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Solving I-fuzzy Bi-matrix Games with I-fuzzy Goals by Resolving Indeterminacy

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Abstract

In this paper, we study I-fuzzy bi-matrix games with I-fuzzy goals. The indeterminacy factor in the I-fuzzy goal of each player is resolved using the Hurwicz optimism-pessimism rule. As a result, such an I-fuzzy bi-matrix game reduces to solving a fuzzy optimization problem with S-shaped membership functions. The latter problem is equivalently converted into its crisp counterpart using the conventional methods available in the literature of the fuzzy optimization. The resultant problem so obtained is devoid of any binary variable. A numerical example is presented to illustrate the proposed approach.

Keywords: bi-matrix fuzzy game, I-fuzzy set, I-fuzzy goals indeterminacy resolving function, piecewise linear membership function

1 Introduction

Atanassov [4, 6, 5] integrated the notion of hesitancy degree in the definition of a fuzzy set, by adding a new component which describes the degree of nonmembership in a given fuzzy set, and called such a set an intuitionistic fuzzy set. While the definition of fuzzy set provides the degree of membership of an element in a given set and its nonmembership degree is understood as one minus its membership degree, the definition of an intuitionistic fuzzy set provides more-or-less independent degree of membership and degree of nonmembership of an element in a given set. The only requirement in latter is that the sum of the two degrees is not greater than one. As a result, an intuitionistic fuzzy set exhibits characteristics of affirmation and negation, as well as hesitation. For instance, in any confronting situation in decision making, beside support or positive response and objection or negative response, there could also be an abstention which indicates hesitation and indeterminacy in response to the situation. Intutionistic fuzzy set, very naturally, model such cases in decision making problems. Quite a few applications of intutionistic fuzzy sets have emerged in recent years in various areas; for instance, we may refer to Atanassov [7], Szmidt and Kacprzyk [24], and many other references cited therein.

Intutionistic fuzzy set had its share of controversy (see, Dubois et al. [11] and Grzegorzewski and MrÓwka [13]) surrounding its nomenclature because similar name had also been used for intuitionistic logic, and the two concepts differ in their mathematical structure and treatment. It obviously makes sense to avoid using same terminology for two different concepts. Hence, as suggested in [11] and [13], Atanassov's intuitionistic fuzzy set is called *Atanassov's I-fuzzy set* or simply an *I-fuzzy set*. Henceforth, in this paper, we shall only be using an I-fuzzy set.

Fuzzy matrix games have been extensively studied in literature. For instance, refer to the good texts by Bector and Chandra [8] and Nishizaki and Sakawa [21]. Aggarwal et al. [2] studied duality for I-fuzzy linear programming problems and applied it to define a solution concept for two person zero sum matrix games with

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I-fuzzy goals. Further, in [1], the authors studied two person zero sum matrix games having fuzzy payoffs and I-fuzzy goals.

Compare to the literature on the fuzzy matrix games, few attempts have been made to study fuzzy bi-matrix games. In [18], Mangasarian and Stone showed that every equilibrium point of a two person nonzero sum game can always be obtained by solving a suitable quadratic programming problem. Maeda [17] studied fuzzy bi-matrix game with fuzzy payoffs and showed the existence of Nash equilibrium in a fuzzy bi-matrix game. Vidyottama et al. [25] established an equivalence of a fuzzy bi-matrix game with fuzzy goal to a crisp nonlinear programming problem and extended this study to fuzzy bi-matrix games with fuzzy payoffs and fuzzy goals. Recently, Nayak and Pal [20] studied an I-fuzzy bi-matrix game with intutionistic fuzzy goals. They defined the notion of a Nash equilibrium solution for such a game on the lines of Nishizaki and Sakawa [21].

On the other hand, several authors have used different membership functions to depict different preferences granularity of decision makers. Hannan [14] interpolated fuzzy sets defining goals and constraints in an optimization problem by piecewise linear concave membership functions and solve the same using a goal programming approach. Nakamura [19], Yang et al. [27] and Inuiguchi et al. [16] proposed different techniques to solve fuzzy optimization problem with piecewise linear (quasi-concave) membership function. In Nakamura's approach [19], a subsidiary piecewise linear function was introduced which separates the whole membership function into a finite number of concave and convex sub-functions. In this way, a piecewise linear membership function is expressed in terms of logical functions and the fuzzy optimization problem is studied by solving a finite number of sub- problems. Yang et al. [27] reformulated a fuzzy linear programming problem with S-shaped membership functions as an integer linear program with binary variables.

The aim of this paper is to study a single objective I-fuzzy bi-matrix game with I-fuzzy goals. In this context, we observe an ambiguity in the recent work of Nayak and Pal [20] in the way I-fuzzy sets are used in defining the membership and the nonmembership functions of the I-fuzzy goals. We define an equilibrium solution of the I-fuzzy bi-matrix game and propose to remove the ambiguity in [20] by making the representation of the membership and the nonmembership functions of the I-fuzzy goals truly in the I-fuzzy spirit. We first resolve the indeterminacy associated with each player's I-fuzzy goal and then showed that solving such a game is equivalent to solving a fuzzy bi-matrix game with piecewise linear S-shaped membership functions. Two optimization problems are formulated to solve the latter game, depending upon the optimistic and pessimistic attitude of the two players, using Inuiguchi et al. [16] and Yang et al. [27] schemes.

The remainder of the paper is organized as follows. Section 2 presents the basic definitions and preliminaries on the intutionistic fuzzy sets. Section 3, describes the solution concept of the I-fuzzy bi-matrix game with I-fuzzy goals. Section 4 presents the two equivalent nonlinear programming problems for obtaining the equilibrium solution of the I-fuzzy bi-matrix game based on the optimistic and the pessimistic approaches of the two players. Section 5 presents a numerical example to illustrate the proposed methodology. Section 6 is concluding remarks.

2 Preliminaries

In this section, we present few definitions on I-fuzzy sets. We also state Yager's [26] indeterminacy resolution principle which transforms an I-fuzzy set into a fuzzy set. We next present the interpretation of an I-fuzzy inequality as explained by Dubey et al. [10]. For all the notations and definitions, we shall be following [4, 6, 5, 10, 26].

Definition 1 ***(I-fuzzy set)*** *Let X be a universal set. An I-fuzzy set (originally called an intuitionistic fuzzy set in [4]) $\tilde{A}$ in X is described by $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle \mid x \in X\}$, where $\mu_{\tilde{A}} : X \to [0,1]$ and $\nu_{\tilde{A}} : X \to [0,1]$ define, respectively, the degree of belonging and the degree of not-belonging of an element $x \in X$ to the set $\tilde{A}$ such that $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$.*

If $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) = 1$, for all $x \in X$, then $\tilde{A}$ degenerates to the standard fuzzy set.

Definition 2 ***(Set theoretic operations in I-fuzzy sets)*** *Let $\tilde{A}$ and $\tilde{B}$ be two I-fuzzy sets in X. Their* *standard union and standard intersection are I-fuzzy sets* $\tilde{C} = \tilde{A} \cup \tilde{B}$, *and* $\tilde{D} = \tilde{A} \cap \tilde{B}$, *defined respectively as*

$$\tilde{C} = \{\langle x,\ \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}\rangle \mid x \in X\}\,;$$
$$\tilde{D} = \{\langle x,\ \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}\rangle \mid x \in X\}\,.$$

The standard negation of an I-fuzzy set $\tilde{A}$, *is an I-fuzzy set*

$$\tilde{A}^c = \{\langle x, \nu_{\tilde{A}}(x), \mu_{\tilde{A}}(x)\rangle \mid x \in X\}\,.$$

Definition 3 ***(Measure of indeterminacy)*** *Let* $\tilde{A}$ *be an I-Fuzzy set in* X. *Then the value* $\pi_{\tilde{A}}(x)$ *given by*

$$\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)\,,\ x \in X,$$

is called the measure of indeterminacy or undecidedness of $x \in \tilde{A}$.

Note that the range of undecidedness of $x \in \tilde{A}$ is an interval $[\mu_{\tilde{A}}(x)\,,\ 1 - \nu_{\tilde{A}}(x)]$, and the measure of its indeterminacy is length of this interval.

Using the Hurwicz's optimism-pessimism criterion [15], for a fixed λ, $\lambda \in [0,1]$, an I-fuzzy set $\tilde{A}$ is transformed into a fuzzy set $\tilde{\mathbb{A}}$ whose membership function is described by

$$f_{\tilde{\mathbb{A}}}(\lambda, x) = (1-\lambda)\mu_{\tilde{A}}(x) + \lambda(1 - \nu_{\tilde{A}}(x))\,,\ \ x \in X\,.$$

Henceforth, we shall be calling this function as *indeterminacy resolving function of* $\tilde{A}$. The parameter λ depicts the outlook of the decision maker towards resolving indeterminacy; $\lambda = 0$, means that the decision maker resolves indeterminacy fully in favor of membership (complete optimism in resolving indeterminacy), while $\lambda = 1$ indicates that the decision maker resolves indeterminacy fully in negation of the nonmembership function (complete pessimism in resolving indeterminacy).

2.1 Decision Making in I-fuzzy Environment

Consider a multiobjective optimization problem with r goals and q constraints. Let the set of goals be $G_\ell,\ \ell = 1, \ldots, r$, and let the set of constraints be $C_k,\ k = 1, \ldots, q$, each of which can be characterized as an I-fuzzy set on the universal set X.

Angelov [3] used the Bellman and Zadeh's extension principle [9] and defined the I-fuzzy decision as follows:

$$\tilde{D} = (\tilde{G}_1 \cap \tilde{G}_2 \cap \ldots \cap \tilde{G}_r) \cap (\tilde{C}_1 \cap \tilde{C}_2 \cap \ldots \cap \tilde{C}_q)$$

with

$$\tilde{D} = \{\langle x,\ \mu_{\tilde{D}}(x),\ \nu_{\tilde{D}}(x)\rangle \mid x \in X\},$$

where

$$\mu_{\tilde{D}}(x) = \min_{\ell,\,k}\left\{\mu_{\tilde{G}_\ell}(x),\ \mu_{\tilde{C}_k}(x)\right\} \quad \text{and} \quad \nu_{\tilde{D}}(x) = \max_{\ell,\,k}\left\{\nu_{\tilde{G}_\ell}(x),\ \nu_{\tilde{C}_k}(x)\right\}.$$

Angelov [3] associated a value function with $\tilde{D}$ as $V_{\tilde{D}}(x) = \mu_{\tilde{D}}(x) - \nu_{\tilde{D}}(x)\,,\ x \in X$, and the optimal solution is defined in the sense of finding an $x^\star \in X$ such that $V_{\tilde{D}}(x^\star) = \max_{x \in X} V_{\tilde{D}}(x)$.

Dubey et al. [10] implemented Yager's [26] idea of resolving indeterminacy in the interval uncertainty represented by I-fuzzy sets in optimization problems. It was observed that this approach can yield a better optimal value for decision making problem than the one proposed in [3]. We briefly describe their [10] decision making approach in I-fuzzy environment.

Let $\lambda \in [0,1]$ be fixed. Associate a fuzzy set $\tilde{\mathbb{D}}$, with an I-fuzzy decision set $\tilde{D}$, having membership function explained as

$$f_{\tilde{\mathbb{D}}}(\lambda, x) = \min_{\ell,\,k}\left\{f_{\tilde{G}_\ell}(\lambda, x),\ f_{\tilde{C}_k}(\lambda, x) \mid x \in X\right\}\,,$$

where $f_{\tilde{G}_\ell}(\lambda, x)$ and $f_{\tilde{C}_k}(\lambda, x)$ are the indeterminacy resolving functions of the I-fuzzy sets representing the ℓ^{th} goal and the k^{th} constraint, respectively. Then, $x^\star \in X$ is an *optimal decision*, if $f_{\tilde{\mathbb{D}}}(\lambda, x^\star) = \max\limits_{x \in X} f_{\tilde{\mathbb{D}}}(\lambda, x)\,,$ that is, $f_{\tilde{\mathbb{D}}}(\lambda, x^\star) \geq f_{\tilde{\mathbb{D}}}(\lambda, x)\,,\ \forall\, x \in X$.

Hence solving an optimization problem with Atanassov's I-fuzzy goal is equivalent to solving the following optimization problem:

$$\begin{array}{ll} \max & \alpha \\ \text{subject to} & f_{\tilde{G}_\ell}(\lambda, x) \geq \alpha, \quad \ell = 1, \dots, r \\ & f_{\tilde{C}_k}(\lambda, x) \geq \alpha, \quad k = 1, \dots, q \\ & 0 \leq \alpha \leq 1, \;\; x \in X. \end{array}$$

2.2 Interpretation of I-fuzzy Inequality $a^T x \gtrsim^{IF} b$

Let $a,\ b \in \mathbb{R}^n$, the $n-$dimensional real space. Though there is no unique way to define an I-fuzzy inequality $a^T x \gtrsim^{IF} b$ but two natural approaches are 'the optimistic approach' and 'the pessimistic approach'. We briefly explain them (see, [2] and [10] for details) in the following.

For a given acceptance tolerance $\hat{p} > 0$, the linear membership function associated with this inequality is described as follows:

$$\mu(a^T x) = \begin{cases} 1, & a^T x \geq b \\ 1 - \dfrac{b - a^T x}{\hat{p}}, & b - \hat{p} \leq a^T x \leq b \\ 0, & a^T x \leq b - \hat{p}\,. \end{cases}$$

Let $\hat{q}$ $(0 < \hat{q} < \hat{p})$ be the tolerance in rejection of the I-fuzzy inequality $a^T x \gtrsim^{IF} b$. Then, we have two approaches, namely, the optimistic approach and the pessimistic approach, to define the nonmembership function of $a^T x \gtrsim^{IF} b$. The linear nonmembership function in optimistic and pessimistic approaches are defined respectively as follows:

$$\nu(a^T x) = \nu_{\text{optimistic}}(a^T x) = \begin{cases} 0, & a^T x \geq b \\ 1 - \dfrac{a^T x - b + \hat{p} + \hat{q}}{\hat{p} + \hat{q}}, & b - \hat{p} - \hat{q} \leq a^T x \leq b \\ 1, & a^T x \leq b - \hat{p} - \hat{q}. \end{cases}$$

$$\nu(a^T x) = \nu_{\text{pessimistic}}(a^T x) = \begin{cases} 0, & a^T x \geq b - \hat{p} + \hat{q} \\ 1 - \dfrac{a^T x - b + \hat{p}}{\hat{q}}, & b - \hat{p} \leq a^T x \leq b - \hat{p} + \hat{q} \\ 1, & a^T x \leq b - \hat{p}. \end{cases}$$

The I-fuzzy inequality $a^T x \lesssim^{IF} b$ is treated equivalent to $(-a)^T x \gtrsim^{IF} (-b)$.

In the section to follow, we present the I-fuzzy bi-matrix game with I-fuzzy goal.

3 I-fuzzy Bi-matrix Game with I-fuzzy Goals: Proposed Models

The following notations are used in the paper:

$\mathbb{R}^n$:	$n-$dimensional real space;
$\mathbb{R}^n_+$:	the non-negative orthant of $\mathbb{R}^n$;
$e^T = (1, \dots, 1)$	vector of 'ones' whose dimension is as per the specific context;
S^m	$= \{x \in \mathbb{R}^m_+ \mid e^T x = 1\}$, the strategy space of Player I;
S^n	$= \{x \in \mathbb{R}^n_+ \mid e^T y = 1\}$, the strategy space of Player II;
A	$m \times n$ real matrix representing payoffs of Player I;
B	$m \times n$ real matrix representing payoffs of Player II.

Before we present our proposed work, we highlight a flaw in the recent work of Nayak and Pal [20] on I-fuzzy bi-matrix game with I-fuzzy goal. Nayak and Pal [20] took the I-fuzzy bi-matrix game as $(S^m,\ S^n,\ A,\ B)$ with I-fuzzy goals. They considered

$$\overline{a} = \max_x \max_y x^T A y = \max_i \max_j a_{ij}\,, \quad \underline{a} = \min_x \min_y x^T A y = \min_i \min_j a_{ij}\,,$$

and define the membership function and the nonmembership function for Player-I as follows:

$$\mu_1(x^TAy) = \begin{cases} 1, & x^TAy \geq \overline{a} \\ \dfrac{x^TAy - \underline{a}}{\overline{a} - \underline{a}}, & \underline{a} \leq x^TAy \leq \overline{a} \\ 0, & x^TAy \leq \underline{a}\,, \end{cases}$$

and

$$\nu_1(x^TAy) = \begin{cases} 1, & x^TAy \leq \underline{a} \\ \dfrac{x^TAy - \overline{a}}{\underline{a} - \overline{a}}, & \underline{a} \leq x^TAy \leq \overline{a} \\ 0, & x^TAy \geq \overline{a}. \end{cases}$$

The membership function and the nonmembership function for Player II are defined on similar lines using the expected payoff x^TBy of Player II.

Here, the noteworthy fact is that $\mu_1(x^TAy) + \nu_1(x^TAy) = 1,\ \forall\ (x,y) \in S^m \times S^n$. Consequently, the study by Nayak and Pal [20] fails to capture the true spirit of I-fuzzy set and reduces to the classical fuzzy case only which had already been carried out by Nishizaki and Sakawa [21].

In this work we took a different approach so as to clearly distinguish the role of I-fuzzy goals in an I-fuzzy bi-matrix game.

Let $V_0 \in \mathbb{R}$ and $W_0 \in \mathbb{R}$ denote the aspiration levels of Player-I and Player-II, respectively. Then, the I-fuzzy bi-matrix game with I-fuzzy goals, denoted by $(IFBG)$, is defined as

$$(IFBG) \equiv (S^m,\ S^n,\ A,\ B,\ V_0,\ \lesssim^{IF},\ W_0,\ \gtrsim^{IF})\,.$$

Here $\gtrsim^{IF}$ and $\lesssim^{IF}$ shall be interpreted in the sense of subsection 2.2.

Definition 4 ***(Equilibrium solution of the game*** $(IFBG)$***)***
A point $(x^\star,\, y^\star) \in S^m \times S^n$ *is called an equilibrium solution of the I-fuzzy bi-matrix game* $(IFBG)$ *if*

$$\begin{aligned} &x^TAy^\star \lesssim^{IF} V_0\,, \quad \forall\, x \in S^m, \\ &x^{\star T}By \lesssim^{IF} W_0\,, \quad \forall\, y \in S^n, \\ &x^{\star T}Ay^\star \gtrsim^{IF} V_0\,, \\ &x^{\star T}By^\star \gtrsim^{IF} W_0\,. \end{aligned}$$

Since S^m and S^n are the convex polytopes, the specific choice of membership and nonmembership functions will lead to the following I-fuzzy nonlinear programming problem:

$$\begin{aligned} (IFNLP) \qquad \text{Find } (x,\, y) \quad &\text{such that} \\ &A_iy \lesssim^{IF} V_0\,, \quad i = 1, \ldots, m, \\ &x^TB_j \lesssim^{IF} W_0\,, \quad j = 1, \ldots, n, \\ &x^TAy \gtrsim^{IF} V_0\,, \\ &x^TBy \gtrsim^{IF} W_0\,, \\ &x \in S^m, \\ &y \in S^n, \end{aligned}$$

where for $A_i\,,\ i = 1, \ldots, m$, and $B_j\,,\ j = 1, \ldots, n$, denote, respectively, the i^{th} row of matrix A and the j^{th} column of matrix B.

4 Equivalent Optimization Models for Problem $(IFNLP)$

We next present two models for problem $(IFNLP)$, depending upon the optimistic/pessimistic interpretation of the I-fuzzy inequalities.

4.1 Model in the Optimistic Framework

A player has an optimistic attitude towards rejection, depicting the state that the player is not in a position to accept a criterion but at the same time does not want to completely reject it. Hence the player has a liberal view for rejection.

To model this, let p_0 and q_0 $(0 < q_0 < p_0)$, respectively, denote the tolerances in the acceptance and the rejection of I-fuzzy inequality $A_i y \lesssim^{IF} V_0$, $i = 1, \ldots, m$, in $(IFNLP)$. Then, the membership and the nonmembership functions for the i^{th} I-fuzzy inequality are, respectively, as follows:

$$\mu_i(A_i y) = \begin{cases} 1, & A_i y \leq V_0 \\ 1 - \dfrac{A_i y - V_0}{p_0}, & V_0 \leq A_i y \leq V_0 + p_0 \\ 0, & A_i y \geq V_0 + p_0, \end{cases}$$

and

$$\nu_i(A_i y) = \begin{cases} 0, & A_i y \leq V_0 \\ 1 + \dfrac{A_i y - (V_0 + p_0 + q_0)}{p_0 + q_0}, & V_0 \leq A_i y \leq V_0 + p_0 + q_0 \\ 1, & A_i y \geq V_0 + p_0 + q_0. \end{cases}$$

The indeterminacy resolving functions $f_i(\lambda, A_i y)$, $i = 1, \ldots, m$, are as follows:

$$f_i(\lambda, A_i y) = \begin{cases} 1, & A_i y \leq V_0 \\ f_{i,1} = 1 + (V_0 - A_i y)\big(\dfrac{p_0 + (1-\lambda)q_0}{p_0(p_0 + q_0)}\big), & V_0 \leq A_i y \leq V_0 + p_0 \\ f_{i,2} = \dfrac{\lambda(V_0 + p_0 + q_0 - A_i y)}{(p_0 + q_0)}, & V_0 + p_0 \leq A_i y \leq V_0 + p_0 + q_0 \\ 0, & A_i y \geq V_0 + p_0 + q_0. \end{cases}$$

Next, let p_1 and q_1 $(0 < q_1 < p_1)$, respectively, be the tolerances in the acceptance and the rejection of the constraint $x^T A y \gtrsim^{IF} V_0$. Then the membership function and the nonmembership functions associated with this constraint are, respectively, described as follows:

$$\mu(x^T A y) = \begin{cases} 0, & x^T A y \geq V_0 - p_1 \\ 1 - \dfrac{V_0 - x^T A y}{p_1}, & V_0 - p_1 \leq x^T A y \leq V_0 \\ 1, & x^T A y \geq V_0, \end{cases}$$

and

$$\nu(x^T A y) = \begin{cases} 1, & x^T A y \leq V_0 - p_1 - q_1 \\ 1 - \dfrac{x^T A y - V_0 + p_1 + q_1}{p_1 + q_1}, & V_0 - p_1 - q_1 \leq x^T A y \leq V_0 \\ 0, & x^T A y \geq V_0. \end{cases}$$

The associated indeterminacy resolving function is as follows:

$$f_{m+1}(\lambda, x^T A y) = \begin{cases} 0, & x^T A y \leq V_0 - p_1 - q_1 \\ f_{m+1,1} = \dfrac{\lambda(x^T A y - (V_0 - p_1 - q_1))}{p_1 + q_1}, & V_0 - p_1 - q_1 \leq x^T A y \leq V_0 - p_1 \\ f_{m+1,2} = 1 + (x^T A y - V_0)\big(\dfrac{p_1 + (1-\lambda)q_1}{p_1(p_1 + q_1)}\big), & V_0 - p_1 \leq x^T A y \leq V_0 \\ 1, & x^T A y \geq V_0. \end{cases}$$

Similarly, let s_0 and t_0 $(0 < t_0 < s_0)$, respectively, be the tolerances in the acceptance and the rejection of I-fuzzy inequalities $x^T B_j \lesssim^{IF} W_0$, $j = 1, \ldots, n$, in $(IFNLP)$. Then the membership and the nonmembership functions for the j^{th} I-fuzzy inequality are respectively as follows:

$$\mu_j(x^T B_j) = \begin{cases} 1, & x^T B_j \leq W_0 \\ 1 - \dfrac{x^T B_j - W_0}{s_0}, & W_0 \leq x^T B_j < W_0 + s_0 \\ 0, & x^T B_j \geq W_0 + s_0, \end{cases}$$

and

$$\nu_j(x^TB_j) = \begin{cases} 0, & x^TB_j \le W_0 \\ 1 + \dfrac{x^TB_j - (W_0+s_0+t_0)}{s_0+t_0}, & W_0 \le x^TB_j \le W_0+s_0+t_0 \\ 1, & x^TB_j \ge W_0+s_0+t_0. \end{cases}$$

The indeterminacy resolving functions $f_{m+1+j}(\eta, x^TB_j),\ j=1,\dots,n$, are as follows:

$$f_{m+1+j}(\eta, x^TB_j) = \begin{cases} 1, & x^TB_j \le W_0 \\ f_{m+1+j,\,1} = 1 + (W_0 - x^TB_j)\big(\dfrac{s_0+(1-\eta)t_0}{s_0(s_0+t_0)}\big), & W_0 \le x^TB_j \le W_0+s_0 \\ f_{m+1+j,\,2} = \dfrac{\eta(W_0+s_0+t_0-x^TB_j)}{(s_0+t_0)}, & W_0+s_0 \le x^TB_j \le W_0+s_0+t_0 \\ 0, & x^TB_j \ge W_0+s_0+t_0\,. \end{cases}$$

Similarly, let s_1 and t_1 $(0 < t_1 < s_1)$, be the tolerances in the acceptance and the rejection of the I-fuzzy inequality $x^TBy \gtrsim^{IF} W_0$. Then, the membership and the nonmembership functions are respectively as follows:

$$\mu(x^TBy) = \begin{cases} 0, & x^TBy \ge W_0 - s_1 \\ 1 - \dfrac{x^TBy - W_0}{s_1}, & W_0 - s_1 \le x^TBy \le W_0 \\ 1, & x^TBy \ge W_0\,, \end{cases}$$

and

$$\nu(x^TBy) = \begin{cases} 1, & x^TBy \le W_0 - s_1 - t_1 \\ 1 - \dfrac{x^TBy - W_0 + s_1 + t_1}{s_1+t_1}, & W_0 - s_1 - t_1 \le x^TBy \le W_0 \\ 0, & x^TBy \ge W_0\,, \end{cases}$$

and the indeterminacy resolving function is as follows:

$$f_{m+n+2}(\eta, x^TBy) = \begin{cases} 0, & x^TBy \le W_0 - s_1 - t_1 \\ f_{m+n+2,\,1} = \dfrac{\eta(x^TBy - (W_0 - s_1 - t_1))}{s_1+t_1}, & W_0 - s_1 - t_1 \le x^TBy \le W_0 - s_1 \\ f_{m+n+2,\,2} = 1 + (x^TBy - W_0)\big(\dfrac{s_1+(1-\eta)t_1}{s_1(s_1+t_1)}\big), & W_0 - s_1 \le x^TBy \le W_0 \\ 1, & x^TBy \ge W_0. \end{cases}$$

Such functions are depicted in Figure 1 and Figure 2.

It is important to note here that the functions $f_i(\lambda, A_iy)$, $f_{m+1}(\lambda, x^TAy)$, $f_{m+1+j}(\eta, x^TB_j)$, and $f_{m+n+2}(\eta, x^TBy)$ are piecewise linear S-shaped functions with convex type break points. Here, we implement the technique of Inuiguchi et al. [16] which convert a piecewise linear membership function with convex break points into a piecewise linear membership function with concave break points. The algorithm in [16] involves the following steps.

Step 1:– Arrange

$$0,\ \frac{\lambda q_0}{p_0+q_0},\ \frac{\lambda q_1}{p_1+q_1},\ \frac{\eta t_0}{t_0+s_0},\ \frac{\eta t_1}{t_1+s_1},\ 1,$$

in ascending order, and let them be re-named as $c_1 = 0,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6 = 1$. For $t = 1,\dots,6$, compute

$$v_p^t = f_p^{-1}(c_t),\ p = 1,\dots,m+n+2.$$

Step 2:– Set $\delta_1^{'} = 1$, and for $\theta = 1,2,3,4$, calculate

$$\delta_{\theta+1}^{'} = \delta_\theta^{'} \min_{1\le p\le m+n+2}\big(\frac{v_p^{\theta+2} - v_p^{\theta+1}}{v_p^{\theta+1} - v_p^{\theta}}\big).$$

Step 3:– Compute

$$\delta_\theta = \frac{\delta_\theta^{'}}{\sum_{\theta=1}^{5}\delta_\theta^{'}},\ \theta = 1,\dots,5.$$

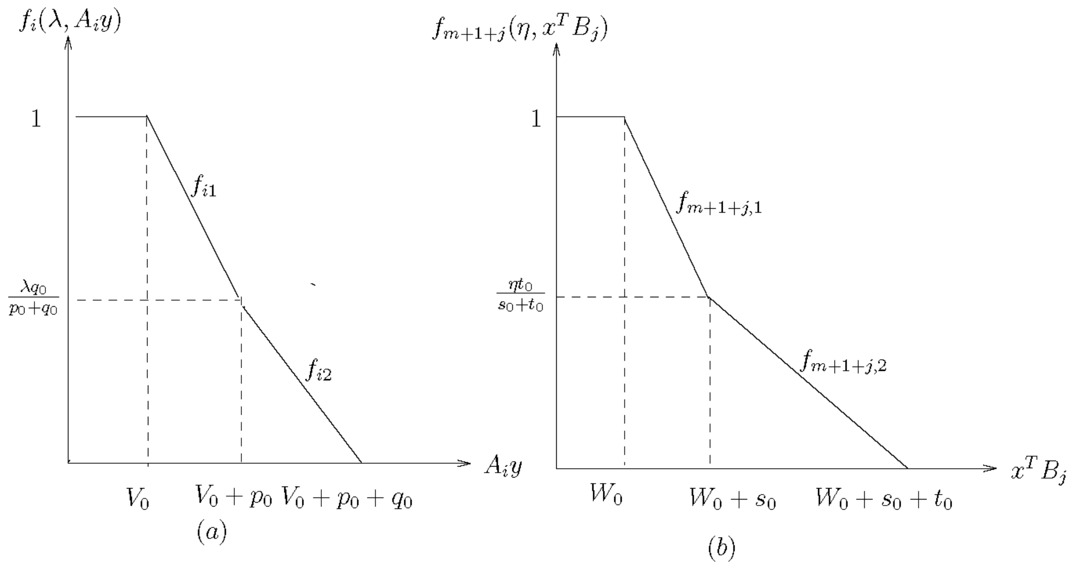

Figure 1: Indeterminacy resolving functions in optimistic approach (a) $A_i y \lesssim^{IF} V_0$ (b) $x^T B_j \lesssim^{IF} W_0$

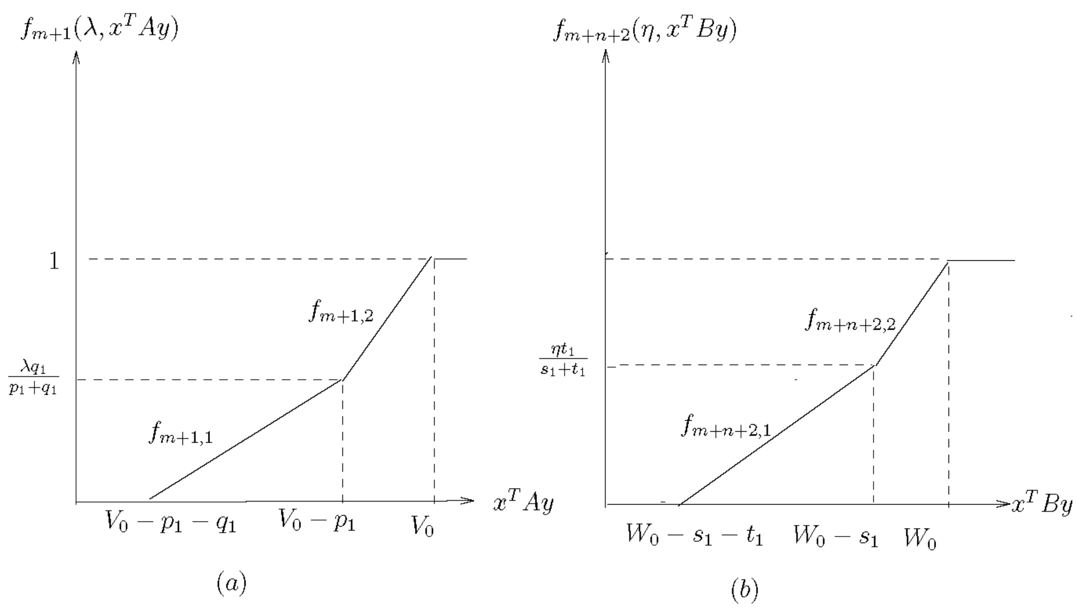

Figure 2: Indeterminacy resolving functions in optimistic approach (a) $x^T Ay \gtrsim^{IF} V_0$ (b) $x^T By \gtrsim^{IF} W_0$

Step 4:– For $p = 1, \ldots, m+n+2$, compute

$$h_p^\theta = \frac{\delta_\theta}{v_p^{\theta+1} - v_p^\theta}, \quad \theta = 1, \ldots, 5,$$

and

$$\hat{f}_p(v_p^t) = \begin{cases} 0, & t = 1 \\ \sum_{\theta=1}^{t-1} \delta_\theta, & 2 \le t \le 6. \end{cases}$$

Step 5:– For $p = 1, 2, \ldots, m, m+2, m+3, \ldots, m+n+1$, define

$$\hat{f}_p(x) = \begin{cases} 0, & x \ge v_p^1 \\ \min\limits_{1 \le \theta \le 5} \big((x - v_p^\theta) h_p^\theta + \hat{f}_p(v_p^\theta)\big), & v_p^6 \le x \le v_p^1 \\ 1, & x \le v_p^6 . \end{cases}$$

And for $p = m+1$ and $m+n+2$, define

$$\hat{f}_p(x) = \begin{cases} 0, & x \le v_p^1 \\ \min\limits_{1 \le \theta \le 5} \big((x - v_p^\theta) h_p^\theta + \hat{f}_p(v_p^\theta)\big), & v_p^1 \le x \le v_p^6 \\ 1, & x \ge v_p^6 . \end{cases}$$

Note that, after applying Inuiguichi et al. [16] algorithm on all the constraints of $(IFNLP)$, the constraints get transformed into concave piecewise linear functions. Subsequently, applying Yang et al. [27] approach, an I-fuzzy optimization problem $(IFNLP)$ is equivalent to the following crisp nonlinear program:

$$\begin{array}{lll} (CENLP)_o & \max & \alpha \\ & \text{subject to} & \dfrac{\delta_\theta}{v_i^{\theta+1} - v_i^\theta}(A_i y - v_i^\theta) + \hat{f}_i(v_i^\theta) \ge \alpha, \ i = 1, \ldots, m, \ \theta = 1, \ldots, 5 \\ & & \dfrac{\delta_\theta}{v_{m+1}^{\theta+1} - v_{m+1}^\theta}(x^T A y - v_{m+1}^\theta) + \hat{f}_{m+1}(v_{m+1}^\theta) \ge \alpha, \quad \theta = 1, \ldots, 5 \\ & & \dfrac{\delta_\theta}{v_{m+1+j}^{\theta+1} - v_{m+1+j}^\theta}(x^T B_j - v_{m+1+j}^\theta) + \hat{f}_{m+1+j}(v_{m+1+j}^\theta) \ge \alpha, \\ & & \qquad\qquad\qquad j = 1, \ldots, n, \ \theta = 1, \ldots, 5 \\ & & \dfrac{\delta_\theta}{v_{m+n+2}^{\theta+1} - v_{m+n+2}^\theta}(x^T B y - v_{m+n+2}^\theta) + \hat{f}_{m+n+2}(v_{m+n+2}^\theta) \ge \alpha, \quad \theta = 1, \ldots, 5 \\ & & x \in S^m, \ y \in S^n, \ \ \alpha \in [0, 1]. \end{array}$$

Summarizing the above discussion, we get that, finding an equilibrium solution of an I-fuzzy bi-matrix game $(IFBG)$ with I-fuzzy goals, in an optimistic framework, requires to solve the crisp nonlinear programming problem $(CENLP)_o$. Also, if $(x^\star, y^\star, \alpha^\star)$ is an optimal solution of $(CENLP)_o$, then $(x^\star, y^\star)$ is an equilibrium solution of $(IFBG)$ and $\alpha^\star$ is the highest degree up to which the aspiration goals V_0 and W_0 for respective players Player I and Player II are met, when both players have taken the optimistic approach.

4.2 Model in the Pessimistic Framework

In this approach, a player has a pessimistic approach towards acceptance of a criterion, amounting to saying that a complete rejection of a criterion does not entail its full acceptance.

Let p_0' and q_0' $(0 < {q'}_0 < {p'}_0)$, respectively, be the tolerances in the acceptance and the rejection of m constraints $A_i y^* \lesssim^{IF} V_0, \ i = 1, \ldots, m$, in $(IFNLP)$. Then, the membership function and the nonmembership function for the i^{th} I-fuzzy inequality $A_i y^* \lesssim^{IF} V_0$ are, respectively, as follows:

$$\mu_i(A_i y) = \begin{cases} 1, & A_i y \le V_0 \\ 1 - \dfrac{A_i y - V_0}{p_0'}, & V_0 \le A_i y \le V_0 + p_0' \\ 0, & A_i y \ge V_0 + p_0' , \end{cases}$$

and

$$\nu_i(A_iy) = \begin{cases} 0, & A_iy \le V_0 + p_0' - q_0' \\ 1 + \dfrac{A_iy - (V_0 + p_0')}{q_0}, & V_0 + p_0' - q_0' \le A_iy \le V_0 + p_0' \\ 1, & A_iy \ge V_0 + p_0'. \end{cases}$$

The indeterminacy resolving functions are as follows:

$$f_i(\lambda, A_iy) = \begin{cases} 1, & A_iy \le V_0 \\ f_{i,1} = 1 + (1-\lambda)\dfrac{(V_0 - A_iy)}{p_0'}, & V_0 \le A_iy \le V_0 + p_0' - q_0' \\ f_{i,2} = \dfrac{\lambda p_0' + (1-\lambda)q_0'}{q_0'}(1 + \dfrac{V_0 - A_iy}{p_0'}), & V_0 + p_0' - q_0' \le A_iy \le V_0 + p_0' \\ 0, & A_iy \ge V_0 + p_0'. \end{cases}$$

Similarly, if s_0' and t_0' $(0 < t_0' < s_0')$, respectively, are the tolerances in the acceptance and the rejection of n constraints $x^TB_j \lesssim^{IF} W_0,\ j = 1, \ldots, n$, in $(IFNLP)$, then the indeterminacy resolving function for the j^{th} constraint $x^TB_j \lesssim^{IF} W_0$ are as follows:

$$f_{m+1+j}(\eta, x^TB_j) = \begin{cases} 1, & x^TB_j \le W_0 \\ f_{m+1+j,1} = 1 + (1-\eta)\dfrac{(W_0 - x^TB_j)}{s_0'}, & W_0 \le x^TB_j \le W_0 + s_0' - t_0' \\ f_{m+1+j,2} = \dfrac{\eta s_0' + (1-\eta)t_0'}{t_0'}(1 + \dfrac{W_0 - x^TB_j}{s_0'}), & W_0 + s_0' - t_0' \le x^TB_j \le W_0 + s_0' \\ 0, & x^TB_j \ge W_0 + s_0'. \end{cases}$$

Here it is to be noted here that $f_i(\lambda, A_iy)$ and $f_{m+1+j}(\eta, x^TB_j)$ are piecewise linear S-shaped functions with concave break points.

Furthermore, let p_1' and q_1' $(0 < q_1' < p_1')$, respectively, be the tolerances in the acceptance and the rejection of the constraint $x^TAy \gtrsim^{IF} V_0$, then the indeterminacy function is as follows:

$$f_{m+1}(\lambda, x^TAy) = \begin{cases} 0, & x^TAy \le V_0 - p_1' \\ f_{m+1,1} = \dfrac{\lambda p_1' + (1-\lambda)q_1'}{q_1'}(1 + \dfrac{x^TAy - V_0)}{p_1'}), & V_0 - p_1' \le x^TAy \le V_0 - p_1' + q_1' \\ f_{m+1,2} = 1 + (1-\lambda)\dfrac{x^TAy - V_0}{p_1'}, & V_0 - p_1' + q_1' \le x^TAy \le V_0 \\ 1, & x^TAy \ge V_0. \end{cases}$$

Again let s_1' and t_1' $(0 < t_1' < s_1')$ be, respectively, the tolerances in the acceptance and the rejection of the last constraint $x^TBy \gtrsim^{IF} W_0$, then the indeterminacy resolving function is as follows:

$$f_{m+n+2}(\eta, x^TBy) = \begin{cases} 0, & x^TBy \le W_0 - s_1' \\ f_{m+n+2,1} = \dfrac{\eta s_1' + (1-\eta)t_1'}{t_1'}(1 + \dfrac{x^TBy - W_0)}{s_1'}), & W_0 - s_1' \le x^TBy \le W_0 - s_1' + t_1' \\ f_{m+n+2,2} = 1 + (1-\eta)\dfrac{x^TBy - W_0}{s_1'}, & W_0 - s_1' + t_1' \le x^TBy \le W_0 \\ 1, & x^TBy \ge W_0. \end{cases}$$

Here $f_{m+1}(\lambda, x^TAy)$ and $f_{m+n+2}(\eta, x^TBy)$ are S-shaped piecewise linear functions with concave break points.

Following the approach of Yang et al. [27], solving the I-fuzzy bi-matrix game $(IFBG)$ is equivalent to solving the following nonlinear programming problem:

$$\begin{aligned}
&\max && \alpha \\
&\text{subject to} && f_{i,1}(\lambda, A_i y) \geq \alpha, \quad i = 1, \dots, m \\
& && f_{i,2}(\lambda, A_i y) \geq \alpha, \quad i = 1, \dots, m \\
& && f_{m+1+j,1}(\eta, x^T B_j) \geq \alpha \quad j = 1, \dots, n \\
& && f_{m+j+1,2}(\eta, x^T B_j) \geq \alpha \quad j = 1, \dots, n \\
& && f_{m+1,1}(\lambda, x^T A y) \geq \alpha \\
& && f_{m+1,2}(\lambda, x^T A y) \geq \alpha \\
& && f_{m+n+2,1}(\eta, x^T B y) \geq \alpha \\
& && f_{m+n+2,2}(\eta, x^T B y) \geq \alpha \\
& && x \in S^m, \ y \in S^n, \ \alpha \in [0, 1],
\end{aligned}$$

or equivalently solving the following problem:

$$(CENLP)_p \qquad \begin{aligned}
&\max && \alpha \\
&\text{subject to} && \\
& && \alpha \leq 1 + (1-\lambda)\frac{(V_0 - A_i y)}{p_0'}, && i = 1, \dots, m \\
& && \alpha \leq \frac{\lambda p_0' + (1-\lambda) q_0'}{q_0'}\Big(1 + \frac{V_0 - A_i y}{p_0'}\Big), && i = 1, \dots, m \\
& && \alpha \leq 1 + (1-\eta)\frac{(W_0 - x^T B_j)}{s_0'}, && j = 1, \dots, n \\
& && \alpha \leq \frac{\eta s_0' + (1-\lambda) t_0'}{t_0'}\Big(1 + \frac{W_0 - x^T B_j}{s_0'}\Big), && j = 1, \dots, n \\
& && \alpha \leq \frac{\lambda p_1' + (1-\lambda) q_1'}{q_1'}\Big(1 + \frac{x^T A y - V_0)}{p_1'}\Big) && \\
& && \alpha \leq 1 + (1-\lambda)\frac{x^T A y - V_0}{p_1'} && \\
& && \alpha \leq \frac{\eta s_1' + (1-\eta) t_1'}{t_1'}\Big(1 + \frac{x^T B y - W_0)}{s_1'}\Big) && \\
& && \alpha \leq 1 + (1-\eta)\frac{x^T B y - W_0}{s_1'} && \\
& && x \in S^m, \ y \in S^n, \ \alpha \in [0, 1]. &&
\end{aligned}$$

From the above discussion, we observe that finding an equilibrium solution of the I-fuzzy bi-matrix game $(IFBG)$, in a pessimistic approach, is equivalent to solving the crisp nonlinear programming problem $(CENLP)_p$ for different values of λ and η. Also, if $(x^\star, y^\star, \alpha^\star)$ is an optimal solution of $(CENLP)_p$, then $(x^\star, y^\star)$ is an equilibrium solution of $(IFBG)$ and $\alpha^\star$ is the maximum degree up to which the aspiration goals V_0 and W_0 of respective players Player I and Player II are met, when both players have taken the pessimistic approach.

Remark 1 *When $q_0 = t_0 = q_1 = t_1 = 0$, the game problem $(IFBG)$, in an I-fuzzy environment, subsumes to a fuzzy bi-matrix game with fuzzy payoffs studied by Vidyottama et al. [25]. In this case, the Step 1 of the algorithm (p. 13) has v_p^t values equal for $t = 1, \dots, 5$ and for $p = 1, \dots, m+n+2$. So, we denote all of them by v_p^1 only and the other distinct value (that is, v_p^6 in Step 1) is then taken as v_p^2. Thus,*

$$v_i^1 = V_0 + p_0\,, \quad v_i^2 = V_0\,, \ \ i = 1, \dots, m,$$

$$v_{m+1}^1 = V_0 - p_1\,, \quad v_{m+1}^2 = V_0\,,$$

$$v_{m+1+j}^1 = W_0 + s_0\,, \quad v_{m+1+j}^2 = W_0\,, \ \ j = 1, \dots, n,$$

$$v_{m+n+2}^1 = W_0 - s_1\,, \quad v_{m+n+2}^2 = W_0.$$

For this special case, problem $(CENLP)_o$ reduces to the following nonlinear programming problem:

$$\begin{array}{ll} max & \alpha \\ subject\ to & 1 - \dfrac{(A_i y - V_0)}{p_0} \geq \alpha \quad i = 1, \ldots, m \\ & 1 + \dfrac{(x^T A y - V_0)}{p_1} \geq \alpha \\ & 1 - \dfrac{(x^T B_j - W_0)}{s_0} \geq \alpha \quad j = 1, \ldots, n \\ & 1 + \dfrac{(x^T B y - W_0)}{s_1} \geq \alpha \\ & x \in S^m,\ y \in S^n,\ \alpha \in [0,1], \end{array}$$

which is same as the equivalent problem obtained for solving the fuzzy bi-matrix game in [25].

5 Numerical Illustration

We present the famous example of Mangasarian and Stone [18] which was also studied by Vidyottama et al. [25] in fuzzy environment.

Example 1 *Consider the two person non zero sum game with payoffs matrices as follows:*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

When we employ the procedure in [25], we get, $V_0 = W_0 = 0.20$. Both player may aspire respective goals values V_0 and W_0 close to 0.20.

Optimistic Approach

Consider the membership and the nonmembership functions of the I-fuzzy bi-matrix game $(IFBG)$ in the optimistic approach with $V_0 = 0.30$, $W_0 = 0.40$. Let the tolerances be $p_0 = 0.20$, $q_0 = 0.10$, $s_0 = 0.25$, $t_0 = 0.15$ and $p_1 = 0.20$, $q_1 = 0.10$, $s_1 = 0.30$, $t_1 = 0.10$. Assuming that $\lambda = \eta = \frac{1}{3}$. After resolving the indeterminacy, we get the piecewise linear indeterminacy resolving functions, with convex break points, as follows:

$$f_1(A_1 y) = \begin{cases} 1, & 2y_1 - y_2 \leq 0.30 \\ 1 + \dfrac{40}{9}(0.30 - 2y_1 + y_2), & 0.30 \leq 2y_1 - y_2 \leq 0.50 \\ \dfrac{10}{9}(0.60 - 2y_1 + y_2), & 0.50 \leq 2y_1 - y_2 \leq 0.60 \\ 0, & 2y_1 - y_2 \geq 0.60, \end{cases}$$

$$f_2(A_2 y) = \begin{cases} 1, & y_1 - y_2 \leq 0.30 \\ 1 + \dfrac{40}{9}(0.30 - (-y_1 + y_2)), & 0.30 \leq -y_1 + y_2 \leq 0.50 \\ \dfrac{10}{9}(0.60 - (-y_1 + y_2)), & 0.50 \leq -y_1 + y_2 \leq 0.60 \\ 0, & y_1 - y_2 \geq 0.60, \end{cases}$$

$$f_4(B_1^T x) = \begin{cases} 1, & x_1 - x_2 \leq 0.40 \\ 1 + \dfrac{7}{2}(0.40 - (x_1 - x_2)), & 0.40 \leq x_1 - x_2 \leq 0.65 \\ \dfrac{5}{6}(0.80 - (x_1 - x_2)), & 0.65 \leq x_1 - x_2 \leq 0.80 \\ 0, & x_1 - x_2 \geq 0.80, \end{cases}$$

$$f_5(B_2^T x) = \begin{cases} 1, & x_1 - 2x_2 \le 0.40 \\ 1 + \dfrac{7}{2}(0.40 - (-x_1 + 2x_2)), & 0.40 \le -x_1 + 2x_2 \le 0.65 \\ \dfrac{5}{6}(0.80 - (-x_1 + 2x_2)), & 0.65 \le -x_1 + 2x_2 \le 0.80 \\ 0, & -x_1 - 2x_2 \ge 0.80. \end{cases}$$

On the same lines, we also have

$$f_3(x^T Ay) = \begin{cases} 0, & x^T Ay \le 0 \\ \dfrac{10}{9}(2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2), & 0 \le x^T Ay \le 0.10 \\ 1 + \dfrac{40}{9}(2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 - 0.30), & 0.10 \le x^T Ay \le 0.30 \\ 1, & x^T Ay \ge 0.30, \end{cases}$$

$$f_6(x^T By) = \begin{cases} 0, & x^T By \le 0 \\ \dfrac{5}{6}(x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2), & 0 \le x^T By \le 0.10 \\ 1 + \dfrac{10}{3}(x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - 0.40), & 0.10 \le x^T By \le 0.40 \\ 1, & x^T By \ge 0.40. \end{cases}$$

Again the indeterminacy resolving functions $f_3(x^T Ay)$ and $f_6(x^T By)$ are piecewise linear S-shaped linear functions with convex break points.

To convert all the aforesaid piecewise linear indeterminacy functions into convex functions by employing Inuiguichi et al. algorithm [16], we follow the procedure laid down in Section 3.1.

Step 1:–

$c_1 = 0,$	$c_2 = 0.083,$	$c_3 = 0.11,$	$c_4 = 0.125,$	$c_5 = 1.$
$v_1^1 = v_2^1 = 0.60,$	$v_1^2 = v_2^2 = 0.525,$	$v_1^3 = v_2^3 = 0.50,$	$v_1^4 = v_2^4 = 0.49,$	$v_1^5 = v_2^5 = 0.30,$
$v_3^1 = 0,$	$v_3^2 = 0.075,$	$v_3^3 = 0.10,$	$v_3^4 = 0.103,$	$v_3^5 = 0.30,$
$v_4^1 = v_5^1 0.80,$	$v_4^2 = v_5^2 = 0.70,$	$v_4^3 = v_5^3 = 0.667,$	$v_4^4 = v_5^4 = 0.650,$	$v_4^5 = v_5^5 = 0.40,$
$v_6^1 = 0,$	$v_6^2 = 0.10,$	$v_6^3 = 0.108,$	$v_6^4 = 0.113,$	$v_6^5 = 0.40.$

Step 2:– Set $\delta_1^{'} = 1$ and obtain $\delta_\theta^{'}$ for $\theta = 2, 3, 4,$

$$\delta_2' = \delta_1^{'} \times \min_{1 \le p \le 6} \left(\frac{v_p^3 - v_p^2}{v_p^2 - v_p^1} \right) = 0.08,$$

$$\delta_3' = \delta_2^{'} \times \min_{1 \le p \le 6} \left(\frac{v_p^4 - v_p^3}{v_p^3 - v_p^2} \right) = 0.0096,$$

$$\delta_4' = \delta_3^{'} \times \min_{1 \le p \le 6} \left(\frac{v_p^5 - v_p^4}{v_p^4 - v_p^3} \right) = 0.1824.$$

Step 3:– Normalizing δ_θ', $\theta = 1, 2, 3, 4$, to obtain

$$\delta_1 = 0.7861,\ \delta_2 = 0.0628,\ \delta_3 = 0.00754,\ \delta_4 = 0.1433.$$

Step 4:– For all $p = 1, \ldots, 6$, we have

$$\hat{f}_p(v_p^1) = 0,\ \hat{f}_p^{'}(v_p^2) = 0.7861,\ \hat{f}_p^{'}(v_p^3) = 0.8489,\ \hat{f}_p^{'}(v_p^4) = 0.8564,\ \hat{f}_p^{'}(v_p^5) = 1.$$

Step 5:– The membership functions are obtained as follows:

$$\hat{f}_1(A_1 y) = \begin{cases} 1, & 2y_1 - y_2 \le 0.30 \\ \min\left(-9.82(2y_1 - y_2) + 5.895,\ -3.14(2y_1 - y_2) + 2.4189,\right. & \\ \left. -0.754(2y_1 - y_2) + 1.2259\right), & 0.30 \le 2y_1 - y_2 \le 0.60 \\ 0, & 2y_1 - y_2 \ge 0.60, \end{cases}$$

$$\hat{f}_2(A_2y)=\begin{cases}1, & -y_1+y_2\leq 0.30\\ \min\left(-9.82(-y_1+y_2)+0.5895,\ -3.14(-y_1+y_2)+2.4189,\right. & \\ \left.-0.754(-y_1+y_2)+1.2259\right), & 0.30\leq -y_1+y_2\leq 0.60\\ 0, & -y_1+y_2\geq 0.60,\end{cases}$$

$$\hat{f}_3(x^TAy)=\begin{cases}0, & x^TAy\leq 0\\ \min\big(10.48(x^TAy),\ 2.512(x^TAy)+0.597, & \\ 0.727(x^TAy)+0.7814\big), & 0\leq x^TAy\leq 0.30\\ 1, & x^TAy\geq 0.30,\end{cases}$$

$$\hat{f}_4(x^TB_1)=\begin{cases}1, & x_1-x_2\leq 0.40\\ \min\left(-7.861(x_1-x_2)+6.288,\ -1.572(x_1-x_2)+1.8851,\right. & \\ \left.-0.754(x_1-x_2)+1.346,\ -0.5732(x_1-x_2)+1.228\right), & 0.40\leq x_1-x_2\leq 0.80\\ 1, & x_1-x_2\geq 0.80,\end{cases}$$

$$\hat{f}_5(x^TB_2)=\begin{cases}1, & -x_1+2x_2\leq 0.40\\ \min\left(-7.861(-x_1+2x_2)+6.288,\ -1.572(-x_1+2x_2)+1.8851,\right. & \\ \left.-0.754(-x_1+2x_2)+1.346,\ -0.5732(-x_1+2x_2)+1.228\right), & 0.40\leq -x_1+2x_2\leq 0.80\\ 0, & -x_1+2x_2\geq 0.80,\end{cases}$$

$$\hat{f}_6(x^TBy)=\begin{cases}0, & x^TBy\leq 0\\ \min\big(7.861(x^TBy),\ 1.508(x^TBy)+0.6880,\ 0.50(x^TBy)+0.794\big), & 0\leq x^TBy\leq 0.40\\ 1, & x^TBy\geq 0.40.\end{cases}$$

The equivalent crisp non-linear program to $(IFNLP)$ is as follows:

$$\begin{array}{ll}\max & \alpha\\ \text{subject to} & -19.64y_1+9.82y_2+5.895\geq\alpha\\ & -6.28y_1+3.14y_2+2.418\geq\alpha\\ & -1.508y_1+0.75y_2+1.2259\geq\alpha\\ & 9.82y_1-9.82y_2+5.895\geq\alpha\\ & 3.14y_1-3.14y_2+2.418\geq\alpha\\ & 0.754y_1-0.754y_2+1.2259\geq\alpha\\ & -7.861x_1+7.861x_2+6.288\geq\alpha\\ & -1.57x_1+1.57x_2+1.8551\geq\alpha\\ & -0.754x_1+0.754x_2+1.346\geq\alpha\\ & -0.5732x_1+0.5732x_2+1.228\geq\alpha\\ & 7.861x_1-15.72x_2+6.288\geq\alpha\\ & 1.57x_1-3.14x_2+1.8551\geq\alpha\\ & 0.754x_1-1.508x_2+1.346\geq\alpha\\ & 0.5732x_1-1.1464x_2+1.228\geq\alpha\\ & 20.96x_1y_1-10.48x_1y_2-10.48x_2y_1+10.48x_2y_2\geq\alpha\\ & 5.024x_1y_1-2.512x_1y_2-2.512x_2y_1+2.512x_2y_2+0.597\geq\alpha\\ & 1.454x_1y_1-0.727x_1y_2-0.727x_2y_1+0.727x_2y_2+0.7814\geq\alpha\\ & 7.86x_1y_1-7.86x_1y_2-7.86x_2y_1+15.72x_2y_2\geq\alpha\\ & 1.508x_1y_1-1.508x_1y_2-1.508x_2y_1+3.016x_2y_2+0.6880\geq\alpha\\ & 0.50x_1y_1-0.50x_1y_2-0.50x_2y_1+x_2y_2+0.794\geq\alpha\\ & x_1+x_2=1,\ x_1,\ x_2\geq 0\\ & y_1+y_2=1,\ y_1,\ y_2\geq 0\\ & 0\leq\alpha\leq 1.\end{array}$$

The optimal solution is $x^\star = (0.4972, 0.5028)$, $y^\star = (0.4304, 0.5696)$ and $\alpha^\star = 0.9375$. Thus, the assumed aspiration level for both players is achieved with 93.75%.

Tables 1, 2, and 3 depict optimal solutions of $(CENLP)_o$ for different combinations of $(\lambda,\ \eta)$.

Table 1: Optimal solutions for players in the optimistic approach

$\lambda = \eta$	$\alpha^\star$	$x_1^\star$	$x_2^\star$	$y_1^\star$	$y_2^\star$
0	0.60	0.5	0.5	0.44	0.56
$\frac{1}{3}$	0.9375	0.4972	0.5708	0.4304	0.5696
$\frac{1}{2}$	0.8591	0.576	0.424	0.438	0.562
$\frac{3}{4}$	0.7637	0.497	0.503	0.434	0.566
1	0.9275	0.5	0.5	0.434	0.566

Table 2: Optimal solutions for players in the optimistic approach when $\lambda = 1/3$ and η varies

η	$\alpha^\star$	$x_1^\star$	$x_2^\star$	$y_1^\star$	$y_2^\star$
$\frac{1}{4}$	0.9011	0.6397	0.3602	0.4635	0.5364
$\frac{1}{3}$	0.9375	0.4972	0.5708	0.4304	0.5696
$\frac{1}{2}$	0.9346	0.5	0.5	0.4414	0.5585
$\frac{2}{3}$	0.9666	0.495	0.505	0.452	0.548
$\frac{3}{4}$	0.8376	0.495	0.505	0.453	0.547

Table 3: Optimal solutions for players in the optimistic approach when $\eta = 1/3$ and λ varies

λ	$\alpha^\star$	$x_1^\star$	$x_2^\star$	$y_1^\star$	$y_2^\star$
$\frac{1}{4}$	0.9650	0.445	0.555	0.254	0.746
$\frac{1}{3}$	0.9375	0.4972	0.5708	0.4304	0.5696
$\frac{1}{2}$	0.9578	0.511	0.489	0.318	0.682
$\frac{2}{3}$	0.7384	0.726	0.274	0.495	0.505
$\frac{3}{4}$	0.8940	0.501	0.499	0.419	0.581

Pessimistic Approach

Consider the I-fuzzy version of the I-fuzzy bi-matrix game with $V_0 = 0.30; W_0 = 0.40$. The tolerances are set as $p_0' = 0.20,\ q_0' = 0.10,\ s_0' = 0.25,\ t_0' = 0.15$ and $p_1' = 0.20,\ q_1' = 0.10,\ s_1' = 0.30,\ t_1' = 0.10$.

Consider the membership functions and the nonmembership functions of the I-fuzzy bi-matrix game in the pessimistic approach with given tolerances, and after resolving the indeterminacy with $\lambda = \eta = \frac{1}{3}$, we get the piecewise linear indeterminacy resolving functions for both players, all with concave break points only, as follows:

$$f_1(A_1y) = \begin{cases} 1, & 2y_1 - y_2 \leq 0.30 \\ 1 + \dfrac{(0.30 - 2y_1 + y_2)}{0.30}, & 0.30 \leq 2y_1 - y_2 \leq 0.40 \\ \dfrac{4}{0.60}(0.50 - 2y_1 + y_2), & 0.40 \leq 2y_1 - y_2 \leq 0.50 \\ 0, & 2y_1 - y_2 \geq 0.50, \end{cases}$$

$$f_2(A_2y) = \begin{cases} 1, & -y_1 + y_2 \leq 0.30 \\ 1 + \dfrac{(0.30 + y_1 - y_2)}{0.30}, & 0.30 \leq -y_1 + y_2 \leq 0.40 \\ \dfrac{4}{0.60}(0.50 + y_1 - y_2), & 0.40 \leq -y_1 + y_2 \leq 0.50 \\ 0, & -y_1 + y_2 \geq 0.50, \end{cases}$$

$$f_4(B_1x) = \begin{cases} 1, & x_1 - x_2 \leq 0.40 \\ 1 + \dfrac{2}{0.75}(0.40 - x_1 + x_2), & 0.40 \leq x_1 - x_2 \leq 0.50 \\ \dfrac{11}{2.25}(0.65 - x_1 + x_2), & 0.50 \leq x_1 - x_2 \leq 0.65 \\ 0, & x_1 - x_2 \geq 0.65, \end{cases}$$

$$f_5(B_2x) = \begin{cases} 1, & -x_1 + 2x_2 \leq 0.40 \\ 1 + \dfrac{2}{0.75}(0.40 + x_1 - 2x_2), & 0.40 \leq -x_1 + 2x_2 \leq 0.50 \\ \dfrac{11}{2.25}(0.65 + x_1 - 2x_2), & 0.50 \leq -x_1 + 2x_2 \leq 0.65 \\ 0, & -x_1 + 2x_2 \geq 0.65. \end{cases}$$

On the same line, we also obtain the following:

$$f_3(x^TAy) = \begin{cases} 0, & x^TAy \leq 0.10 \\ \dfrac{4}{0.60}(2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 - 0.10), & 0.10 \leq x^TAy \leq 0.20 \\ 1 + \dfrac{1}{0.30}(2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 - 0.30), & 0.20 \leq x^TAy \leq 0.30 \\ 1, & x^TAy \geq 0.30, \end{cases}$$

$$f_6(x^TBy) = \begin{cases} 0, & x^TBy \leq 0.10 \\ \dfrac{5}{2.70}(x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - 0.10), & 0.10 \leq x^TBy \leq 0.20 \\ 1 + \dfrac{2}{0.90}(x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - 0.40), & 0.20 \leq x^TBy \leq 0.40 \\ 1, & x^TBy \geq 0.40. \end{cases}$$

The corresponding nonlinear program $(CENLP)_p$ to be solved is given as follows:

$$\begin{array}{ll}
\max & \alpha \\
\text{subject to} & -2y_1 + y_2 + 0.60 \geq 0.30\alpha \\
& -8y_1 + 4y_2 + 2 \geq 0.60\alpha \\
& y_1 - y_2 + 0.60 \geq 0.30\alpha \\
& 4y_1 - 4y_2 + 2 \geq 0.60\alpha \\
& -2x_1 + 2x_2 + 1.55 \geq 0.75\alpha \\
& -11x_1 + 11x_2 + 7.15 \geq 2.25\alpha \\
& 2x_1 - 4x_2 + 1.55 \geq 0.75\alpha \\
& 11x_1 - 22x_2 + 7.15 \geq 2.25\alpha \\
& 8x_1y_1 - 4x_1y_2 - 4x_2y_1 + 4x_2y_2 - 0.40 \geq 0.60\alpha \\
& 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 \geq 0.30\alpha \\
& 5x_1y_1 - 5x_1y_2 - 5x_2y_1 + 10x_2y_2 - 0.50 \geq 2.70\alpha \\
& 2x_1y_1 - 2x_1y_2 - 2x_2y_1 + 4x_2y_2 + 0.10 \geq 0.90\alpha \\
& x_1 + x_2 = 1 \\
& y_1 + y_2 = 1 \\
& \alpha \leq 1.
\end{array}$$

The optimal solution is $x^\star = (0.4825, 0.5174)$, $y^\star = (0.3312, 0.6687)$ and $\alpha^\star = 0.4774$. Thus, the assumed aspiration level for both players is achieved with 47.74%. Table 4 depicts optimal solutions for Player I and Player II for different combinations of $\lambda = \eta$.

Table 4: Optimal solutions for players in the pessimistic approach

$\lambda = \eta$	$\alpha^\star$	$x_1^\star$	$x_2^\star$	$y_1^\star$	$y_2^\star$
$\frac{1}{4}$	0.3125	0.4723	0.5276	0.40	0.60
$\frac{1}{3}$	0.4773	0.4825	0.5174	0.3312	0.6687
$\frac{1}{2}$	0.7999	0.5000	0.5000	0.44	0.56
$\frac{2}{3}$	0.4453	0.4756	0.5243	0.2769	0.7230
$\frac{3}{4}$	0.7918	0.4939	0.5060	0.3797	0.6202

We can easily work out the optimal solutions (equilibrium) for both players for different other values of λ and η in the pessimistic situation.

Both nonlinear problems $(CENLP)_o$ and $(CENLP)_p$ are solved on LINDO solver on a Window 64 bits platform.

6 Concluding Remarks

In this paper, we studied the I-fuzzy bi-matrix games with I-fuzzy goals. The earlier study on fuzzy bi-matrix games by Vidyottama et al. [25] falls as a special case of our study. Also, we have improvised the study of Nayak and Pal [20] by providing a true I-fuzzy framework to the I-fuzzy bi-matrix games. We presented two models to compute equilibrium solution of the I-fuzzy bi-matrix game when both players either adopt the optimistic approach or the pessimistic approach in describing the nonmembership functions for their respective goals. Their attitude in resolving the indeterminacy in their respective I-fuzzy goals are captured by parameters λ (for Player I) and η (for Player II) varying independently in the interval $[0, 1]$. The advantage with the proposed scheme, especially in the optimistic framework, is that the resultant optimization problems does not involve any binary/integer variables, which is often the case with S-shaped piecewise linear membership functions. In both the approaches, the equivalent crisp optimization problems are nonlinear programming problems which can easily be solved on any of the available solvers. However, beside the two parameters λ and η for the two players' indeterminacy resolving interests, these nonlinear programming problems involve certain tolerance parameters, consequently the equilibrium solution of the game is sensitive to the changes in these parameters.

The other two natural optimization models could be when one player possesses an optimistic approach in describing the nonmembership function for the I-fuzzy goal while the other player has a pessimistic approach for doing the same. These optimization models can easily be worked out on similar lines. Moreover, in recent literature, one found some studies on fuzzy bi-matrix games involving different types of payoffs. For example, Gao [12] used uncertain variables and uncertainty theory to study uncertain bi-matrix games, Roy and Mula [22] used bifuzzy numbers to investigate bifuzzy bi-matrix games, while Seikh et al. [23] used triangular intuitionistic fuzzy numbers in bi-matrix games. The present study can easily be extended in future to include different types of uncertainties in the pay-offs besides the fuzzy goals.

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References

[1] Aggarwal, A., Dubey, D., Mehra, A., and S. Chandra, Application of Atanassov's I-fuzzy set theory to matrix games with fuzzy goals and fuzzy payoffs, *Fuzzy Information and Engineering*, vol.4, no.4, pp.401–414, 2012.

[2] Aggarwal, A., Mehra, A., and S. Chandra, Application of linear programming with I-fuzzy sets to matrix games with I-fuzzy goals, *Fuzzy Optimization and Decision Making*, vol.11, no.4, pp.465–480, 2012.

[3] Angelov, P.P., Optimization in an intuitionistic fuzzy environment, *Fuzzy Sets and Systems*, vol.86, pp.299–306, 1997.

[4] Atanassov, K.T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol.20, pp.87–96, 1986.

[5] Atanassov, K.T., New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol.61, pp.137–142, 1986.

[6] Atanassov, K.T., More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol.33, pp.37–45, 1989.

[7] Atanassov, K.T., *Intuitionistic Fuzzy Sets*, Springer, Heidelberg, 1999.

[8] Bector, C.R., and S. Chandra, *Fuzzy Mathematical Programming and Fuzzy Matrix Games*, Springer-Verlag, Berlin, Heidelberg, 2005.

[9] Bellman, R.E., and L.A. Zadeh, Decision making in fuzzy environment, *Management Sciences*, vol.17, pp.141–164, 1970.

[10] Dubey, D., Chandra, S., and A. Mehra, Fuzzy linear programming under interval uncertainity based on IFS representation, *Fuzzy Sets and Systems*, vol.188, pp.68–87, 2012.

[11] Dubois, D., Gottwald, S., Hajek, P., Kacprzyk, J., and H. Prade, Terminological difficulties in fuzzy set theory the case of intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol.156, pp.485–491, 2005.

[12] Gao, J., Uncertain bi-matrix game with applications, *Fuzzy Optimization and Decision Making*, vol.12, no.1, pp.65–78, 2013.

[13] Grzegorzewski, P., and E. MrÓwka, Some notes on Atanassov's multiobjective fuzzy sets, *Fuzzy Sets and Systems*, vol.156, pp.492–495, 2005.

[14] Hannan, E.L., Linear programming with multiple fuzzy goals, *Fuzzy Sets and Systems*, vol.6, pp.235–246, 1981.

[15] Hurwicz, L., A criterion for decision-making under uncertainity, Technical report, Cowels Commission, 1952.

[16] Inuiguchi, M., Ichihashi, H., and Y. Kume, A solution algorithm for fuzzy linear programming with piecewise linear membership functions, *Fuzzy Sets and Systems*, vol.34, pp.15–31, 1990.

[17] Maeda, T., On characterization of equilibirum strategy of the bimatrix game with fuzzy payoff, *Journal of Mathematical Analysis and Applications*, vol.251, pp.885–896, 2000.

[18] Mangasarian, O.L., and H. Stone, Two person non zero sum game and quadratic programming, *Journal of Mathematical Analysis and Applications*, vol.9, pp.348–355, 1964.

[19] Nakamura, K., Some extensions of fuzzy linear programming, *Fuzzy Sets and Systems*, vol.14, pp.211–229, 1984.

[20] Nayak, P.K., and M. Pal, Bimatrix games with intiutionstic fuzzy goals, *Iraninan Journal of Fuzzy Systems*, vol.7, no.1, pp.65–79, 2010.

[21] Nishizaki, I., and M. Sakawa, *Fuzzy and Multiobjective Games for Conflict Resolution*, Physica-Verleg, Heidelberg, 2001.

[22] Roy, S.K., and P. Mula, Bi-matrix game in bi-fuzzy environment, *Journal of Uncertainty Analysis and Applications*, vol.1, no.11, 2013, doi:10.1186/2195-5468-1-11.

[23] Seikh, M.R., Nayak, P.K., and M. Pal, Solving bi-matrix games with pay-offs of triangular intuitionistic fuzzy numbers, *European Journal of Pure and Applied Mathematics*, vol.8, no.2, pp.153–171, 2015.

[24] Szmidt, E., and J. Kacprzyk, Remarks on some application of intuitionistic fuzzy sets in decision making, *Notes on IFS*, vol.2, pp.2–31, 1996.

[25] Vidyottama, V., Chandra, S., and C.R. Bector, Bi-matrix game with fuzzy goals and fuzzy payoffs, *Fuzzy Optimization and Decision Making*, vol.3, pp.327–344, 2004.

[26] Yager, R., Some aspects of intuitionistic fuzzy sets, *Fuzzy Optimization and Decision Making*, vol.8, pp.67–90, 2009.

[27] Yang, T., Ignizio, J.P., and H.J. Kim, Fuzzy programming with nonlinear membership functions: piecewise linear approximation, *Fuzzy Sets and Systems*, vol.41, pp.39–53, 1991.