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Approximate Equilibrium Situations with Possible Concessions in Finite Noncooperative Game by Sampling Irregularly Fundamental Simplexes as Sets of Players' Mixed Strategies

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Abstract

An approximation technique for solving finite noncooperative game is represented. By this technique, the approximate solution is found in two stages. Firstly, fundamental simplexes as sets of players' mixed strategies are converted into finite lattices. Probabilities on simplex are allowed to be sampled irregularly for the conversion generalization. Requirements imposed on the simplexes' conversion are stated purposely to maintain the approximation quality. Secondly, approximate Nash equilibrium strategies are searched on the finite set of situations in mixed strategies. In order not to obtain empty solution, the corresponding payoff inequalities are relaxed. Algorithms for searching both the approximate Nash equilibria and approximate strong Nash equilibria are general-schemed. By adjusting magnitudes of relaxations, a unique (strong) Nash equilibrium can be got. The represented approximation technique is a constructive practice-oriented development of the finite noncooperative game and models based on it. Universality of the approximation technique is determined with the generalized sampling and controlled equilibration, and the general-schemed search algorithms.

Keywords: finite noncooperative game, fundamental simplex, finite lattice, approximate (strong) Nash equilibria, search algorithm

1 Introduction

For predicting and controlling interest interaction processes through discrete time, an appropriate model [29, 13] is the noncooperative game (NCG). NCG fits rationally for resources allocation [29, 5, 32]. The most essential NCG-modeling is for enterprise interaction in macro- and microeconomics [20, 4], for discharging bulky queries and queues in maintenance and servicing [5, 31, 9], for predicting and controlling [29, 2] socio-ecological processes (events), and, specifically, for decision-making under non-predictable uncertainties [19, 11]. NCG proposes equilibrium or efficient situations with the players' strategies therein to dissolve controversy and competition. Using these strategies theoretically ensures rationality, fairness, equilibrium, and relative utility.

For practicing, the strategy support would better contain probabilities having one or two digits after decimal point [18, 26]. Otherwise, relative statistical frequency (RSF) is not going to be reached needing thousands of cycles of that the interaction conditions would repeat. RSF is inseparably linked with practicing the game solutions, if their elements are not pure strategies. This is, why infinite NCGs (INCGs) nonetheless are tried to be brought to finite NCGs (FNCGs) [26, 17, 21, 3, 1]. FNCG, unlike INCG, has only finite supports of strategies. If infinite, the set of pure strategies really used at least once is of zero measure. Eventually, infinite support strategy (ISS) exists just mathematically, and is not implemented.

Decimal point digit limitation (DPDL) in FNCG solution strategies (FNCGSS) and INCG inapplicability (INCGI) clarify essentiality of NCGs. Only those FNCGs are essential for non-theoretical application whose FNCGSS has DPDL. And INCGI urges to map INCG into FNCG, fulfilling finite approximation. The approximation quality is defined by conditions and requirements of that mapping.

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2 Approximate Solutions of FNCG in Mixed Strategies

When FNCG is solved, FNCGSS occur pure and mixed. Solution pure strategies are found or calculated easily by straight search. Solution mixed strategies (SMS) are found either exactly [29, 13, 20, 30, 10] or approximately [17, 21, 3, 8, 12, 16]. Exact SMS are not always searchable [29, 3, 8]. Because of DPDL is violated often, the found exact SMS may come inapplicable.

Motivation of the approximation ensues also from the following. There is no universal algorithmic approach for finding Nash equilibrium (NE) strategies [29]. An exception is for dyadic games being the simplest FNCGs [29, 20, 27, 15]. Dyadic FNCG between two players is solved trivially and exactly as the bimatrix game. Dyadic FNCG with three players takes some technique of visualization of the cube of situations in mixed strategies [29, 20, 22]. For more than three players, dyadic FNCG is solved purely in analytics [29, 27]. However, there is a lot of classes of dyadic games where SMS contain probabilities being irrational numbers [29, 20]. Such exact SMS are inapplicable (DPDL is obviously violated). They need to be approximated anyway.

For approximating FNCGSS, the start of searching should be simpler for not violating DPDL and speed up the search. Therefore, sets of players' mixed strategies should be sampled. The sampling will map infinite fundamental simplexes as sets of players' mixed strategies into finite sets. And a main difficulty is the sampling step selection, because quality of FNCG finite approximation and FNCGSS search speedup are contrary.

Number of situations in mixed strategies in an FNCG, not violating DPDL, is finite ever. So, number of approximate solutions of an FNCG in mixed strategies, not violating DPDL, is finite also. This finiteness allows straight search, and non-theoretical application of an approximate solution.

3 Goal and Congruent Tasks

The final goal is to state equilibrium situations approximated over the finitely sampled fundamental simplexes. Dealing with DPDL, possible concessions for saving equilibria are admissible. Before converting fundamental simplexes as sets of players' mixed strategies into finite sets, FNCG notations are to be stated. As the simplexes' conversion is tied to the sampling step selection, conditions of the sampling should be grounded. For generalization of the sampling, points on fundamental simplex will be allowed to be selected irregularly. And the completing task is a conception of searching approximate equilibrium situations with possible concessions. In fact, the order of these goal and congruent tasks gives the work organization.

4 FNCG Notations

FNCG of $N \in \mathbb{N} \setminus \{1\}$ players is a tuple

$$\left\langle \left\{ X_n = \left\{ x_{nm} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}, \left\{ \mathbf{K}_n = \left[k_J^{\langle n \rangle} \right]_{\mathscr{F}} \right\}_{n=1}^{N} \right\rangle \tag{1}$$

by the n-th player's pure strategies set X_n and its payoff $\mathscr{F}$-matrix $\mathbf{K}_n$ at $\mathscr{F} = \bigtimes_{r=1}^{N} M_r$ and $M_r \in \mathbb{N} \setminus \{1\}$ for set of indices

$$J = \left\{ j_i \right\}_{i=1}^{N}, \quad j_i \in \left\{ \overline{1, M_i} \right\} \quad \forall\, i = \overline{1, N}\,. \tag{2}$$

The n-th player's set of mixed strategies is $(M_n - 1)$-dimensional fundamental simplex

$$\mathcal{P}_n = \left\{ \mathbf{P}_n = \left[p_m^{\langle n \rangle} \right]_{1 \times M_n} \in \mathbb{R}^{M_n} : p_m^{\langle n \rangle} \in [0;\, 1],\, \sum_{m=1}^{M_n} p_m^{\langle n \rangle} = 1 \right\}. \tag{3}$$

The n-th player's expected payoff in the situation $\left\{ \mathbf{P}_i \right\}_{i=1}^{N} \in \bigtimes_{r=1}^{N} \mathcal{P}_r$ is

$$v_n\left(\left\{ \mathbf{P}_i \right\}_{i=1}^{N} \right) = \sum_{j_q = \overline{1, M_q},\; q = \overline{1, N}} \left(k_J^{\langle n \rangle} \cdot \prod_{r=1}^{N} p_{j_r}^{\langle r \rangle} \right). \tag{4}$$

Without practical peculiarities, FNCG notations (1) — (4) are common [29, 21, 14]. A peculiar question is how to calculate the payoff (4) rapidly when N increases, and N-dimensional matrices' format $\mathscr{F}$ enlarges. Irregularity of selecting points on fundamental simplexes will enable a partial control over the payoff (4) calculation. Thus, when converting fundamental simplexes as sets of players' mixed strategies into finite sets, the sampling step is customizable for a purpose of, particularly, reducing calculations.

5 Conversion of Fundamental Simplexes as Sets of Players' Mixed Strategies into Finite Lattices

Let s_{nm} be the number of points selected across m-th dimension of simplex (3). Inasmuch as a pure strategy itself must be selected, $s_{nm} \in \mathbb{N} \setminus \{1\}$. Sum of components of any selected M_n-dimensional point on fundamental simplex (3) must be 1. Therefore, only $M_n - 1$ numbers $\{s_{nm}\}_{m=1}^{M_n-1}$ define those points selected on fundamental simplex (3). Then simplex (3) is converted into finite lattice

$$\bar{\mathcal{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) = \left\{\bar{\mathbf{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) = \left[\bar{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}\right)\right]_{1\times M_n} \in \mathcal{P}_n : \bar{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}\right) = \pi_m^{\langle n\rangle}(h),\right.$$
$$\left.\pi_m^{\langle n\rangle}(1) = 0,\ \pi_m^{\langle n\rangle}(s_{nm}) = 1,\ \pi_m^{\langle n\rangle}(h) < \pi_m^{\langle n\rangle}(h+1)\ \forall\, h = \overline{1, s_{nm}-1} \text{ by } m = \overline{1, M_n}\right\} \subset \mathcal{P}_n . \tag{5}$$

For probability $\bar{p}_{M_n}^{\langle n\rangle}\left(s_{nM_n}^{-1}\right)$, the assignment $\bar{p}_{M_n}^{\langle n\rangle}\left(s_{nM_n}^{-1}\right) = \pi_{M_n}^{\langle n\rangle}(h)$ is formal, because

$$\bar{p}_{M_n}^{\langle n\rangle}\left(s_{nM_n}^{-1}\right) = 1 - \sum_{m=1}^{M_n-1} \bar{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}\right) \tag{6}$$

virtually. So, the formally retained number s_{nM_n} depends on what numbers $\{s_{nm}\}_{m=1}^{M_n-1}$ are assigned and which probabilities $\left\{\pi_m^{\langle n\rangle}(h)\right\}_{m=1}^{M_n-1}$ are selected at every $h = \overline{1, s_{nm}}$.

Usually, the sampling step is constant. But sometimes specific probabilities are desired to be included into lattice. They may be close to probabilities in an equilibrium situation, if such situation appears known. Besides, we may want to select some of those probabilities which are close (as far as we know it) to probabilities in an equilibrium situation. Anyway, customization of the sample step due to properties in (5) is a universal multipurpose factor of fundamental simplexes' conversion into finite lattices. It is generalization of the sampling. But improper conversion (poorly selected probabilities for the lattice) badly influences on the approximation quality. Requirements imposed on the simplexes' conversion follow.

6 Conditions Grounding the Conversion of Fundamental Simplexes into Finite Lattices

The conversion of fundamental simplexes into finite lattices should correspond to discrete continuity meaning connectedness of the players' expected payoffs. The content is that as the player changes its mixed strategy minimally then any player's payoff changes no more than for a given value, specific for this player. Strictly speaking, for any $\bar{\mathbf{P}}_q^{\langle 0\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \in \bar{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)$ such that

$$\rho_{\mathbb{R}^{M_q}}\left(\bar{\mathbf{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right), \bar{\mathbf{P}}_q^{\langle 0\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right) = \left\|\bar{\mathbf{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) - \bar{\mathbf{P}}_q^{\langle 0\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\| =$$
$$= \min_{\substack{\bar{\mathbf{P}}_q^{*}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \in \bar{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right),\ \bar{\mathbf{P}}_q^{**}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \in \bar{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \\ \bar{\mathbf{P}}_q^{*}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \neq \bar{\mathbf{P}}_q^{**}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)}} \left\|\bar{\mathbf{P}}_q^{*}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) - \bar{\mathbf{P}}_q^{**}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\| \tag{7}$$

there should be

$$\left| v_n\left(\left\{\overline{\mathbf{P}}_i\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right) - v_n\left(\left\{\left\{\overline{\mathbf{P}}_i\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N} \setminus \left\{\overline{\mathbf{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}\right\} \cup \left\{\overline{\mathbf{P}}_q^{\langle 0\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}\right)\right| \leqslant \alpha_n \quad (8)$$

for some $\alpha_n > 0$ by $n = \overline{1, N}$ and $q = \overline{1, N}$. Naturally, the norm in (7) is Euclidean $\mathbb{R}^{M_q}$-norm.

If condition (8) is violated for some q, the q-th player should rearrange its probabilities in lattice $\overline{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)$ at least in a dimension. If this fails (for all dimensions), another possibility is to increase one (or more) of $M_q - 1$ numbers $\left\{s_{qm}\right\}_{m=1}^{M_q-1}$. These actions should recur until condition (8) at (7) holds by $n = \overline{1, N}$ and $q = \overline{1, N}$ (fundamental simplexes are converted into finite lattices properly).

7 Searching Approximate Equilibrium Situations with Possible Concessions

Let $U_n = \left|\overline{\mathcal{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right|$ and enumerate elements of the lattice (5) as

$$\left\{\overline{\mathbf{P}}_n^{\langle u_n\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) = \left[\overline{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}; u_n\right)\right]_{1\times M_n}\right\}_{u_n=1}^{U_n} = \overline{\mathcal{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right). \quad (9)$$

Arrangement of elements in (9) is that the first element is

$$\overline{\mathbf{P}}_n^{\langle 1\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) = \left[\overline{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}; 1\right)\right]_{1\times M_n} \text{ by } \overline{p}_1^{\langle n\rangle}\left(s_{n1}^{-1}; 1\right) = 1 \text{ and } \overline{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}; 1\right) = 0 \quad \forall\, m = \overline{2, M_n} \quad (10)$$

corresponding to the n-th player's first pure strategy x_{n1}. And the last element in (9) is

$$\overline{\mathbf{P}}_n^{\langle U_n\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) = \left[\overline{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}; U_n\right)\right]_{1\times M_n}$$

$$\text{by } \overline{p}_m^{\langle n\rangle}\left(s_{nm}^{-1}; U_n\right) = 0 \quad \forall\, m = \overline{1, M_n - 1} \text{ and } \overline{p}_{M_n}^{\langle n\rangle}\left(s_{nM_n}^{-1}; U_n\right) = 1 \quad (11)$$

corresponding to the n-th player's last pure strategy x_{nM_n}. The arrangement in (9) may sometimes imply that pure strategies are ranked, but factually it is not limited to anything.

In the mixed extension of FNCG (1), at least an NE situation $\left\{\mathbf{P}_i^*\right\}_{i=1}^{N} \in \bigtimes_{r=1}^{N} \mathcal{P}_r$ satisfying inequalities

$$v_n\left(\left\{\left\{\mathbf{P}_i^*\right\}_{i=1}^{N} \setminus \left\{\mathbf{P}_n^*\right\}\right\} \cup \left\{\mathbf{P}_n\right\}\right) \leqslant v_n\left(\left\{\mathbf{P}_i^*\right\}_{i=1}^{N}\right) \quad \forall\, \mathbf{P}_n \in \mathcal{P}_n \text{ and } \forall\, n = \overline{1, N} \quad (12)$$

exists [29]. The finite lattice

$$\mathcal{L}\left(\left\{\left\{s_{rm}^{-1}\right\}_{m=1}^{M_r-1}\right\}_{r=1}^{N}\right) = \bigtimes_{r=1}^{N} \overline{\mathcal{P}}_r\left(\left\{s_{rm}^{-1}\right\}_{m=1}^{M_r-1}\right) \subset \bigtimes_{r=1}^{N} \mathcal{P}_r \quad (13)$$

may not contain this NE situation. Generally, the set of NE situations on the finite lattice (13) may turn out empty. But if we relax the inequalities (12) stated for the finite lattice (13) as

$$v_n\left(\left\{\left\{\overline{\mathbf{P}}_i^{\langle u_i^*\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N} \setminus \left\{\overline{\mathbf{P}}_n^{\langle u_n^*\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right\}\right\} \cup \left\{\overline{\mathbf{P}}_n^{\langle u_n\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right\}\right) \leqslant v_n\left(\left\{\overline{\mathbf{P}}_i^{\langle u_i^*\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right)$$

$$\forall\, u_n = \overline{1, U_n} \text{ and } \forall\, n = \overline{1, N} \quad (14)$$

at $u_i^* \in \left\{\overline{1, U_n}\right\}$ for each player individually, then we get

$$v_n\left(\left\{\left\{\overline{\mathbf{P}}_i^{\langle u_i^*\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N} \setminus \left\{\overline{\mathbf{P}}_n^{\langle u_n^*\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right\}\right\} \cup \left\{\overline{\mathbf{P}}_n^{\langle u_n\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right\}\right) \leqslant v_n\left(\left\{\overline{\mathbf{P}}_i^{\langle u_i^*\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right) + \beta_n$$

$$\forall\, u_n = \overline{1, U_n} \text{ and } \forall\, n = \overline{1, N} \quad (15)$$

by some relaxations $\beta_n \geqslant 0$. For the n-th player, the relaxation β_n is its concession for holding equilibrium.

Definition 1. A node

$$\left\{ \overline{\mathbf{P}}_i^{\langle u_i^* \rangle} \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \in \mathcal{L} \left(\left\{ \left\{ s_{rm}^{-1} \right\}_{m=1}^{M_r - 1} \right\}_{r=1}^{N} \right) \tag{16}$$

of the finite lattice (13) is called $\{\beta_n\}_{n \in B}$-conceded NE situation in FNCG (1) for a set

$$B = \left\{ q \in \left\{ \overline{1, N} \right\} : \beta_q > 0 \right\} \tag{17}$$

if the inequalities (15) are true.

A $\{\beta_n\}_{n \in B}$-conceded NE situation (16) becomes classic NE situation if the set (17) is empty. If the set (17) is nonempty, and there is a $\{\beta_n\}_{n \in B}$-conceded NE situation (16), then this situation is approximate equilibrium one. Obviously, concessions by nonempty set (17) are needed heavier for strong equilibrium approximation.

Definition 2. A node (16) of the finite lattice (13) is called $\{\gamma_C\}_{C \subset \{\overline{1, N}\}}$-conceded strong NE situation for coalitions $C \subset \left\{ \overline{1, N} \right\}$ in FNCG (1) if inequalities

$$\sum_{n \in C} v_n \left(\left\{ \left\{ \overline{\mathbf{P}}_i^{\langle u_i^* \rangle} \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \setminus \left\{ \overline{\mathbf{P}}_q^{\langle u_q^* \rangle} \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \right\}_{q \in C} \right\} \cup \left\{ \overline{\mathbf{P}}_q^{\langle u_q \rangle} \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \right\}_{q \in C} \right) \leqslant$$

$$\leqslant \sum_{n \in C} v_n \left(\left\{ \overline{\mathbf{P}}_i^{\langle u_i^* \rangle} \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \right) + \gamma_C \quad \forall\, u_q = \overline{1, U_q} \quad \text{for} \quad q \in C \quad \text{and} \quad \forall\, C \subset \left\{ \overline{1, N} \right\} \tag{18}$$

are true by coalition C relaxation $\gamma_C \geqslant 0$.

A $\{\gamma_C\}_{C \subset \{\overline{1, N}\}}$-conceded strong NE situation (16) becomes classic strong NE situation if $\gamma_C = 0 \quad \forall\, C \subset \left\{ \overline{1, N} \right\}$. Classically, if there is a strong NE situation in FNCG (1) then there is an NE situation definitely [29]. When strong equilibrium is approximated, a $\{\gamma_C\}_{C \subset \{\overline{1, N}\}}$-conceded strong NE situation does not always mean that a $\{\beta_n\}_{n \in B}$-conceded NE situation exists. This is because concessions $\{\beta_n\}_{n \in B}$ may be taken too small. In other words, approximate NE situation and approximate strong equilibrium situation are advisable to be coherent in their concessions.

Theorem 1. If a node (16) of the finite lattice (13) is $\{\gamma_C\}_{C \subset \{\overline{1, N}\}}$-conceded strong NE situation, then by

$$\beta_n \geqslant \gamma_C \quad \forall\, C \subset \left\{ \overline{1, N} \right\} \quad \text{for} \quad |C| = 1 \quad \text{and} \quad \forall\, n = \overline{1, N}, \tag{19}$$

this node is $\{\beta_n\}_{n \in B}$-conceded NE situation.

Proof. Condition (19) brings the inequalities (18), for single player coalitions, to the view (15). By the theorem pre-condition, inequalities (18) are true so that inequalities (15) are true as well. Consequently, the spoken $\{\gamma_C\}_{C \subset \{\overline{1, N}\}}$-conceded strong NE situation is $\{\beta_n\}_{n \in B}$-conceded NE situation. The theorem has been proved.

Denote the set of $\{\beta_n\}_{n \in B}$-conceded NE situations in FNCG (1) by $\mathcal{N}\left(\{\beta_n\}_{n \in B} \right)$. If the set of the n-th player's β_n-conceded acceptable situations is $\mathcal{A}_n$, then the set $\mathcal{N}\left(\{\beta_n\}_{n \in B} \right)$ can be found just by the finite intersection:

$$\mathcal{N}\left(\{\beta_n\}_{n \in B} \right) = \bigcap_{r=1}^{N} \mathcal{A}_r \, . \tag{20}$$

The set $\mathcal{A}_n$ is searched by the generalized algorithm in Figure 1, where a is a counter of elements in the set $\mathcal{A}_n$ and the set S_n is a currently stored subset of the already found β_n-conceded acceptable situations.

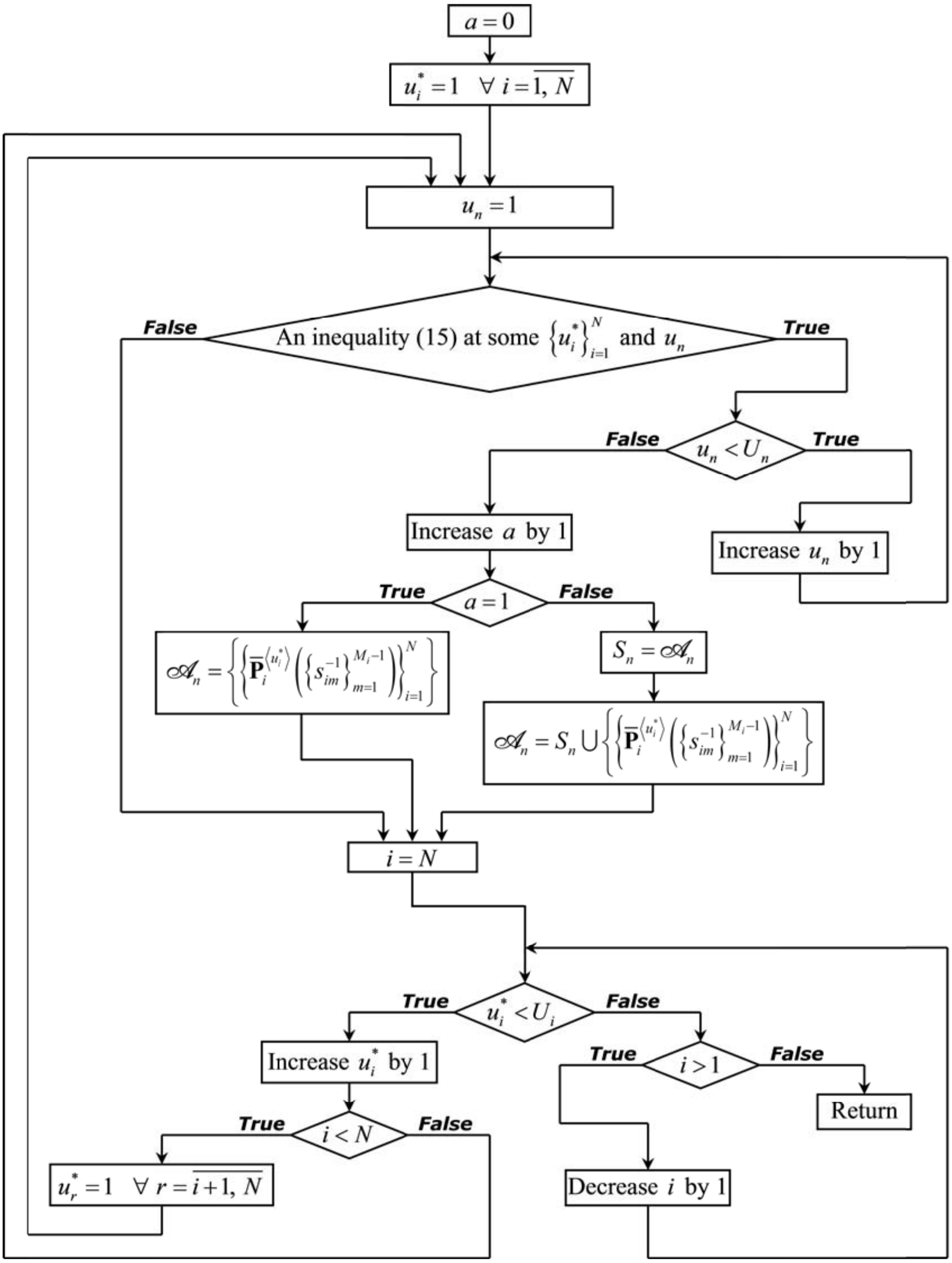

Figure 1: Generalized algorithm for searching the set $\mathscr{A}_n$

In Figure 2, the generalized algorithm for searching the set $\mathscr{N}\left(\left\{\beta_n\right\}_{n\in B}\right)$ in FNCG (1) is represented, referring to the algorithm in Figure 1. However, this algorithm is ineffective. A speedup algorithm for searching the set $\mathscr{N}\left(\left\{\beta_n\right\}_{n\in B}\right)$ in FNCG (1) is represented in Figure 3, where A and A_n are currently stored nonempty intersections of the first n sets $\left\{\mathscr{A}_r\right\}_{r=1}^n$, found by the algorithm in Figure 1. By the algorithm in Figure 3, the set $\mathscr{N}\left(\left\{\beta_n\right\}_{n\in B}\right)$ is searched successively. As soon as the intersection $\bigcap_{r=1}^{n} \mathcal{A}_r$ occurs empty, concessions $\{\beta_n\}_{n\in B}$ are taken greater at once. If $\mathcal{N}\left(\{\beta_n\}_{n\in B}\right) = \varnothing$ there is no necessity to have all the sets $\{\mathcal{A}_r\}_{r=1}^{N}$ for making the intersection (20).

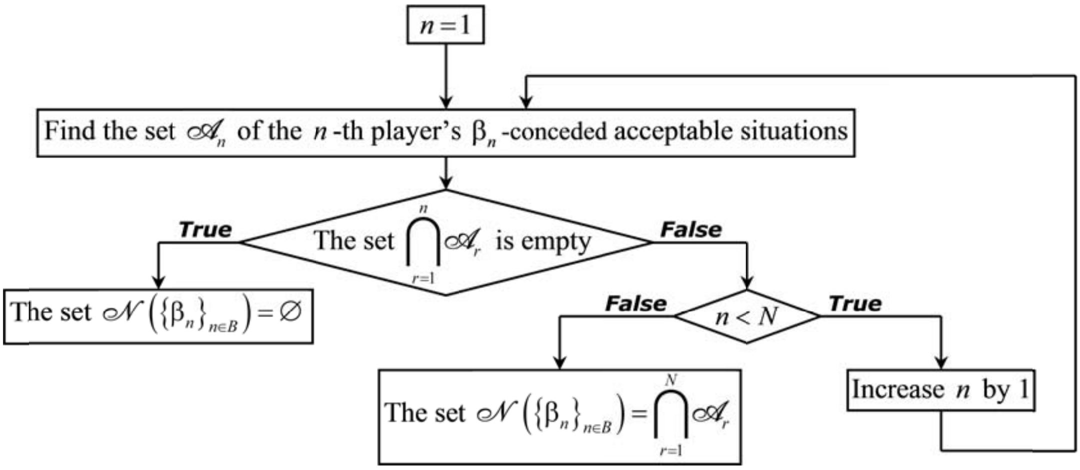

Figure 2: Generalized algorithm for searching the set $\mathcal{N}\left(\{\beta_n\}_{n\in B}\right)$ in FNCG (1)

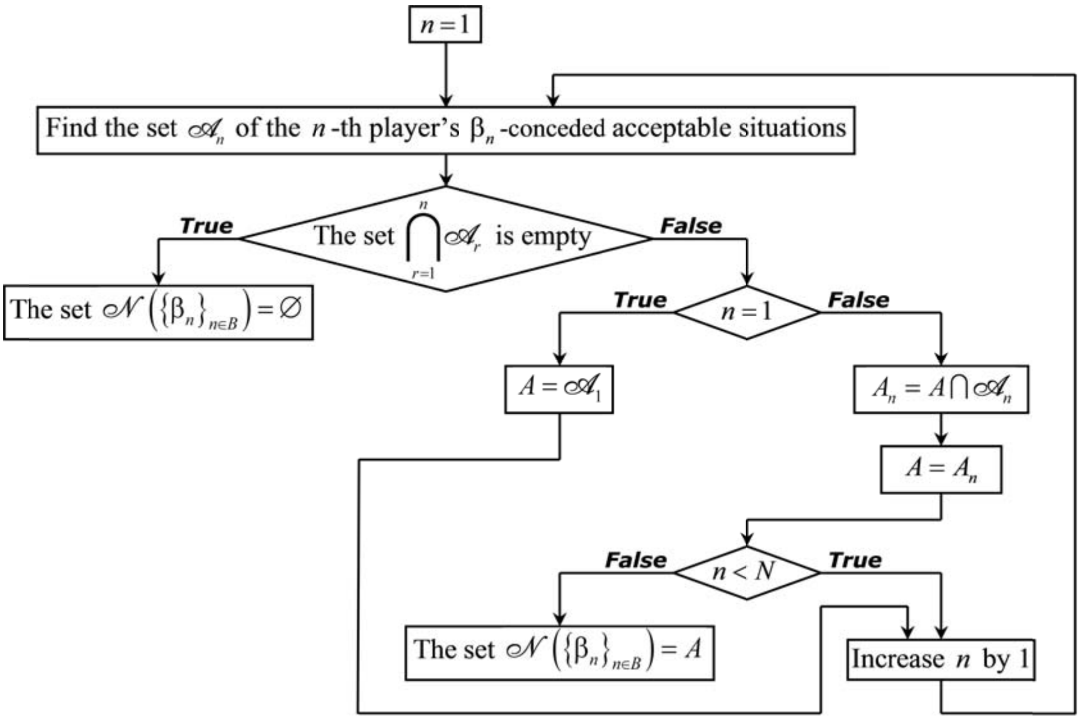

Figure 3: A speedup algorithm for searching the set $\mathcal{N}\left(\{\beta_n\}_{n\in B}\right)$ in FNCG (1)

In FNCG (1), the set of $\{\gamma_C\}_{C\subset\{\overline{1, N}\}}$-conceded strong NE situations denoted by $\mathcal{S}\left(\{\gamma_C\}_{C\subset\{\overline{1, N}\}}\right)$ is searched successively as well. Similarly to the existing acceptable situations' set $\mathcal{A}_n$ for the n-th player, the set of $\gamma_{H(q_{\text{cltn}})}$-

conceded acceptable situations $\mathscr{A}_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$ for the q_{cltn}-th coalition $H\left(q_{\text{cltn}}\right) \subset \left\{\overline{1, N}\right\}$ exists likewise. Namely, the set $\mathscr{A}_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$ consists of all the nodes (16) satisfying the $\sum_{q \in H\left(q_{\text{cltn}}\right)} U_q$ inequalities in (18) for a coalition $C = H\left(q_{\text{cltn}}\right)$. Number of all coalitions is

$$T_{\text{cltn}} = \sum_{q=1}^{N} \frac{N!}{q!\left(N-q\right)!} \tag{21}$$

by q players in a coalition. Hence, the set

$$\mathscr{S}\left(\left\{\gamma_C\right\}_{C \subset \left\{\overline{1, N}\right\}}\right) = \bigcap_{q_{\text{cltn}}=1}^{T_{\text{cltn}}} \mathscr{A}_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle} \tag{22}$$

should be searched by the algorithm in Figure 4, similar to Figure 3, where A_{cltn} and $A_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$ are currently stored nonempty intersections of the first q_{cltn} sets $\left\{\mathscr{A}_{r_{\text{cltn}}}^{\langle \mathscr{S} \rangle}\right\}_{r_{\text{cltn}}=1}^{q_{\text{cltn}}}$. And the set $\mathscr{A}_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$ is searched by the algorithm in Figure 5, where a_{cltn} is a counter of elements in the set $\mathscr{A}_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$ and the set $S_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$ is a currently stored subset of the already found $\gamma_{H\left(q_{\text{cltn}}\right)}$-conceded acceptable situations.

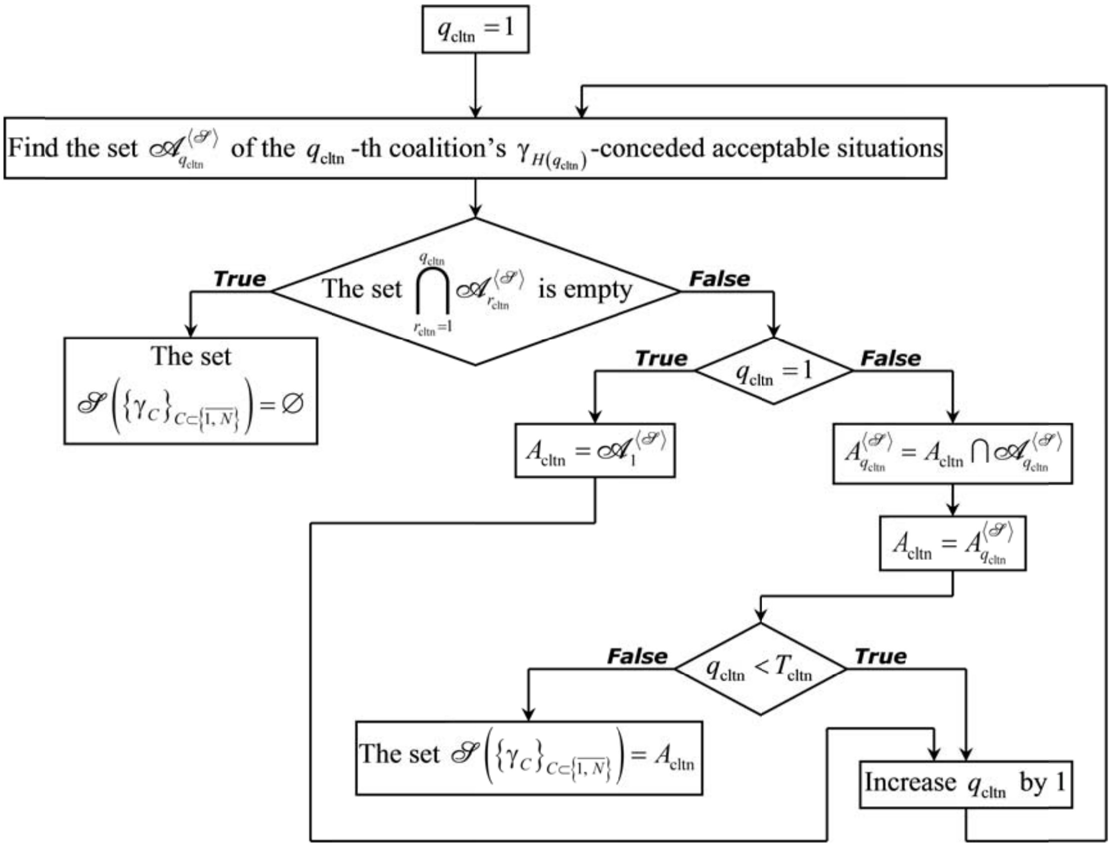

Figure 4: An algorithm for searching the set $\mathscr{S}\left(\left\{\gamma_C\right\}_{C \subset \left\{\overline{1, N}\right\}}\right)$ in FNCG (1)

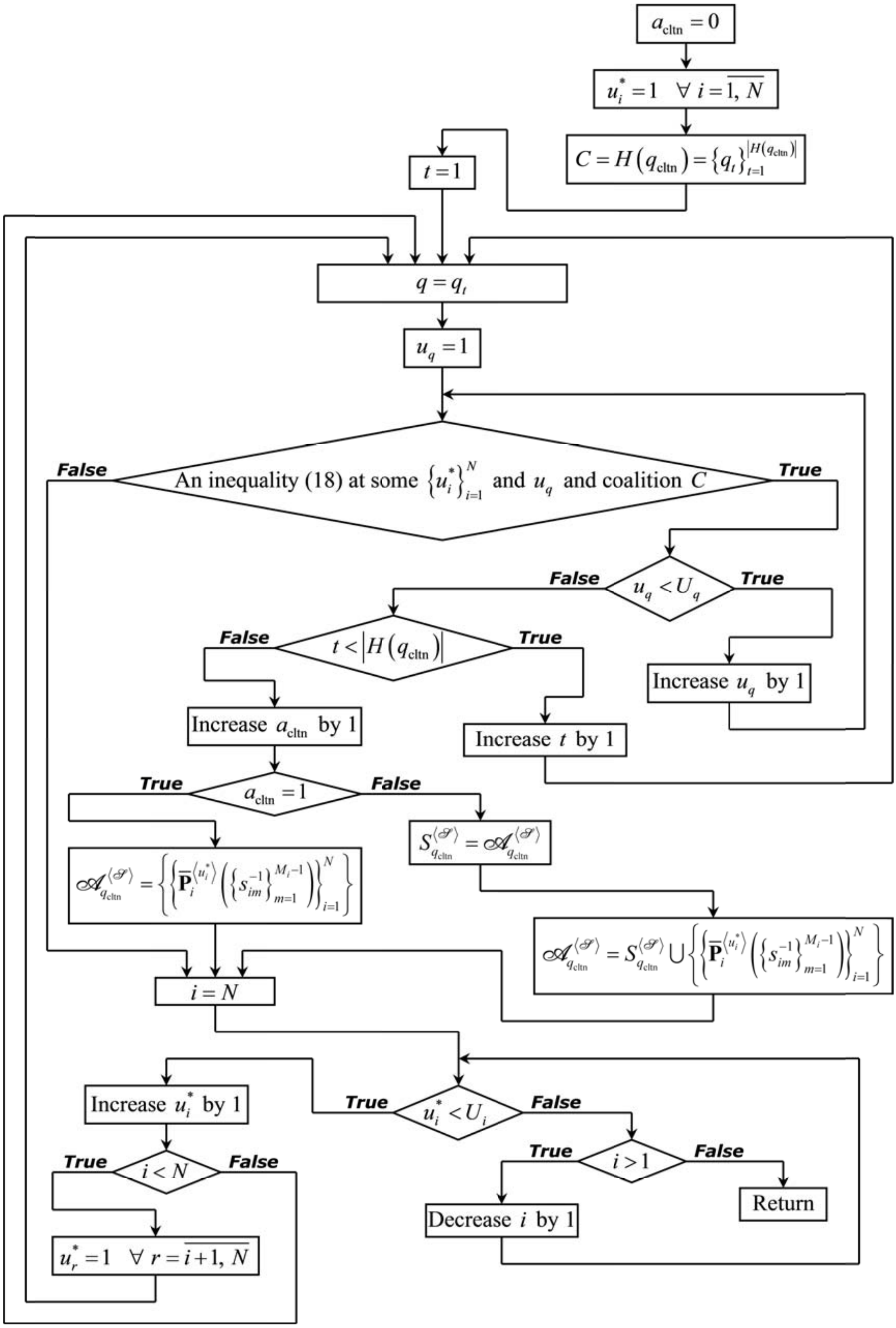

Figure 5: Generalized algorithm for searching the set $\mathscr{A}_{q_{\text{cltn}}}^{\langle \mathscr{S} \rangle}$

Reasonable enumeration of elements in the set $\left\{\mathscr{A}_{q_{\text{cltn}}}^{\langle\mathscr{S}\rangle}\right\}_{q_{\text{cltn}}=1}^{T_{\text{cltn}}}$ speeds up the algorithm for searching the set $\mathscr{S}\left(\left\{\gamma_C\right\}_{C\subset\left\{\overline{1,N}\right\}}\right)$ in Figure 4. The enumeration must start with a coalition $C=H(1)$, for which the condition of $\sum\limits_{q\in H(1)} U_q$ inequalities in (18) is of the highest risk to fail, i. e. at least one of those inequalities turns out false. A relevant definition comes.

Definition 3. A node (16) of the finite lattice (13) is called $\left\{\gamma_{C_j(k)}\right\}_{j=1}^{T_{\text{cltn}}(k)}$-conceded k-strong NE situation for

$$T_{\text{cltn}}(k)=\frac{N!}{k!(N-k)!}$$

coalitions of k players in FNCG (1) if inequalities

$$\sum_{n\in C_j(k)} v_n\left(\left\{\left\{\overline{\mathbf{P}}_i^{\langle u_i^*\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\setminus\left\{\overline{\mathbf{P}}_q^{\langle u_q^*\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}_{q\in C_j(k)}\right\}\cup\left\{\overline{\mathbf{P}}_q^{\langle u_q\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}_{q\in C_j(k)}\right)\leqslant$$

$$\leqslant\sum_{n\in C_j(k)} v_n\left(\left\{\overline{\mathbf{P}}_i^{\langle u_i^*\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right)+\gamma_{C_j(k)}$$

$$\forall\, u_q=\overline{1,U_q}\ \text{ for }\ q\in C_j(k)\ \text{ and }\ \forall\, C_j(k)\subset\left\{\overline{1,N}\right\}\ \text{ by }\ \left|C_j(k)\right|=k \tag{23}$$

are true by coalition $C_j(k)$ relaxation $\gamma_{C_j(k)}\geqslant 0$.

Theorem 2. If a node (16) of the finite lattice (13) is $\left\{\gamma_{C_j(k)}\right\}_{j=1}^{T_{\text{cltn}}(k)}$-conceded k-strong NE situation $\forall\, k=\overline{1,N}$ then this node is $\left\{\gamma_C\right\}_{C\subset\left\{\overline{1,N}\right\}}$-conceded strong NE situation.

Proof. All possible coalitions are all nonempty subsets of the set $\left\{\overline{1,N}\right\}$ whose covering is of N subcoverings:

$$\left\{\overline{1,N}\right\}=\bigcup_{k=1}^{N}\bigcup_{j=1}^{T_{\text{cltn}}(k)} C_j(k).$$

Due to the theorem pre-condition, N sets of the inequalities (23) by $k=\overline{1,N}$ cover the set of the inequalities (18). The theorem has been proved.

Clearly, $\left\{\gamma_{C_j(1)}\right\}_{j=1}^{N}$-conceded 1-strong NE situation is actually $\left\{\gamma_{C_n(1)}\right\}_{n\in B}$-conceded NE situation by the set

$$B=\left\{q\in\left\{\overline{1,N}\right\}:\gamma_{C_q(1)}>0\right\}.$$

Considering $\left\{\beta_n\right\}_{n\in B}$-conceded NE situations and $\left\{\gamma_C\right\}_{C\subset\left\{\overline{1,N}\right\}}$-conceded strong NE situations, those $\left\{\gamma_{C_j(k)}\right\}_{j=1}^{T_{\text{cltn}}(k)}$-conceded k-strong NE situations by $k=\overline{2,N}$ are intermediate. And it is well-known that strong equilibria commonly are of lesser likelihood. The corollary is the least likelihood of k-strong equilibria is at $\left\{\gamma_{C_1(N)}\right\}$-conceded N-strong NE situations (when all the players make up the single coalition). Hence, coalitions $\left\{H\left(q_{\text{cltn}}\right)\right\}_{q_{\text{cltn}}=1}^{T_{\text{cltn}}}$ for searching the set $\left\{\mathscr{A}_{q_{\text{cltn}}}^{\langle\mathscr{S}\rangle}\right\}_{q_{\text{cltn}}=1}^{T_{\text{cltn}}}$ are recommended to be enumerated according to the number of players in the coalition in descending order, i. e. the first coalition is such that $\left|H(1)\right|=N$, and the last N coalitions are such that $\left\{\left|H\left(q_{\text{cltn}}\right)\right|=1\right\}_{q_{\text{cltn}}=T_{\text{cltn}}-N+1}^{T_{\text{cltn}}}$. If there are two coalitions and more by the same number of players within, then the lesser counter number is at the coalition having the smaller concession.

8 Ways of Speedup in Searching Conceded NE Situations and Conceded Strong NE Situations

There are five main ways to speed up searching both $\{\beta_n\}_{n\in B}$-conceded NE situations and $\{\gamma_C\}_{C\subset\{\overline{1,\, N}\}}$-conceded strong NE situations:

1. The players are re-numbered so that the smaller concession corresponds to the lesser player's number. The coalitions $\left\{H\left(q_{\text{cltn}}\right)\right\}_{q_{\text{cltn}}=1}^{T_{\text{cltn}}}$ are numbered likewise: by the same number of players within the coalition, the smaller concession corresponds to the lesser coalition's number.
2. While calculating expected payoffs (4), the multidimensional matrix multiplication is parallelized [28, 7].
3. Concessions are adjusted not great to obtain the single equilibrium or a few ones.
4. For condition (8), magnitudes $\{\alpha_n\}_{n=1}^{N}$ are adjusted starting with great ones.
5. Numbers $\{s_{nm}\}_{m=1}^{M_n-1}$ are assigned through adjusting them by increment. The adjustment starts at the primitive set $\left\{\{s_{nm}=5\}_{m=1}^{M_n-1}\right\}_{n=1}^{N}$ or about that.

For testing the possible concessions along with magnitudes $\{\alpha_n\}_{n=1}^{N}$ and numbers $\{s_{nm}\}_{m=1}^{M_n-1}$ for the sampling, matrices whose elements are random and normalized values fit well [23, 24]. Randomization can issue from standard normal distribution. Uniform distribution is used subsequently.

9 Discussible Items

Sampling and approximation in NCG-modeling are motivated by INCGI. In real applications of modeling interest interaction processes, an ISS must be sampled, deliberately or during the interaction. Losses in accuracy are unavoidable, but they can be controlled with relaxations.

Relaxations $\{\beta_n\}_{n\in B}$ and $\{\gamma_C\}_{C\subset\{\overline{1,\, N}\}}$ are bound to approximation, and they are consequence of the sampling. The sampling is a particular consequence of DPDL which allows an RSF to converge to the corresponding probability in FNCGSS. DPDL surely influences on magnitudes of relaxations. And FNCGSS are realizable owing to DPDL. Nevertheless, full realizability is always depending on how many cycles the interaction conditions would repeat.

Both irregularity of selecting points on fundamental simplexes and DPDL ensure faster calculation of the payoff (4). Faster calculation comes from that there are no infinite repeating decimals within the approximated FNCGSS, and DPDL allows to switch over to single precision [7, 28]. This saves memory and disk space. However, this works fine if probability is not, say, $1/3$ or $3/7$, and payoffs have DPDL.

Convenience of sampling irregularly fundamental simplexes consists in that some probabilities are extraneous or insignificant. For instance, as probability increases, we may sample denser, and probabilities closer to 0 are selected sparser. Besides, if an exact NE strategy is known then sampling points are aspired to be rounded closer to probabilities in the known exact NE strategy.

The proved theorems prompt how to organize searching efficiently. If strong equilibria are of interest, then they are to be checked first amenably to Theorem 1. And according to Theorem 2, the search process is recommended to be split into N parts. But note that the assertion about that $\left\{\gamma_{C_j(k)}\right\}_{j=1}^{T_{\text{cltn}}(k)}$-conceded k-strong NE situation is of greater likelihood than $\left\{\gamma_{C_j(k+1)}\right\}_{j=1}^{T_{\text{cltn}}(k+1)}$-conceded $(k+1)$-strong NE situation by $k\in\left\{\overline{1,\, N-1}\right\}$ is purely probabilistic. It relies on the concessions are not very scattered. If relaxations $\left\{\gamma_{C_j(k+1)}\right\}_{j=1}^{T_{\text{cltn}}(k+1)}$ are taken big, and relaxations $\left\{\gamma_{C_j(k)}\right\}_{j=1}^{T_{\text{cltn}}(k)}$ are taken far less, then the set of $\left\{\gamma_{C_j(k+1)}\right\}_{j=1}^{T_{\text{cltn}}(k+1)}$-conceded $(k+1)$-strong NE situations is going to be nonempty, while the set of $\left\{\gamma_{C_j(k)}\right\}_{j=1}^{T_{\text{cltn}}(k)}$-conceded k-strong NE situations may turn out empty.

10 Conclusion

Equilibrium situations approximated over the finitely sampled fundamental simplexes are stated in two stages. On the initial stage, fundamental simplexes as sets of players' mixed strategies are converted into finite lattices. Player-specified probabilities are included into these lattices. And conversion of fundamental simplexes into finite lattices is conditioned by connectedness of the players' expected payoffs, what is expressed by the inequality (8) at (7) for the n-th player. On the second stage, approximate equilibrium situations with possible concessions are searched.

While searching, algorithms in Figure 3 and Figure 4 shall perform efficiently on average. In peculiar cases, e. g. when concessions are very scattered or some of them are null, re-numbering the players and coalitions precedes. For bimatrix games, other algorithms provided for searching pure strategy equilibria are available. Their usage will be not over two $M_1 \times M_2$-matrices, but over two $U_1 \times U_2$-matrices.

Inequalities (14) are the particular case of inequalities (15). Thus, the $\{\beta_n\}_{n \in B}$-conceded NE situation coming with (15) is a generalization of the classic NE situation issuing from (14). Concurrently, the $\{\gamma_C\}_{C \subset \{\overline{1,N}\}}$-conceded strong NE situation is a generalization of the classic strong NE situation. These generalizations are rather practice-oriented than mathematical theorization.

The represented approximation technique is a constructive development of FNCG-modeling. It is essential for NCG-modeling also inasmuch as an NCG is usually transformed into FNCG. The approximation constructivity is in non-restricted possible concessions and irregularity of probabilities which are included into the finite lattices. Aside from the generalized sampling and controlled equilibration, universality of the approximation technique proceeds from the algorithms for searching the sets $\mathscr{N}\left(\{\beta_n\}_{n \in B}\right)$ and $\mathscr{S}\left(\{\gamma_C\}_{C \subset \{\overline{1,N}\}}\right)$ general-schemed in Figure 3 and Figure 4. Finally, adjustment of concessions makes the problem of NCG unique solution [6, 25] removable.

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