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Moment Analysis of Uncertain Stationary Independent Increment Processes

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Abstract

Uncertain process is initialized for modelling the evolution of uncertain phenomena. An uncertain process is said to have independent increments if its increments are independent uncertain variables whenever the time intervals do not overlap, and have stationary increments if its increments are identically distributed uncertain variables whenever the time intervals have the same length. Then stationary independent increment process is a type of uncertain process whose increments are not only independent but also stationary. Moment is an important numerical characteristic of an uncertain stationary independent increment process. This paper aims at investigating the k-th moment of a stationary independent increment process.

Keywords: uncertainty variable, uncertain process, stationary independent increment process, moment

1 Introduction

Probability theory is an axiomatic mathematical system for dealing with frequency by use of probability measure. The probability measure satisfies three axioms which are normality, nonnegativity and countable additivity. When applying probability, we usually assume that cumulative probability distribution is close enough to the real frequency. That is, only we obtain sufficient historical data can we use it to deal with frequency. Unfortunately, the data cannot always be derived because of technical, economic or some other reasons. For example, we cannot exactly know the bearing capacity of an bridge being used and we cannot know the lifetime of a lamp in use. At this time, we should rely on domain experts' belief degrees about the chances that the possible events may happen. Due to the conservativeness of estimations, a big gap exists between the belief degree and real frequency. If we insist on applying probability to modeling belief degrees, there will produce a counterintuitive result which was specified in Liu [8]. Naturally, we should use another type of mathematical tool to resolve this problem. In fact, it is uncertainty theory.

Uncertainty theory was founded by Liu [4] as a new branch of axiomatic mathematics for describing belief degrees by using uncertain measure. As a counterpart of probability measure, the uncertain measure satisfies normality, duality, subadditivity and product axioms, while they are different from the ones in probability theory. Then the concept of uncertain variable was put forward by Liu [4] for modeling an uncertain quantity. Additionally, in order to dealing with the operations between different uncertain variables, Liu [6] presented the product axiom. It is very different from probability's, because the former is the minimum of all uncertain measures and the latter is the product of all probability measures. For describing an uncertain variable, Liu [4] put forward a concept of uncertainty distribution. Following that, Peng and Iwamura [13] gave a sufficient and necessary condition for the uncertainty distribution of an uncertain variable. As an important characteristic to describe uncertain variables, k-th moment was introduced by Liu [4]. Sheng and Kar [14] gave some formulas to calculate moments by using of inverse uncertainty distribution. All in all, under the framework of the uncertainty theory, a number of researchers obtained some important and useful results such as [2, 3, 11, 12].

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Uncertain process proposed by Liu [5] for modeling the evolution of uncertain phenomena. It is essentially a sequence of uncertain variables indexed by time. It is an independent increment process if the increments are independent uncertain variables whenever the time intervals do not overlap. In addition, it is a stationary increment process if all increments identically distributed uncertain variables whenever the time intervals have the same length. Furthermore, an uncertain process is an independent stationary increment process presented by Liu [5] if it has not only independent increments but also stationary increments. Liu [7] studied the expected value of an uncertain stationary increment process and Chen [1] investigated the variance of an uncertain stationary increment process. As an extension, we study the k-th moment and k-th central moment of an uncertain stationary increment process. The rest of this paper is organized as follows. In Section 2, some useful fundamental concepts and properties concerning uncertain variables and uncertain processes will be reviewed. Then Section 3 will be devoted to studying the k-th moment and k-th central moment of an uncertain stationary increment process. Finally, we make a brief conclusion in Section 4.

2 Preliminaries

In this section, we will introduce some fundamental concepts and properties concerning uncertain variables and uncertain processes.

2.1 Uncertain Variable

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event and assigned a number $\mathcal{M}\{\Lambda\}$ to indicate the belief degree that we believe Λ will happen. In order to deal with belief degrees rationally, Liu [4] suggested the following three axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Definition 1. (Liu [4]) *The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.*

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Furthermore, the product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by Liu [6] as follows:

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrary events chosen from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2. (Liu [4]) *An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 3. (Liu [4]) *Suppose ξ is an uncertain variable. Then the uncertainty distribution of ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

An uncertainty distribution $\Phi(x)$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0,1)$. Inverse uncertainty distribution plays an important role in the operation of independent uncertain variables.

The operational law of independent uncertain variables was given by Liu [7] in order to calculate the uncertainty distribution of a strictly increasing or a decreasing function with respect to independent uncertain variables. Before introducing the operational law, the concept of independence of uncertain variables is presented as follows.

Definition 4. (Liu [6]) *The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Theorem 1. (Liu [7]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an uncertainty distribution

$$\Phi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \left(\min_{1\le i\le m} \Phi_i(x) \wedge \min_{m+1\le i\le n} (1-\Phi_i(x)) \right).$$

Theorem 2. (Liu [7]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

The expected value is the average value of an uncertain variable in the sense of uncertain measure and it represents the size of an uncertain variable.

Definition 5. (Liu [4]) *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \le x\}\mathrm{d}x$$

provided that at least one of the two integrals is finite.

Theorem 3. (Liu [4]) *Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x).$$

Definition 6. (Liu [4]) *Let ξ be an uncertain variable and let k be a positive integer. Then $E[\xi^k]$ is called the k-th moment of ξ.*

Theorem 4. (Liu [4]) *Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k\mathrm{d}\Phi(x).$$

Definition 7. (Liu [4]) *Let ξ be an uncertain variable and let k be a positive integer. Then $E[(\xi - E[\xi])^k]$ is called the k-th central moment of ξ.*

Theorem 5. (Sheng and Kar [14]) *Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be a positive integer. If the expected value $E[\xi]$ exists, then the k-th central moment of ξ is*

$$E[\xi^k] = \int_{-\infty}^{+\infty} (x - E[\xi])^k \mathrm{d}\Phi(x).$$

2.2 Uncertain Process

Uncertain process is initialized by Liu [5] to model the evolution of uncertain phenomena. Virtually, it is a sequence of uncertain variables vary with time.

Definition 8. (Liu [5]) *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let T be a totally ordered set (e.g. time). An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B at each time t.*

For describing an uncertain process well, Liu [10] proposed a concept of uncertainty distribution. In fact, an uncertainty distribution of an uncertain process is a sequence of uncertainty distributions of uncertain variables indexed by time.

Definition 9. (Liu [10]) *An uncertain process X_t is said to have an uncertainty distribution $\Phi_t(x)$ if the uncertain variable X_t has the uncertainty distribution $\Phi_t(x)$ at each time t.*

From the definition, it is clear that the uncertainty distribution of uncertain process is a surface instead of a curve.

Definition 10. (Liu [10]) *Uncertain processes $X_{1t}, X_{2t}, \ldots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \ldots, t_k$, the uncertain vectors*

$$\boldsymbol{\xi_i} = (X_{it_1}, X_{it_2}, \cdots, X_{it_k}),\ i = 1, 2, \ldots, n$$

are independent, i.e., for any k-dimensional Borel sets $B_1, B_2, \ldots, B_n$, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\boldsymbol{\xi_i} \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi_i} \in B_i\}.$$

Definition 11. (Liu [5]) *An uncertain process X_t is said to have independent increments if*

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \ldots, t_k$ are any times with $t_1 < t_2 < \cdots < t_k$.

An uncertain process X_t is said to have stationary increments if for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Definition 12. (Liu [5]) *An uncertain process X_t is said to be a stationary independent increment process if it has not only stationary increments but also independent increments.*

Theorem 6. (Chen [1]) *Suppose X_t is an uncertain stationary independent increment process. Then X_t and $(1-t)X_0 + tX_1$ are identically distributed uncertain variables for any time $t \ge 0$.*

As a special type of uncertain process, an uncertain stationary independent process also has some special properties for its expected value and variance. First the expected value of an uncertain stationary independent process is linear function of time. And the variance of an uncertain stationary independent process is proportional to the square of time. The detailed contents are given as follows.

Theorem 7. (Liu [7]) *Suppose X_t is an uncertain stationary independent increment process. Then there exists two real numbers a and b such that*

$$E[X_t] = a + bt$$

for any time $t \geq 0$.

Theorem 8. (Liu [7]) *Suppose X_t is an uncertain stationary independent increment process with an initial value 0. Then we have*

$$E[X_{t+s}] = E[X_t] + E[X_s]$$

for any times t and s.

Theorem 9. (Chen [1]) *Suppose X_t is an uncertain stationary independent increment process with a crisp initial value X_0. Then Then there exists a real numbers b such that*

$$V[X_t] = bt^2$$

for any time $t \geq 0$.

Theorem 10. (Chen [1]) *Suppose X_t is an uncertain stationary independent increment process with a crisp initial value X_0. Then we have*

$$\sqrt{V[X_{t+s}]} = \sqrt{V[X_t]} + \sqrt{V[X_s]}$$

for any times t and s.

3 Main Results

As we all know, moment is an important numerical characteristic of an uncertain variable or an uncertain stationary independent increment process. In other words, for describing an uncertain variable or an uncertain stationary independent increment process well, we should study the moment with our energy and time. Liu [7] investigated the expected value of an uncertain stationary independent increment process. Then Chen [1] studied the variance of an uncertain stationary independent increment process. In this section we study the k-th moment and k-th central moment of an uncertain stationary increment process, which is extensions of the above existing literatures.

Theorem 11. *Let X_t be an uncertain stationary independent increment process with a crisp initial value X_0. Then there exists a real number a such that*

$$E[(X_t - EX_t)^k] = at^k$$

for any time $t \geq 0$.

Proof. From Theorem 6, we know that X_t and $(1-t)X_0 + tX_1$ are identically distributed uncertain variables. Since X_0 is a constant, we have

$$\begin{aligned} E[(X_t - EX_t)^k] &= E[(1-t)X_0 + tX_1 - E((1-t)X_0 + tX_1)]^k \\ &= E[(1-t)X_0 + tX_1 - E(1-t)X_0 - E(tX_1)]^k \\ &= E[tX_1 - tE(tX_1)]^k \\ &= t^k E[(X_1 - EX_1)^k]. \end{aligned}$$

Hence this theorem holds for $a = E[(X_1 - EX_1)^k]$. □

Remark 1. If we set $k = 2$, then the result in [1] is obtained. That is, the above theorem is an extension of existing result in [1].

Example 1. Let X_t be a stationary independent increment process and $X_t \sim \mathcal{L}(0, at)$ with a crisp initial value X_0. Then

$$E[(X_t - EX_t)^4] = \frac{a^4}{80}t^4 \tag{1}$$

for any time $t \geq 0$. In fact,

$$E[(X_1 - EX_1)^4] = \int_0^a \left(x - \frac{a}{2}\right)^4 \mathrm{d}\frac{x}{a} = \frac{a^4}{80},$$

so X_t has the 4-th moment (1) by Theorem 11.

Example 2. Let X_t be a stationary independent increment process and $X_t \sim \mathcal{Z}(0, at, bt)$ with a crisp initial value X_0. Then

$$E[(X_t - EX_t)^3] = \frac{b(3b^2 - 8ab - 4a^2)}{128} t^3 \tag{2}$$

for any time $t \geq 0$. In fact,

$$E[(X_1 - EX_1)^3] = \int_0^a \left(x - \frac{2a+b}{4}\right)^3 \mathrm{d}\frac{x}{a} + \int_a^b \left(x - \frac{2a+b}{4}\right)^3 \mathrm{d}\frac{x+b-2a}{2(b-a)} = \frac{b(3b^2 - 8ab - 4a^2)}{128},$$

so X_t has the 3-th moment (2) by Theorem 11.

Example 3. Let X_t be a stationary independent increment process and $X_t \sim \mathcal{N}(0, at)$ with a crisp initial value X_0. Then

$$E[(X_t - EX_t)^2] = a^2 t^2 \tag{3}$$

for any time $t \geq 0$. In fact,

$$E[(X_1 - EX_1)^2] = \int_{-\infty}^{+\infty} x^2 \mathrm{d}\left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}a}\right)\right)^{-1} = a^2,$$

so X_t has the 2-th moment (3) by Theorem 11.

Theorem 12. *Let X_t be an uncertain stationary independent increment process with a crisp initial value X_0. Then for any times s and t, we have*

$$\sqrt[k]{E[(X_{s+t} - EX_{s+t})^k]} = \sqrt[k]{E[(X_s - EX_s)^k]} + \sqrt[k]{E[(X_t - EX_t)^k]}.$$

Proof. It follows from Theorem 11 that there exists a real number a such that

$$E[X_t - EX_t]^k = at^k, \ a = E[(X_1 - EX_1)^k]$$

for any time $t \geq 0$. When k is an even number, $a > 0$, so $\sqrt[k]{a}$ is meaningful for any real number k. Since X_0 is a constant, we have

$$\sqrt[k]{E[(X_{s+t} - EX_{s+t})^k]} = \sqrt[k]{a}(s+t) = \sqrt[k]{a}s + \sqrt[k]{a}t = \sqrt[k]{E[(X_s - EX_s)^k]} + \sqrt[k]{E[(X_t - EX_t)^k]}.$$

Thus the proof is finished. □

Remark 2. If we set $k = 2$, then the result in [1] is also obtained. That is, the existing result in [1] is a special case of the above theorem.

Example 4. Let X_t be a stationary independent increment process and $X_t \sim \mathcal{Z}(0, at, bt)$ with a crisp initial value X_0. Then

$$\sqrt[3]{E[(X_{s+t} - EX_{s+t})^3]} = \sqrt[3]{E[(X_s - EX_s)^3]} + \sqrt[3]{E[(X_t - EX_t)^3]} \tag{4}$$

for any time $s, t \geq 0$. In fact,

$$\begin{aligned}\sqrt[3]{E[(X_{s+t} - EX_{s+t})^3]} &= \sqrt[3]{\frac{b(3b^2 - 8ab - 4a^2)}{128}(s+t)^3} \\ &= \sqrt[3]{\frac{b(3b^2 - 8ab - 4a^2)}{128}}s + \sqrt[3]{\frac{b(3b^2 - 8ab - 4a^2)}{128}}t \\ &= \sqrt[3]{E[(X_s - EX_s)^3]} + \sqrt[3]{E[(X_t - EX_t)^3]},\end{aligned}$$

so (4) holds by Theorem 12.

Theorem 13. *Let X_t be an uncertain stationary independent increment process with an initial value 0. Then there exists a real number a such that*

$$E[X_t^k] = at^k$$

for any time $t \geq 0$.

Proof. According to Theorem 6, we know that X_t and $X_0+t(X_1-X_0))$ are identically distributed uncertain variables. Since $X_0=0$, we have

$$E[X_t^k]=E[X_0+t(X_1-X_0)]^k=t^kE[X_1^k].$$

Hence this theorem holds for $a=E[X_1^k]$. □

Example 5. Let X_t be a stationary independent increment process and $X_t\sim\mathcal{L}(0,at)$ with an initial value 0. Then

$$E[X_t^4]=\frac{a^4}{5}t^4 \tag{5}$$

for any time $t\geq 0$. In fact,

$$E[X_1^4]=\int_0^a x^4\mathrm{d}\frac{x}{a}=\frac{a^4}{5},$$

so X_t has the 4-th moment (5) by Theorem 13.

Example 6. Let X_t be a stationary independent increment process and $X_t\sim\mathcal{Z}(0,at,bt)$ with an initial value 0. Then

$$E[X_t^3]=\frac{3a^3+a^2b+ab^2+b^3}{8}t^3 \tag{6}$$

for any time $t\geq 0$. In fact,

$$E[X_1^3]=\int_0^a x^3\mathrm{d}\frac{x}{a}+\int_a^b x^3\mathrm{d}\frac{x+b-2a}{2(b-a)}=\frac{3a^3+a^2b+ab^2+b^3}{8},$$

so X_t has the 3-th moment (6) by Theorem 13.

Example 7. Let X_t be a stationary independent increment process and $X_t\sim\mathcal{N}(0,at)$ with an initial value 0. Then

$$E[X_t^2]=a^2t^2 \tag{7}$$

for any time $t\geq 0$. In fact,

$$E[X_1^2]=\int_{-\infty}^{+\infty}x^2\mathrm{d}\left(1+\exp\left(\frac{-\pi x}{\sqrt{3}a}\right)\right)^{-1}=a^2,$$

so X_t has the 2-th moment (7) by Theorem 13.

Theorem 14. *Let X_t be an uncertain stationary independent increment process with an initial value* 0. *Then for any times s and t, we have*

$$\sqrt[k]{E[X_{s+t}^k]}=\sqrt[k]{E[X_s^k]}+\sqrt[k]{E[X_t^k]}.$$

Proof. It follows from Theorem 13 that there exist a real number a such that $E[X_t]^k=at^k$ for any time $t\geq 0$. Since X_0 is a constant, we have

$$\sqrt[k]{E[(X_{s+t})^k]}=\sqrt[k]{a}(s+t)=\sqrt[k]{a}s+\sqrt[k]{a}t=\sqrt[k]{E[(X_s)^k]}+\sqrt[k]{E[(X_t)^k]}.$$

Thus we obtain the theorem. □

Example 8. Let X_t be a stationary independent increment process and $X_t\sim\mathcal{Z}(0,at,bt)$ with an initial value 0. Then

$$E[X_{s+t}^3]=\sqrt[3]{E[X_s^3]}+\sqrt[3]{E[X_t^3]} \tag{8}$$

for any time $t\geq 0$. In fact,

$$\begin{aligned}\sqrt[3]{E[(X_{s+t}^3]}&=\sqrt[3]{\frac{3a^3+a^2b+ab^2+b^3}{8}(s+t)^3}\\&=\sqrt[3]{\frac{3a^3+a^2b+ab^2+b^3}{8}}s+\sqrt[3]{\frac{3a^3+a^2b+ab^2+b^3}{8}}t\\&=\sqrt[3]{E[X_s^3]}+\sqrt[3]{E[X_t^3]},\end{aligned}$$

so (8) holds by Theorem 14.

Theorem 15. *Let X_t be an uncertain stationary independent increment process with a crisp initial value X_0. Then there exist two real numbers a and b such that*

$$E[X_t^k] = t^k \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$$

for any time $t \geq 0$.

Proof. Since

$$\begin{aligned} E[X_t^k] &= E[(X_t + EX_t - EX_t)^k] = E\left[\sum_{i=0}^{k} \binom{k}{i} (X_t - EX_t)^i (EX_t)^{k-i}\right] \\ &= \sum_{i=0}^{k} \binom{k}{i} E[(X_t - EX_t)^i EX_t^{k-i}] \\ &= \sum_{i=0}^{k} \binom{k}{i} t^i E[(X_1 - EX_1)^i] t^{k-i} (EX_1)^{k-i} \\ &= t^k \sum_{i=0}^{k} \binom{k}{i} E[(X_1 - EX_1)^i] E[X_1^{k-i}]. \end{aligned}$$

Hence this theorem holds for $a^i = E[(X_1 - EX_1)^i], b = E[X_1^{k-i}]$. □

Example 9. Let X_t be a stationary independent increment process and $X_t \sim \mathcal{Z}(0, at, bt)$ with a crisp initial value X_0. Then we have

$$\begin{aligned} E[(X_t^3] &= t^3 \sum_{i=0}^{3} \binom{3}{i} E[(X_1 - EX_1)^i] E[X_1^{3-i}] \\ &= t^3 \left[\frac{3a^3 + a^2 b + ab^2 + b^3}{8} + 3 \times \frac{-b}{8} \times \frac{3a^2 - 2ab + b^2}{6} + 3 \times \frac{12a^2 - 8ab + 13b^2}{96} \times \frac{2a + b}{4} \right. \\ &\quad \left. + \frac{3b^3 - 8ab^2 - 4a^2 b}{128}\right] \\ &= \frac{72a^3 - 16a^2 b + 39ab^2 + 24b^3}{128} t^3 \end{aligned}$$

for any time $t \geq 0$ by Theorem 14.

4 Conclusion

As an extension of expected value and variance of an uncertain stationary increment process, this paper mainly studied the k-th moment and k-th central moment. Then some examples are given to indicate how to calculate k-th moment and k-th central moment.

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