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Fuzzy Order Statistics based on $\alpha-$pessimistic

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Abstract

In this paper, a new method for developing crisp order statistics for fuzzy random variables is proposed. First, a notation of fuzzy random variables is expressed and then the $\alpha-$pessimistic value of fuzzy random variables is used to extend the crisp order statistics. For this purpose, the concepts of fuzzy cumulative distribution function and fuzzy empirical cumulative distribution are defined. The proposed method is simple and different from those existing in the literature.

Keywords: order statistics, fuzzy random variable, $\alpha-$pessimistic, fuzzy (empirical) cumulative distribution function

1 Introduction

The theory of fuzzy sets and systems has been widely employed in many pattern analysis applications. It is well known tool for formulation and analysis of imprecise and subjective concepts. Fuzzy concepts have been discussed by many authors, including Deschrijver and Kerre [10], Watanabe and Imaizumi [30], Arnold [4, 7], Holena [12, 13], Taheri and Behboodian [25], Buckley [5, 6], Viertl [29], Thompson and Geyer [26], Taheri and Arefi [24], Akbari and Rezaei [2], Parchami et al. [20], Korner's [14], Montenegro et al. [18], Gonzalez et al. [11], Akbari and Rezaei [3].

Order statistics are indispensable in many statistical procedure. Therefore, order statistics is one of the most studied problems. For situation that we deal with fuzzy random variables, fuzzy order statistics can be important and useful in application. Fuzzy order statistics based on $\alpha-$cuts has been discussed by Akbari and Rezaei [1].

Our aim in the present article is to develop fuzzy order statistics using $\alpha-$pessimistic and its cumulative distribution function . The proposed method is simple and completely different from that mentioned above. This method is different because we apply the $\alpha-$pessimistic value of fuzzy random variables to extend the crisp order statistics.

Several ranking methods have been proposed in the literature. See Cheng [9], Modarres and Sadi-Nezhad [17], Nojavan and Ghazanfari [19] and Yao and Wu [32] and Chachi and Taheri [8]. These methods are based on different definitions and our method in the present paper is using $\alpha-$pessimistic.

The paper is organize in the following way: in Section 2, some basic concepts of canonical fuzzy numbers, fuzzy random variables and fuzzy cumulative distribution functions are described. In Section 3, fuzzy order statistics using $\alpha-$pessimistic are introduced. Finally, in Section 4, the obtained results are summarized.

2 Preliminaries

If $\mathcal{X}$ (on real line $\mathcal{R}$) be a universal space, then a fuzzy subset $\widetilde{x}$ of X is defined by its membership function $\mu_{\widetilde{x}} : X \to [0,1]$. When there exist $x \in calX$ such that $\mu_{\widetilde{x}}(x) = 1$ we call $\widetilde{x}$ is called a normal fuzzy set. Also $\widetilde{x}$ is called a convex fuzzy set if $\mu_{\widetilde{x}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\widetilde{x}}(x)\ ,\ \mu_{\widetilde{x}}(y))$ for all $\lambda \in [0,1]$.

The $\alpha-$cut set of $\widetilde{x}$ is denoted by $\widetilde{x}_{[\alpha]} = \{x : \mu_{\widetilde{x}}(x) \geq \alpha\}$ where $\widetilde{x}_{[0]}$ is the closure of the set $\{x : \mu_{\widetilde{x}}(x) > 0\}$. The the fuzzy set $\widetilde{x}$ is called a fuzzy number if $\widetilde{x}$ is normal convex fuzzy set and its $\alpha-$cut sets, is bounded

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$\forall\alpha \neq 0$, and $\widetilde{x}$ is called a closed fuzzy number if $\widetilde{x}$ is fuzzy number and its membership function $\mu_{\widetilde{x}}$ is upper semicontinues, and $\widetilde{x}$ is called a bounded fuzzy number if $\widetilde{x}$ is a fuzzy number and its membership function $\mu_{\widetilde{x}}$ has compact support.

Let $\widetilde{x}$ be a closed and bounded fuzzy number with $x^L_{[\alpha]} = \inf\{x : x \in \widetilde{x}_{[\alpha]}\}$ and $x^U_{[\alpha]} = \sup\{x : x \in \widetilde{x}_{[\alpha]}\}$ and its membership function be strictly increasing on the interval $[x^L_{[\alpha]}, x^L_{[1]}]$ and strictly decreasing on the interval $[x^U_{[1]}, x^U_{[\alpha]}]$, then $\widetilde{x}$ is called canonical fuzzy number.

The imprecision or vagueness can be treated by means of a particular kind of fuzzy numbers, the LR-fuzzy numbers. These numbers are useful in practice because they can be characterized by means of three real numbers: the center, the left spread, and the right spread. The term LR is due to the left (L) and the right (R) shape of the membership function referred to the fuzzy set. We denote the LR fuzzy number by $\widetilde{x} = (l, x, r)_{LR}$ where $x \in \mathcal{R}$ is the central value, $l \in \mathcal{R}^+$, $r \in \mathcal{R}^+$ are left and right spreads, respectively. Also $L : \mathcal{R}^+ \to [0, 1]$ and $R : \mathcal{R}^+ \to [0, 1]$, with $L(0) = R(0) = 1$, are decreasing left and right shape functions with the following membership function

$$\mu_{\tilde{x}}(t) = \begin{cases} L(\frac{x-t}{l}) & \text{if} \quad t \leq x \\ R(\frac{t-x}{r}) & \text{if} \quad t \geq x. \end{cases}$$

The α-cut of $\widetilde{x}$ can be obtain by

$$\widetilde{x}_{[\alpha]} = [x^L_{[\alpha]}, x^U_{[\alpha]}] = [n - L^{-1}(\alpha)l, n + R^{-1}(\alpha)r], \quad \alpha \in [0, 1].$$

2.1 Fuzzy Canonical Number and Fuzzy Random Variable

Let $x \in \mathcal{R}$ be a real number. $\widetilde{x}$ is a fuzzy real number induced by the real number x if for membership function $\mu_{\widetilde{x}}(r)$, $\mu_{\widetilde{x}}(x) = 1$ and $\mu_{\widetilde{x}}(r) < 1$ for $r \neq x$. Suppose $\mathcal{F}(\mathcal{R})$ is the set of all fuzzy real numbers induced by the real number system $\mathcal{R}$ and $\widetilde{x}_1 \sim \widetilde{x}_2$ iff $\widetilde{x}_1$ and $\widetilde{x}_2$ are induced by the same real number x. The relation $\sim$ on $\mathcal{F}(\mathcal{R})$ is defined and then $\sim$ is an equivalence relation, which induces the equivalence classes $[\widetilde{x}] = \{\widetilde{a} : \widetilde{a} \sim \widetilde{x}\}$. The quotient set $\mathcal{F}(\mathcal{R})/\sim$ is the set of all equivalence classes and we call $\mathcal{F}(\mathcal{R})/\sim$ as the fuzzy real number system. In practice, only one element $\widetilde{x}$ from each equivalence class $[\widetilde{x}]$ is taken to from the fuzzy real number system $(\mathcal{F}(\mathcal{R})/\sim)$ that is,

$$(\mathcal{F}(\mathcal{R})/\sim) = \{\widetilde{x} : \widetilde{x} \in [\widetilde{x}], \widetilde{x} \ \ is\ the\ only\ element\ from\ [\widetilde{x}]\}.$$

When the fuzzy real number system $(\mathcal{F}(\mathcal{R})/\sim)$ consists all of canonical fuzzy real numbers then we call $(\mathcal{F}(\mathcal{R})/\sim)$ as the canonical fuzzy real number system.

In the following, an index to compare fuzzy number $\widetilde{a} \in \mathcal{F}(\mathcal{R})$ and crisp value $x \in \mathcal{R}$ is introduced. Then we used the index for defining a new notion of fuzzy random variable.

Definition 1 (Liu [16]) *Let $\widetilde{a} \in \mathcal{F}(\mathcal{R})$ and $x \in \mathcal{R}$. The index*

$$C : \mathcal{F}(\mathcal{R}) \times \mathcal{R} \longrightarrow [0, 1],$$

which is defined by

$$C\{\widetilde{a} \leq x\} = \frac{\sup_{y \leq x} \mu_{\widetilde{a}}(y) + 1 - \sup_{y > x} \mu_{\widetilde{a}}(y)}{2},$$

shows the credibility degree that $\widetilde{a}$ is less than or equal to x. Similarly, $C\{\widetilde{a} > x\} = 1 - C\{\widetilde{a} \leq x\}$ shows the credibility degree that $\widetilde{a}$ is greater than x.

Definition 2 *Let $\widetilde{a} \in \mathcal{F}(\mathcal{R})$ and $\alpha \in [0, 1]$, then*

$$\widetilde{a}_\alpha = \inf\{x \in \widetilde{A}[0] : C\{\widetilde{a} \leq x\} \geq \alpha\},$$

is called the α-pessimistic value of $\widetilde{a}$. It is evident that $\widetilde{a}_\alpha$ is a non-decreasing function of $\alpha \in (0, 1]$ (Peng and Liu [21]).

Example 1 *Let $\widetilde{x} = (l, x, r)_{LR}$ be a LR-fuzzy number and $x \in \mathcal{R}$, then*

$$C\{\widetilde{x} \leq t\} = \begin{cases} \frac{1}{2}L(\frac{x-t}{l}), & t \leq x \\ 1 - \frac{1}{2}R(\frac{t-x}{r}), & t \geq x. \end{cases}$$

The α-pessimistic values of $\widetilde{x}$ can be obtained as follows

$$\widetilde{x}_\alpha = \begin{cases} x - lL^{-1}(2\alpha), & 0.0 < \alpha \leq 0.5 \\ x + rR^{-1}(2(1-\alpha)), & 0.5 \leq \alpha \leq 1.0. \end{cases}$$

For example, consider the triangular fuzzy number $\widetilde{A} = (l, a, r)_T$, then we have

$$C\{\widetilde{A} \leq t\} = \begin{cases} 0, & t < a - l \\ \frac{t-a+l}{2l}, & a - l \leq t \leq a \\ \frac{t-a+r}{2r}, & a \leq t \leq a + r \\ 1, & t > a + r \end{cases}$$

and

$$\widetilde{A}_\alpha = \begin{cases} a - l(1 - 2\alpha), & 0.0 < \alpha \leq 0.5 \\ a - r(1 - 2\alpha), & 0.5 \leq \alpha \leq 1.0. \end{cases}$$

Lemma 1 *Suppose that $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathcal{R})$ and λ is a real number. Then*

1. $(\widetilde{a} \oplus \widetilde{b})_\alpha = \widetilde{a}_\alpha + \widetilde{b}_\alpha$.
2. $(\lambda \otimes \widetilde{b})_\alpha = \lambda \times \widetilde{a}_\alpha$.

Consider a random experiment is described on the probability space $(\Omega, \mathcal{A}, P)$, where Ω is a set of all possible outcomes of the experiment, $\mathcal{A}$ is a $\sigma-$algebra of subsets of Ω and P is a probability measure on the measurable space $(\Omega, \mathcal{A})$.

Definition 3 *A fuzzy random variable is a function $\widetilde{X} : \Omega \to \mathcal{F}(\mathcal{R})$ where*

$$\{(\omega, x) : \omega \in \Omega\ ,\ x \in \widetilde{X}_{[\alpha]}(\omega)\} \in\ \mathcal{F} \times \mathcal{B} \qquad \forall \alpha \in [0, 1],$$

where $\mathcal{B}$ is a $\sigma-$algebra of open sets.

Therefore, all $\alpha-$cuts of $\widetilde{X}$ are compact convex random set and moreover, the above definition is equivalent to the definitions by Puri and Ralescu [23], and Kwakernaak [15].

Lemma 2 *Let $\mathcal{F}(\mathcal{R})$ be a canonical fuzzy real number system. Then $\widetilde{X}$ is a fuzzy random variable iff $X^L_{[\alpha]}$ and $X^U_{[\alpha]}$ are random variables for all $\alpha \in [0, 1]$.*

One can easily show that the following relations are held between the definition of fuzzy random variable proposed in the present paper and Puri and Ralescu definition

$$\widetilde{X}_\alpha = \begin{cases} X^L_{[2\alpha]}, & 0.0 < \alpha \leq 0.5 \\ X^U_{[2(1-\alpha)]}, & 0.5 \leq \alpha \leq 1.0 \end{cases}$$

and

$$\widetilde{X}_{[\alpha]} = [\widetilde{X}_{\frac{\alpha}{2}}, \widetilde{X}_{1-\frac{\alpha}{2}}], \qquad \alpha \in (0, 1].$$

Lemma 3 *$\widetilde{X} : \Omega \to \mathcal{F}(\mathcal{R})$ is a fuzzy random variable (Definiteness 3) iff the crisp-valued mapping $\widetilde{X}_\alpha : \Omega \to \mathcal{R}$ is a crisp-valued random variable on $(\Omega, \mathcal{A}, P)$.*

2.2 Fuzzy Cumulative Distribution Function

In this subsection, we introduce the cumulative distribution function for a fuzzy random variable.

Definition 4 *The fuzzy cumulative distribution function (fcdf) of fuzzy random variables $\widetilde{X}$ at the $x \in \mathcal{R}$ is defined as fuzzy set $\widetilde{F}_{\widetilde{X}}(x)$ with the following membership function:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup_{0\leq\alpha\leq 1} \alpha I_{\widetilde{F}_{\widetilde{X}}(x)[\alpha]}(y), \qquad y \in [0,1],$$

where I is indicator function and $\alpha-$cuts are defined as follows:

$$\begin{aligned}\widetilde{F}_{\widetilde{X}}(x)[\alpha] &= [\inf_{\beta\geq\alpha} F_{\tilde{X}_\beta}(x), \sup_{\beta\geq\alpha} F_{\tilde{X}_\beta}(x)]\\ &= [F_{\tilde{X}_1}(x), F_{\tilde{X}_\alpha}(x)]\\ &= [P(\tilde{X}_1 \leq x), P(\tilde{X}_\alpha \leq x)].\end{aligned}$$

Lemma 4 *Suppose $\widetilde{X}$ is a fuzzy random variable. If $\widehat{F}_{\tilde{X}_\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[\tilde{X}_{i\alpha},\infty)}(x)$, then*

$$P(\sup_{x\in\mathcal{R}} |F_{\tilde{X}_\alpha}(x) - \widehat{F}_{\tilde{X}_\alpha}(x)| \to 0) = 1.$$

Definition 5 *Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables. We say that $\widetilde{X}$ and $\widetilde{Y}$ are independent iff $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are independent, for all $\alpha \in [0,1]$.*

Definition 6 *We say $\widetilde{X}$ and $\widetilde{Y}$ are identically distributed iff $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are identically, for all $\alpha \in (0,1]$.*

Definition 7 *We say $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_n)$ is a fuzzy random sample iff $\widetilde{X}_i$'s are independent and identically distributed.*

Example 2 *Let $\widetilde{Y} = X \oplus \widetilde{a}$, where X is a crisp standard normal random variable, i.e. $X \sim N(0,1)$, and $\widetilde{a} \in \mathcal{F}(\mathcal{R})$ is a constant fuzzy number. For example, suppose $\widetilde{a} = (l,a,r)_{LR}$. Then, we have*

$$\widetilde{X}_\alpha = \begin{cases} X + a - lL^{-1}(2\alpha) \sim N(a - lL^{-1}(2\alpha), 1), & 0.0 < \alpha \leq 0.5\\ X + a + rR^{-1}(2(1-\alpha)) \sim N(a + rR^{-1}(2(1-\alpha), 1), & 0.5 \leq \alpha \leq 1.0.\end{cases}$$

Moreover,

$$\begin{aligned}F_{\widetilde{X}_\alpha}(x) &= P(\widetilde{X}_\alpha \leq x)\\ &= \begin{cases}\Phi(x - a + lL^{-1}(2\alpha)), & 0.0 < \alpha \leq 0.5\\ \Phi(x - a - rR^{-1}(2(1-\alpha)), & 0.5 \leq \alpha \leq 1.0,\end{cases}\end{aligned}$$

where Φ is the cumulative distribution function of the standard normal random variable.

For $\widetilde{a} = (0,1,1)_T$, we easily obtain $P(\widetilde{X}_\alpha \leq x) = \Phi(x + 1 - 2\alpha)$. In Figure 1, the fuzzy cumulative distribution function is shown.

Example 3 *Suppose that, based on a fuzzy random sample of size $n = 70$, the triangular fuzzy numbers presented in Table 1 are observed. According to Lemma 4,*

$$\widehat{F}_{\tilde{X}_\alpha}(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[\tilde{X}_{i\alpha},\infty)}(x),$$

and the fuzzy empirical cumulative distribution function of this fuzzy random sample is shown in Figure 2.

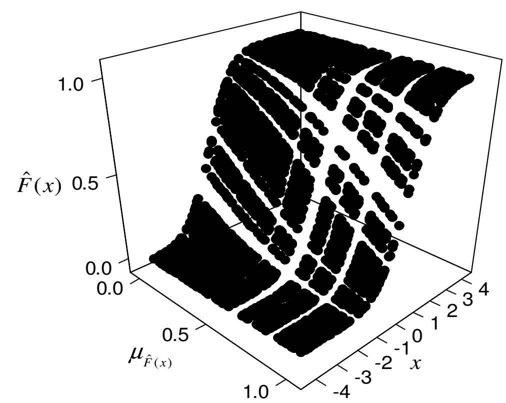

Figure 1: The fuzzy cumulative distribution function in Example 2

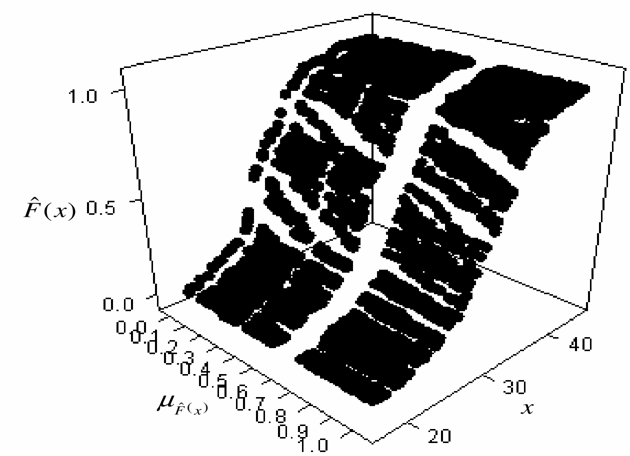

Figure 2: The fuzzy empirical cumulative distribution function in Example 3

3 Fuzzy Order Statistics

In this section, a method for order statistics based on a fuzzy random sample is introduced and then we exhibit its fcdf.

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics based on a random sample $X_1, X_2, ..., X_n$, where X_i have the cumulative distribution function $F(x)$.

Now, we define the fuzzy order statistic $\widetilde{X}_{r:n}$ as follows:

Definition 8 *We define the membership function of the r-th fuzzy order statistic $\widetilde{X}_{r:n}$, as*

$$\mu_{\widetilde{X}_{r:n}}(y) = \sup_{0 \leq \alpha \leq 1} \alpha I_{[X^L_{r:n[\alpha]}, X^U_{r:n[\alpha]}]}(y),$$

where

$$\begin{aligned} X^L_{r:n[\alpha]} &= \inf_{\alpha \leq \beta \leq 1} \widetilde{X}_{r:n\beta} \\ &= \widetilde{X}_{r:n\alpha}, \qquad r = 1, 2, ..., n, \end{aligned}$$

Table 1: Fuzzy data in Example 3

l	x	r	l	x	r
1.24523	25.2909	1.72864	1.37160	31.0334	4.03625
2.94728	29.9799	3.18712	2.90042	34.8163	0.90158
1.68255	33.0093	1.14269	1.81006	20.0322	3.35532
0.61048	29.2955	0.96326	3.27032	27.0381	0.67943
0.23180	21.9519	0.59364	0.74194	33.8341	2.85218
0.04679	19.8198	0.25364	2.59981	34.6934	1.39951
3.76874	28.7143	0.27734	2.62457	29.2538	1.89902
1.01127	29.7808	4.00663	2.42268	33.5577	2.18337
0.50682	28.3500	1.43101	3.82563	30.3166	4.10782
3.22431	30.2153	4.84203	2.04337	28.5156	1.42498
0.11132	30.8110	2.01869	0.18114	29.9150	1.22128
0.46526	33.1391	2.25508	2.04712	34.4876	0.89168
0.01467	30.4639	2.75841	2.35620	29.9235	1.13474
4.53909	33.3049	4.45906	0.84717	20.5256	2.31540
2.15167	27.1056	3.96002	3.85216	28.4781	1.14261
2.08485	28.9761	4.08542	2.86292	32.3726	1.31776
1.83341	24.4571	1.14221	4.00407	35.1992	0.00319
4.69474	31.6310	4.46000	4.42257	28.7133	1.01304
2.97855	31.6484	4.89763	0.81213	39.2002	4.91557
0.31238	32.5505	0.67419	0.98426	28.7888	0.13256
1.42086	30.1117	2.97482	0.95078	33.1867	2.00250
1.37918	36.5644	2.16975	1.76919	35.7839	3.01427
0.51162	26.2165	0.04566	1.79940	27.8816	1.39269
2.69751	24.9689	1.05614	0.40069	39.8767	1.30138
0.73571	34.6551	3.12708	2.83817	26.6043	0.49662
4.47939	28.4524	2.87889	2.64286	34.2790	4.69371
2.56010	33.7513	3.12593	1.25291	28.8742	1.56941
3.54952	29.0574	4.05796	4.83258	33.3633	0.59551
4.23356	38.6393	0.47553	4.09266	40.8332	2.15452
3.17130	29.0497	2.71479	4.09861	29.8499	0.00353
3.75628	21.4551	0.33281	1.82383	35.6830	4.50212
4.46469	34.0926	2.10675	3.96161	25.3679	4.47904
1.73810	28.6539	0.30479	0.62442	24.6808	2.55573
4.88462	30.3194	3.38772	1.40866	32.9128	1.54763
4.36423	30.4591	1.17304	3.26236	22.1262	2.60108

and

$$\begin{aligned} X^U_{r:n[\alpha]} &= \sup_{\alpha\le\beta\le 1} \widetilde{X}_{r:n\beta} \\ &= \widetilde{X}_{r:n1}, \qquad r = 1,2,...,n. \end{aligned}$$

Example 4 *According to Example 3, the membership function of fuzzy order statistics is shown in Figure 3.*

Lemma 5 *From Definition 8, and for any $\alpha \in [0,1]$, we have*

$$\textbf{1)} \quad X^L_{1:n[\alpha]} \le X^L_{2:n[\alpha]} \le \cdots \le X^L_{n:n[\alpha]}.$$

$$\textbf{2)} \quad X^U_{1:n[\alpha]} \le X^U_{2:n[\alpha]} \le \cdots \le X^U_{n:n[\alpha]}.$$

$$\textbf{3)} \quad X^L_{r:n[\alpha]} \le X^U_{r:n[\alpha]}, \qquad r = 1,2,...,n.$$

Definition 9 *We define the membership function of the fuzzy cumulative distributions of the r-th fuzzy order statistic $\widetilde{X}_{r:n}$, denoted by $\widetilde{F}_r(x)$ at $x \in \mathcal{R}$, as the following:*

$$\mu_{\widetilde{F}_r(x)}(y) = \sup_{0\le\alpha\le 1} \alpha I_{\widetilde{F}_r(x)[\alpha]}(y).$$

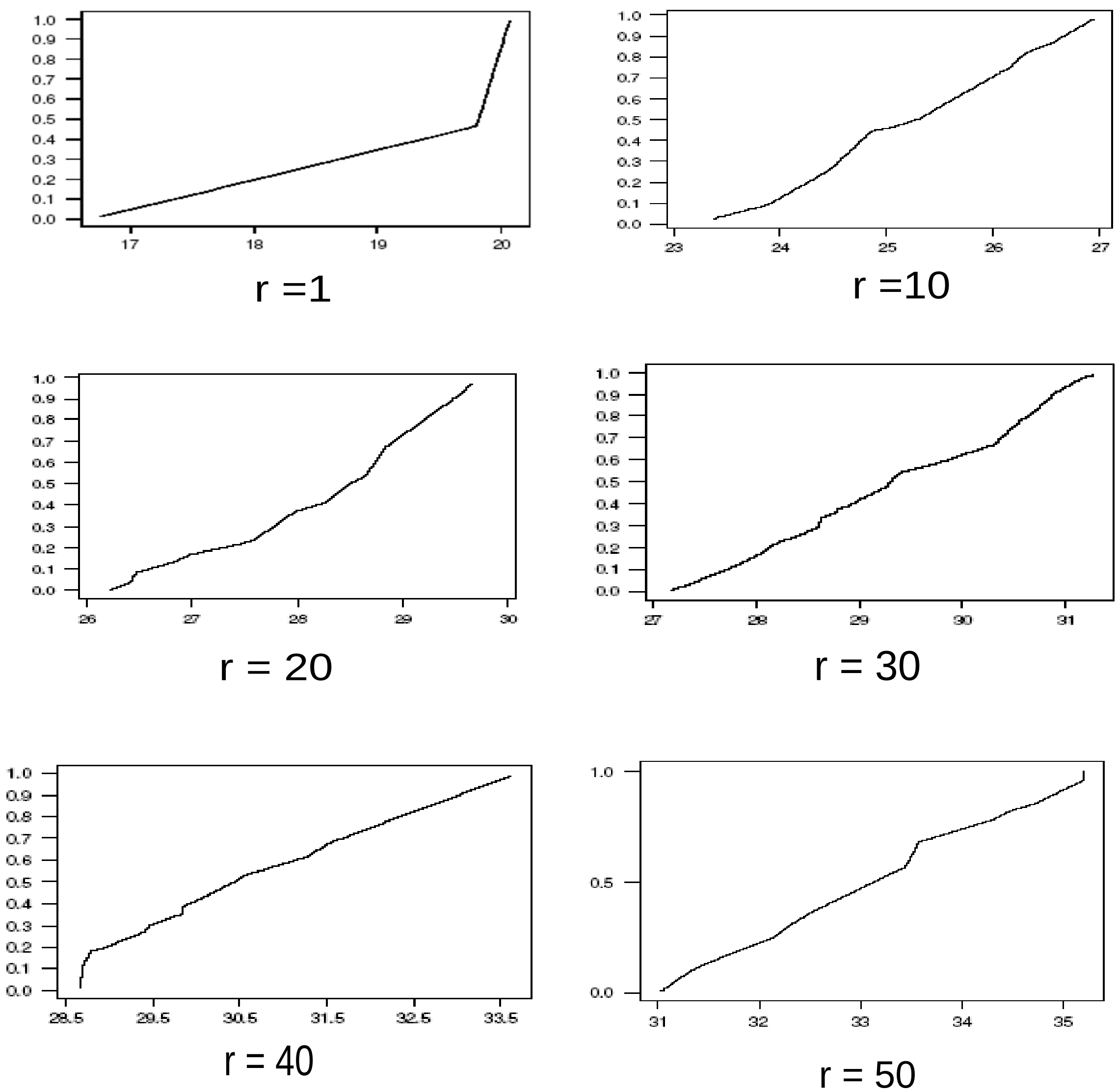

Figure 3: The fuzzy order statistics in Example 4

By Definition 4, we have

$$\begin{aligned}\widetilde{F}_r(x)[\alpha] &= [\inf_{\alpha\le\beta\le 1} F_{\widetilde{X}_{r:n\beta}}(x), \sup_{\alpha\le\beta\le 1} F_{\widetilde{X}_{r:n\beta}}(x)\\ &= [F_{\widetilde{X}_{r:n1}}(x), F_{\widetilde{X}_{r:n\alpha}}(x)]\\ &= [P(\widetilde{X}_{r:n1}\le x), P(\widetilde{X}_{r:n\alpha}\le x)], \qquad \alpha\in(0,1],\end{aligned}$$

where

$$\begin{aligned}P(\widetilde{X}_{r:n1}\le x) &= \sum_{j=r}^{n}\frac{n!}{j!(n-j)!}F^{j}_{\widetilde{X}_1}(x)(1-F_{\widetilde{X}_1}(x))^{n-j},\\ P(\widetilde{X}_{r:n\alpha}\le x) &= \sum_{j=r}^{n}\frac{n!}{j!(n-j)!}F^{j}_{\widetilde{X}_\alpha}(x)(1-F_{\widetilde{X}_\alpha}(x))^{n-j}, \qquad \alpha\in(0,1].\end{aligned}$$

Example 5 *According to Example 3 the fuzzy emetical cumulative distribution of order statistics for $r = n$ is shown in Figure 4.*

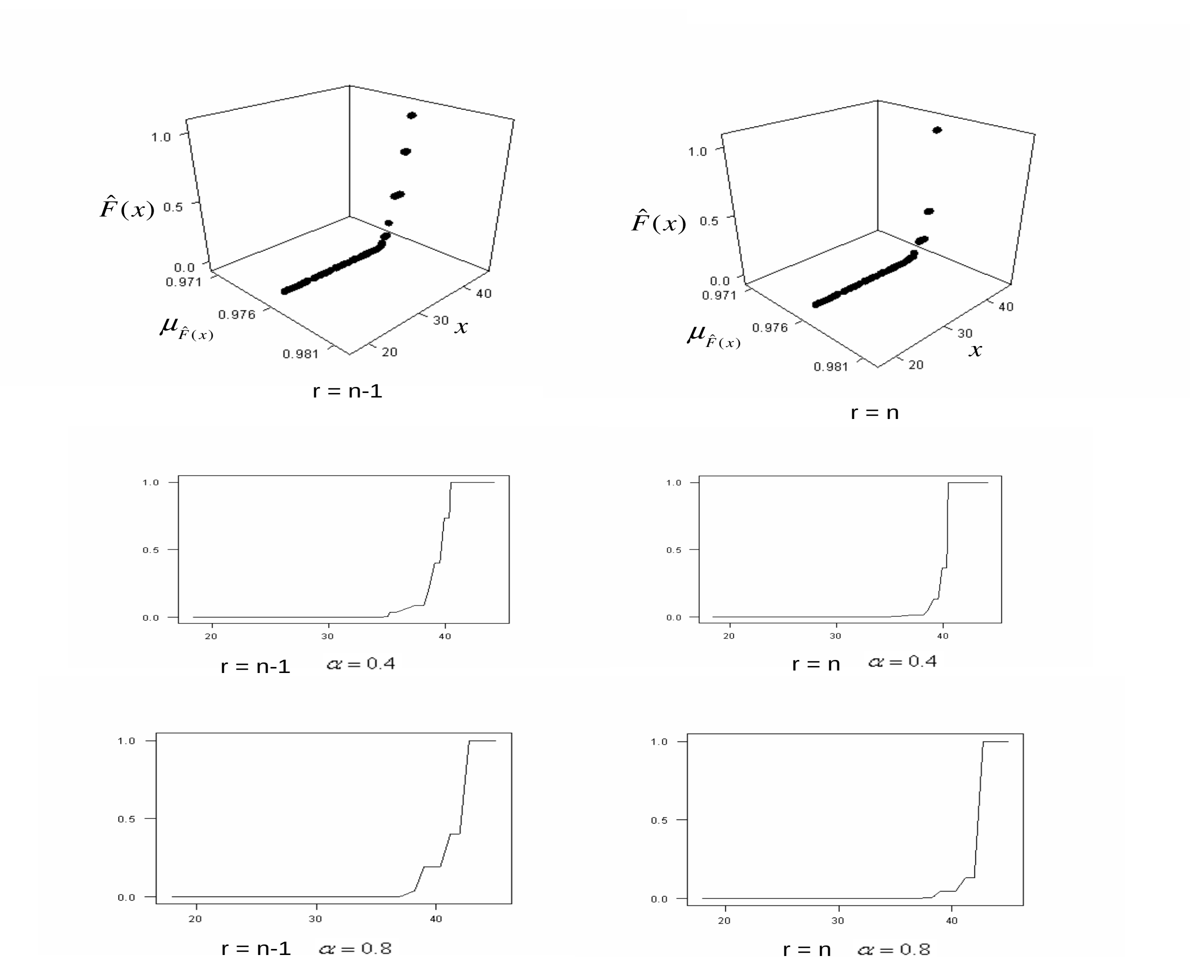

Figure 4: The emetical cumulative distribution function in Example 3

4 Conclusions

In this paper, a technique in order to obtain the distribution of order statistics using fuzzy random variables based on $\alpha-$pessimistic approach have introduced. Extension of the proposed method to stochastic orders between fuzzy order statistics and fuzzy record statistics is a potential area for the future work.

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