of an uncertain random variable belongs to uncertain variables. Furthermore, a random variable has no term of uncertain variable. Hence if the uncertain random variable degenerates to random variable, then the partial entropy is zero. When an uncertain random variable degenerates to an uncertain variable, the partial quadratic entropy becomes the entropy in Definition 4.

Example 1. Suppose that $\tau \sim \mathcal{L}(0,1)$ and $\eta \sim U(0,1)$ where $\mathcal{L}(0,1)$ is linear uncertain variable and $U(0,1)$ is uniform random variable. Then the partial quadratic entropy of $\xi = \eta + \tau$ is

$$PQ[\xi] = \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} S(F(x,y))\mathrm{d}x\mathrm{d}\Psi(y) = \int_0^1 \int_0^1 [(x-y)(1-x-y)]\mathrm{d}x\mathrm{d}y = \frac{1}{6}.$$

Theorem 8. *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. If function f is measurable , then*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has partial quadratic entropy

$$PQ[\xi] = \int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha, y_1, \cdots, y_m)(2\alpha - 1)d\alpha d\Psi_1(y_1)\cdots d\Psi_m(y_m).$$

Proof. It is clear that $S(\alpha)$ is a derivable function with $S'(\alpha) = 1 - 2\alpha$. Since

$$S(F(x, y_1, \cdots, y_m)) = \int_0^{F(x,y_1,\cdots,y_m)} S'(\alpha)\mathrm{d}\alpha = \int_{F(x,y_1,\cdots,y_m)}^1 -S'(\alpha)\mathrm{d}\alpha,$$

we have

$$\begin{aligned}
PQ[\xi] &= \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} S(F(x, y_1, \cdots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&= \int_{\mathbb{R}^m} \int_{-\infty}^{0} S(F(x, y_1, \cdots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&\quad + \int_{\mathbb{R}^m} \int_0^{\infty} S(F(x, y_1, \cdots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&= \int_{\mathbb{R}^m} \int_{-\infty}^{0} \int_0^{F(x,y_1,\cdots,y_m)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&\quad - \int_{\mathbb{R}^m} \int_0^{\infty} \int_{F(x,y_1,\cdots,y_m)}^1 S'(\alpha)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

It follows from Fubini's theorem that

$$\begin{aligned}
PQ[\xi] &= \int_{\mathbb{R}^m} \int_0^{F(0,y_1,\cdots,y_m)} \int_{F^{-1}(\alpha,y_1,\cdots,y_m)}^0 S'(\alpha)\mathrm{d}x\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&\quad - \int_{\mathbb{R}^m} \int_{F(0,y_1,\cdots,y_m)}^1 \int_0^{F^{-1}(\alpha,y_1,\cdots,y_m)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&= \int_{\mathbb{R}^m} \int_0^{F(0,y_1,\cdots,y_m)} F^{-1}(\alpha, y_1, \cdots, y_m)S'(\alpha)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&\quad - \int_{\mathbb{R}^m} \int_{F(0,y_1,\cdots,y_m)}^1 F^{-1}(\alpha, y_1, \cdots, y_m)S'(\alpha)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&= -\int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha, y_1, \cdots, y_m)S'(\alpha)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m) \\
&= \int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha, y_1, \cdots, y_m)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m).
\end{aligned}$$

□

Example 2. Suppose that $\tau \sim \mathcal{L}(0,1)$ and $\eta \sim U(0,1)$ where $\mathcal{L}(0,1)$ is linear uncertain variable and $U(0,1)$ is uniform random variable. Then the partial quadratic entropy of $\xi = \eta + \tau$ is

$$\begin{aligned} PQ[\xi] &= \int_{\mathbb{R}^m} \int_0^1 F^{-1}(\alpha, y)\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= \int_0^1 \int_0^1 (\alpha + y)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}y \\ &= \frac{1}{6}. \end{aligned}$$

Remark 2: The results derived by using Definition 10 and Theorem 8 are same. And it is convenient for us to use the inverse distribution, so this theorem is necessary.

Theorem 9. *Let τ be an uncertain variable with uncertainty distribution function Φ and η be a random variable with probability distribution function Ψ. If $\xi = \eta + \tau$, then*

$$PQ[\xi] = Q[\tau].$$

Proof. It is obvious that $F^{-1}(\alpha, y) = \Phi^{-1}(\alpha) + y$, therefore by using Theorem 8, we obtain

$$\begin{aligned} PQ[\xi] &= \int_{\mathbb{R}} \int_0^1 F^{-1}(\alpha, y)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi(y) = \int_{\mathbb{R}} \int_0^1 (\Phi^{-1}(\alpha) + y)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= \int_{\mathbb{R}} \int_0^1 \Phi^{-1}(\alpha)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi(y) + \int_{\mathbb{R}} \int_0^1 y(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= Q[\tau]. \end{aligned}$$

□

Remark 3: The partial quadratic entropy of the sum of uncertain variables and random variables is the quadratic entropy of uncertain variables.

Example 3. *Suppose that $\tau \sim \mathcal{L}(0,1)$ and $\eta \sim U(1,2)$ where $\mathcal{L}(0,1)$ is linear uncertain variable and $U(1,2)$ is uniform random variable. Then the partial quadratic entropy of $\xi = \eta + \tau$ is*

$$\begin{aligned} PQ[\xi] &= \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} S(F(x,y))dxd\Psi(y) = \int_1^2 \int_0^1 (y + \alpha)(2\alpha - 1)d\alpha d(y - 1) \\ &= \frac{1}{6} = \int_0^1 x(1 - x)dx = Q[\tau]. \end{aligned}$$

Theorem 10. *Let τ be an uncertain variable with uncertainty distribution function Φ and η be a random variable with probability distribution function Ψ. If $\xi = \eta\tau$, then $PQ[\xi] = Q[\tau]E[\eta]$.*

Proof. It is obvious that $F^{-1}(\alpha, y) = \Phi^{-1}(\alpha)y$, therefore by using Theorem 8, we obtain

$$\begin{aligned} PQ[\xi] &= \int_{\mathbb{R}} \int_0^1 F^{-1}(\alpha, y)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi(y) = \int_{\mathbb{R}} \int_0^1 y\Phi^{-1}(\alpha)(2\alpha - 1)\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= \int_0^1 \Phi^{-1}(\alpha)(2\alpha - 1)\mathrm{d}\alpha \int_{\mathbb{R}} y\mathrm{d}\Psi(y) \\ &= Q[\tau]E[\eta]. \end{aligned}$$

□

Remark 4: The partial quadratic entropy of the product of an uncertain variable and an random variable is the product of quadratic entropy of the uncertain variable and the expected value of the random variable.

Example 4. *Suppose that $\tau \sim \mathcal{L}(0,1)$ and $\eta \sim U(1,2)$ where $\mathcal{L}(0,1)$ is linear uncertain variable and $U(1,2)$ is uniform random variable. Then the partial quadratic entropy of $\xi = \eta + \tau$ is*

$$\begin{aligned} PQ[\xi] &= \int_{\mathbb{R}^m} \int_{-\infty}^{\infty} S(F(x,y))dxd\Psi(y) = \int_1^2 \int_0^1 y\alpha(2\alpha-1)d\alpha d(y-1) \\ &= \frac{1}{4} = \int_0^1 \alpha(1-\alpha)d\alpha \int_1^2 yd(y-1) = Q[\tau]E[\eta]. \end{aligned}$$

Theorem 11. *Let $\eta_1, \eta_2, \ldots, \eta_n$ be independent random variables and $\tau_1, \eta_2, \ldots, \eta_n$ be independent uncertain variables. Also, suppose that*

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \ldots, \xi_n = f_n(\eta_n, \tau_n)$$

If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to x_{m+1}, $x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has partial quadratic entropy

$$\begin{aligned} PQ[\xi] = \int_{\mathbb{R}^n} \int_0^1 & f(F_1^{-1}(\alpha, y_1), F_2^{-1}(\alpha, y_2) \cdots, F_m^{-1}(\alpha, y_m), F_{m+1}^{-1}(1-\alpha, y_{m+1}), \cdots, \\ & F_n^{-1}(1-\alpha, y_n))(2\alpha - 1)d\alpha d\Psi_1(y_1) \cdots d\Psi_n(y_n). \end{aligned}$$

where $F_i^{-1}(\alpha, y_i)$ is the inverse uncertainty distribution of uncertain variable $f_i(y_i, \tau_i)$ for any real number y_i, $i = 1, 2, \ldots, n$

Proof. Since $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, it follows from Theorem 2 that

$$F^{-1}(\alpha, y_1, \cdots, y_n) = f(F_1^{-1}(\alpha, y_1), \cdots, F_m^{-1}(\alpha, y_m), F_{m+1}^{-1}(1-\alpha, y_{m+1}), \cdots, F_n^{-1}(1-\alpha, y_n)).$$

By invoking Theorem 8, the proof is complete. □

Theorem 12. *Let η_1 and η_2 be random variables with probability distribution functions Ψ_1 and Ψ_2 respectively, and τ_1 and τ_2 be uncertain variables with uncertainty distribution functions Φ_1 and Φ_2, respectively. If $\xi_1 = \eta_1 + \tau_1$ and $\xi_2 = \eta_2 + \tau_2$, then*

$$PQ[\xi_1 \xi_2] = Q[\tau_1 \tau_2] + E[\eta_1]Q[\tau_2] + E[\eta_2]Q[\tau_1].$$

Proof. It is clear that $F_1^{-1}(\alpha, y_1) = y_1 + \Phi_1^{-1}(\alpha)$ and $F_2^{-1}(\alpha, y_2) = y_2 + \Phi_2^{-1}(\alpha)$. By using Theorem 11, we have

$$\begin{aligned} PQ[\xi_1\xi_2] &= \int_{\mathbb{R}^2} \int_0^1 F^{-1}(\alpha, y_1, y_2)(1-2\alpha)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2) \\ &= \int_{\mathbb{R}^2} \int_0^1 F_1^{-1}(\alpha, y_1)F_2^{-1}(\alpha, y_2)(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2) \\ &= \int_{\mathbb{R}^2} \int_0^1 \{y_1 + \Phi_1^{-1}(\alpha)\}\{y_2 + \Phi_2^{-1}(\alpha)\}(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2) \\ &= \int_0^1 \Phi_1^{-1}(\alpha)\Phi_2^{-1}(\alpha)(2\alpha-1)\mathrm{d}\alpha + \int_{\mathbb{R}} \int_0^1 y_1\Phi_2^{-1}(\alpha)(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1) \\ &\quad + \int_{\mathbb{R}} \int_0^1 y_2\Phi_1^{-1}(\alpha)(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_2(y_2) \\ &= Q[\tau_1\tau_2] + E[\eta_1]Q[\tau_2] + E[\eta_2]Q[\tau_1]. \end{aligned}$$

□

Example 5. Suppose that $\tau_1 \sim \mathcal{L}(0,1)$, $\tau_2 \sim \mathcal{L}(1,2)$, $\eta_1 \sim U(0,1)$, and $\eta_2 \sim U(1,2)$ where $\mathcal{L}(0,1)$, $\mathcal{L}(1,2)$ are linear uncertain variables and $U(0,1)$, $U(1,2)$ are uniform random variables, respectively. If $\xi_1 = \eta_1 + \tau_1$ and

$\xi_2 = \eta_2 + \tau_2$, then $\xi_1\xi_2$ has the partial quadratic entropy

$$\begin{aligned}PQ[\xi_1 - 2\xi_2] &= Q[\tau_1\tau_2] + E[\eta_1]Q[\tau_2] + E[\eta_2]Q[\tau_1]\\ &= \int_0^1 \alpha(1+\alpha)(2\alpha-1)\mathrm{d}\alpha + \int_0^1 x\mathrm{d}x \int_0^1 (1+\alpha)(2\alpha-1)\mathrm{d}\alpha\\ &\quad + \int_1^2 x\mathrm{d}(x-1)\int_0^1 \alpha(2\alpha-1)\mathrm{d}\alpha\\ &= \frac{2}{3}.\end{aligned}$$

Theorem 13. *Let η_1 and η_2 be random variables with probability distribution functions Ψ_1 and Ψ_2 respectively, and τ_1 and τ_2 be uncertain variables with uncertainty distribution functions Φ_1 and Φ_2, respectively. If $\xi_1 = \eta_1\tau_1$ and $\xi_2 = \eta_2\tau_2$, then*

$$PQ\left[\frac{\xi_1}{\xi_2}\right] = Q\left[\frac{\tau_1}{\tau_2}\right] E[\eta_1]E\left[\frac{1}{\eta_2}\right].$$

Proof. By using a similar method of Theorem 12 and independence of random variables, the proof is straightforward. □

Example 6. Suppose that $\tau_1 \sim \mathcal{L}(0,1)$, $\tau_2 \sim \mathcal{L}(1,2)$, $\eta_1 \sim U(0,1)$, and $\eta_2 \sim U(1,2)$ where $\mathcal{L}(0,1),\ \mathcal{L}(1,2)$ are linear uncertain variables and $U(0,1),\ U(1,2)$ are uniform random variables, respectively. If $\xi_1 = \eta_1\tau_1$ and $\xi_2 = \eta_2\tau_2$, then ξ_1/ξ_2 has the partial quadratic entropy

$$\begin{aligned}PQ\left[\frac{\xi_1}{\xi_2}\right] &= Q\left[\frac{\xi_1}{\xi_2}\right] E[\eta_1]E\left[\frac{1}{\xi_2}\right]\\ &= \int_0^1 \frac{\alpha}{2-\alpha}(2\alpha-1)\mathrm{d}\alpha \int_0^1 x\mathrm{d}x \int_1^2 \frac{1}{x}\frac{1}{x^2}\mathrm{d}x\\ &= \frac{3}{8}(5\ln 2 - 2).\end{aligned}$$

Theorem 14. *Let η_1 and η_2 be independent random variables and τ_1 and τ_2 be independent uncertain variables. Also suppose that $\xi_1 = f_1(\eta_1,\tau_1)$ and $\xi_2 = f_2(\eta_2,\tau_2)$. Then for any real numbers a and b, we have*

$$PQ[a\xi_1 + b\xi_2] = |a|PQ[\xi_1] + |b|PQ[\xi_2].$$

Proof. STEP 1: We prove $PQ[a\xi_1] = |a|PQ[\xi_1]$. If $a > 0$, then the inverse uncertainty distribution of $af_1(\tau_1, y_1)$ is

$$F^{-1}(\alpha, y_1) = aF_1^{-1}(\alpha, y_1),$$

where $F_1^{-1}(\alpha, y_1)$ is the inverse uncertainty distribution of $f_1(\tau_1, y_1)$. It follows from Theorem 11 that

$$PQ[a\xi_1] = a\int_{\mathbb{R}}\int_0^1 F_1^{-1}(\alpha, y_1)(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1) = |a|PQ[\xi_1].$$

If $a < 0$, then the inverse uncertainty distribution of $af_1(\tau_1, y_1)$ is

$$F^{-1}(\alpha, y_1) = aF_1^{-1}(1-\alpha, y_1).$$

It follows from Theorem 11 that

$$\begin{aligned}PQ[a\xi_1] &= a\int_{\mathbb{R}}\int_0^1 F_1^{-1}(1-\alpha, y_1)(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\\ &= a\int_{\mathbb{R}}\int_1^0 F_1^{-1}(\alpha, y_1)(1-2\alpha)\mathrm{d}(-\alpha)\mathrm{d}\Psi_1(y_1)\\ &= -a\int_{\mathbb{R}}\int_0^1 F_1^{-1}(\alpha, y_1)(2\alpha-1)\mathrm{d}(\alpha)\mathrm{d}\Psi_1(y_1)\\ &= |a|PQ[\xi_1].\end{aligned}$$

STEP 2: We prove $PQ[\xi_1+\xi_2]=PQ[\xi_1]+PQ[\xi_2]$. Note that the inverse uncertainty distribution of $f_1(\tau_1,y_1)+f_2(\tau_2,y_2)$ is

$$F^{-1}(\alpha,y_1,y_2)=F_1^{-1}(\alpha,y_1)+F_2^{-1}(\alpha,y_2).$$

It follows from Theorem 11 that

$$PQ[\xi_1+\xi_2]=\int_{\mathbb{R}^2}\int_0^1 (F_1^{-1}(\alpha,y_1)+F_2^{-1}(\alpha,y_2))(2\alpha-1)\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)=PQ[\xi_1]+PQ[\xi_2].$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$PQ[a\xi_1+b\xi_2]=PQ[a\xi_1]+PQ[b\xi_2]=|a|PQ[\xi_1]+|b|PQ[\xi_2].$$

The theorem is proved. □

The above theorem implies that the partial quadratic entropy of uncertain random variables has the property of positive linearity.

Example 7. *Suppose that $\tau_1\sim\mathcal{L}(0,1)$, $\tau_2\sim\mathcal{L}(1,2)$, $\eta_1\sim U(0,1)$, and $\eta_2\sim U(1,2)$ where $\mathcal{L}(0,1)$, $\mathcal{L}(1,2)$ are linear uncertain variables and $U(0,1)$, $U(1,2)$ are uniform random variables, respectively. If $\xi_1=\eta_1+\tau_1$ and $\xi_2=\eta_2\tau_2$, then $\xi_1-2\xi_2$ has the partial quadratic entropy*

$$\begin{aligned}PQ[\xi_1-2\xi_2] &= PQ[\xi_1]+2PQ[\xi_2]\\ &= \int_0^1\int_0^1 (y+\alpha)(2\alpha-1)d\alpha dy+2\int_1^2\int_1^2 y(1+\alpha)(2\alpha-1)d\alpha d(y-1)\\ &= \frac{31}{2}.\end{aligned}$$

4 Conclusions

This paper studied some properties of partial quadratic entropy of uncertain random variables. We first introduced a definition of partial quadratic entropy for uncertain random variables. Then we proved a formula to calculate the partial quadratic entropy for uncertain random variables by using inverse distribution. Based on the definition and formula, several properties were derived such as positive linearity. The study of properties of other partial entropies such as cross entropy are potential works for future research.

References

[1] Bhandari, D., and N.R. Pal, Some new information for fuzzy sets, *Information Science*, vol.67, no.3, pp.209–228, 1993.

[2] Chen, X.W., and W. Dai, Maximum entropy principle for uncertain variables, *International Journal of Fuzzy Systems*, vol.13, no.3, pp.232–236, 2011.

[3] Dai, W., Quadratic entropy of uncertain variables, *Information: An International Interdisciplinary Journal*, 2012, http://orsc.edu.cn/online/100707.pdf.

[4] Dai, W., and X.W. Chen, Entropy of function of uncertain variables, *Mathematical and Computer Modelling*, vol.55, nos.3-4, pp.754–760, 2012.

[5] De Luca, A., and S. Termini, A definition of nonprobabilitistic entropy in the setting of fuzzy sets theory, *Information and Control*, vol.20, pp.301–312, 1972.

[6] Gao, R., Milne methods for solving uncertain differential equations, *Applied Mathematics and Computation*, vol.274, pp.774–785, 2016.

[7] Gao, R., and Y.H. Sheng, Law of large numbers for uncertain random variables with different chance distributions, *Journal of Intelligent and Fuzzy Systems*, vol.31, pp.1227–1234, 2016 .

[8] Gao, R., Sun, Y., and D.A. Ralescu, Order statistics of uncertain random variables with application to k-out-of-n system, *Fuzzy Optimization and Decision Making*, doi: 10.1007/s10700-016-9245-9.

[9] Hou, Y.C., Subadditivity of chance measure, *Journal of Uncertainty Analysis and Applications*, vol.2, article 14, 2014.

[10] Jaynes, E., Information theory and statistical mechanics, *Phisical Reviews*, vol.106, no.4, pp.620–630, 1957.

[11] Kaufmann, A., *Introduction to the Theory of Fuzzy Subsets*, vol.1, Academic Press, New York, 1975.

[12] Kosko, B., Fuzzy entropy and conditioning, *Information Sciences*, vol.40, pp.165–174, 1986.

[13] Kruse, R., and K. Meyer, *Statistics with Vague Data,* Reidel Publishing Company, Dordrecht, 1987.

[14] Kwakernaak, H., Fuzzy random variables-1: definitions and theorems, *Information Sciences*, vol.15, pp.1–29, 1978.

[15] Kwakernaak, H., Fuzzy random variables-2: algorithms and examples for the discrete case, *Information Sciences*, vol.17, pp.713–720, 1979.

[16] Liu, B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[17] Liu, B., Some research problems in uncertainty theory, *Journal of Uncertain Systems*, vol.3, no.1, pp.3–10, 2009.

[18] Liu, B., *Theory and Practice of Uncertain Programming*, 2nd Edition, Springer, Berlin, 2009.

[19] Liu, B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.

[20] Liu, B., Why is there a need for uncertainty theory? *Journal of Uncertain Systems*, vol.6, no.1, pp.3–10, 2012.

[21] Liu, Y.H., Uncertain random variables: a mixture of uncertainty and randomness, *Soft Computing*, vol.17, no.4, pp.625–634, 2013.

[22] Liu, Y.H., Uncertain random programming with applications, *Fuzzy Optimization and Decision Making*, vol.12, no.2, pp.153–169, 2013.

[23] Liu, Y.K., and B. Liu, Fuzzy random variables: a scalar expected value operator, *Fuzzy Optimization and Decision Making*, vol.2, no.2, pp.143–160, 2003.

[24] Liu, Y.K., and B. Liu, Fuzzy random programming with equilibrium chance constraints, *Information Sciences*, vol.170, pp.363–395, 2005.

[25] Puri, M., and D. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, vol.114, pp.409–422, 1986.

[26] Shannon, C., A mathematical theory of communication, *Bell System Techcnical Journal*, vol.27, pp.373–423, 1948.

[27] Sheng, Y.H., Shi, G., and D.A. Ralescu, Entropy of uncertain random variables with application to minimum spanning tree problem, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, pp.1–17, 2015.

[28] Yager, R.R., On measures of fuzziness and negation, part I: membership in the unit interval, *International Journal of General Systems*, vol.5, pp.221–229, 1979.

[29] Yao, K., Gao, J.W., and W. Dai, Sine entropy for uncertain variables, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.21, no.5, pp.1–11, 2013.

[30] Zadeh, L.A., Fuzzy sets, *Information and Control*, vol.8, pp.338–353, 1965.