Journal of Uncertain Systems
Vol.10, No.3, pp.163-180, 2016
Online at: www.jus.org.uk

Fuzzy Hukuhara Delta Differential and Applications to Fuzzy Dynamic Equations on Time Scales

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Received 4 August 2014; Revised 22 May 2016

Abstract

Using the concept of Hukuhara difference, in this paper we introduce a class of new derivatives called Hukuhara delta derivative and a class of new integrals called Hukuhara delta integral for fuzzy set-valued functions on time scales. Moreover, some corresponding properties of Hukuhara delta derivative and Hukuhara delta integral are discussed. Furthermore, sufficient conditions are established for the existence and uniqueness of solution to the fuzzy dynamic equations on time scales with the help of Banach contraction principle.

Keywords: fuzzy differential equations, Δ_H-differential, Δ_H-integral, time scales

1 Introduction

1.1 Dynamic Equations on Time Scales

The theory of time scales was introduced by Stefan Hilger [9, 10], which attracted the attention of many researchers due to the ability of this theory to model many real world problems as the dynamical systems include both continuous and discrete nature simultaneously. For fundamental results in the theory of time scales, we refer Agarwal et al. [1, 2], Bohner et al. [4, 5, 3], Guseinov [7], Lakshmikantham et al. [15]. Recently, Hashmi et al. [8] studied the existence and uniqueness of the solution for the dynamic systems on time scales with uncertain parameters.

1.2 Fuzzy Set Theory and Its Applications

The rise and development of new fields such as general system theory, robotics, artificial intelligence and language theory, force us to be engaged in specifying imprecise notions. When a real world problem is transformed into a deterministic model, we cannot ususally be sure that the model is perfect. Using fuzzy set theory, we can model the meaning of vague notions and also certain kinds of human reasoning. Fuzzy set theory and its applications have been developed by Kaleva [12, 13], Lakshmikantham et al. [14], Liu et al. [16], Murty et al. [18, 19, 20, 21, 22], Puri et al. [23, 24], You [27].

1.3 Motivation

The differential and integral calculus for multivalued functions on time scales using Hukuhara difference was introduced and developed by Hong [11]. Later, Lupulescu [17] studied the differentiability and integrability for the interval-valued functions on time scales using generalized Hukuhara difference (gH-difference). Further, Fard et al. [6] studied the calculus of fuzzy-number-valued functions on time scales using gH-difference. Furthermore, Vasavi et al. [25] studied the fundamental properties of fuzzy set-valued functions on time scales using generalized delta derivative (Δ_g-derivative) with Hukuhara difference. Recently, Vasavi et al. [26] studied the properties of second type Hukuhara delta derivative (Δ_{SH}-derivative) for fuzzy set-valued functions

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on time scales and established the existence and uniqueness of solutions for the fuzzy dynamic equations on time scales. The main aim of this paper is to fill the gap in the literature of fuzzy time scale calculus [6, 25, 26]. In this context, we introduce fuzzy Hukuhara delta derivative and integral for fuzzy set-valued functions on time scales using Hukuhara difference.

1.4 Structure of the Paper

The paper is organized as follows. In Section 2, some basic definitions and results related to fuzzy calculus as well as time scale calculus are presented. In Section 3, we introduce and study the properties of new class of derivatives called Hukuhara delta derivative (Δ_H-derivative). In Section 4, we introduce Hukuhara delta integral (Δ_H-integral) and studied some properties. In Section 5, we establish sufficient conditions for the existence and uniqueness of solutions for the fuzzy dynamic equations on time scales.

2 Preliminaries

Firstly, we recall some results related to fuzzy calculus. Let $\mathbb{P}_k(\mathbb{R}^n)$ denotes the family of all nonempty compact convex subsets of $\mathbb{R}^n$. Define the addition and scalar multiplication in $\mathbb{P}_k(\mathbb{R}^n)$ as usual. Then $\mathbb{P}_k(\mathbb{R}^n)$ is a commutative semigroup [13] under addition, which satisfies the cancellation law. Moreover, if $\alpha, \beta \in \mathbb{R}$ and $A, B \in \mathbb{P}_k(\mathbb{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1.A = A,$$

and if $\alpha, \beta \geq 0$ then $(\alpha+\beta)A = \alpha A + \beta A$. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

where $||.||$ denotes the Euclidean norm in $\mathbb{R}^n$ (or) equivalently, we define the Hausdorff distance between A and B as

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$, $d(b,A) = \inf_{a\in A}\|a-b\|$. Then $(P_K(\mathbb{R}^n), d_H)$ becomes a complete and seperable metric space [13]. Define

$$\mathbb{E}^n = \{u : \mathbb{R}^n \to [0,1]/u \ \text{ satisfies(i)-(iv)below}\},$$

where

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) the closure of $\{x \in \mathbb{R}^n/u(x) > 0\}$, denoted by $[u]^0$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R}^n/u(x) \geq \alpha\}$, then from (i)-(iv) it follows that the α-level set $[u]^\alpha \in \mathbb{P}_K(\mathbb{R}^n)$ for all $0 < \alpha \leq 1$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function then according to Zadeh's extension principle we can extend $g : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^n$ by

$$g(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[g(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$, for all $u, v \in \mathbb{E}^n$, $0 < \alpha \leq 1$ and g is continuous. For addition and scalar multiplication, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, [ku]^\alpha = k[u]^\alpha, \text{ where } u, v \in \mathbb{E}^n, \ k \in \mathbb{R}, \ 0 \leq \alpha \leq 1.$$

Theorem 2.1. ([13]) *If $u \in \mathbb{E}^n$, then*

(i) $[u]^\alpha \in \mathbb{P}_k(R^n)$ *for all* $0 \leq \alpha \leq 1$,

(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,

(iii) If $\{\alpha_k\} \subset [0,1]$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then*

$$[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}.$$

Conversely, if $\{A^\alpha / 0 \leq \alpha \leq 1\}$ *is a family of subsets of* $\mathbb{R}^n$ *satisfying(i)-(iii),then there exists* $u \in \mathbb{E}^n$ *such that*

$$[u]^\alpha = A^\alpha, \quad for \quad 0 < \alpha \leq 1, \ and$$

$$[u]^0 = cl\left\{ \bigcup_{0<\alpha\leq 1} A^\alpha \right\} \subset A^0, \ where\ cl\ denotes\ the\ closure\ of\ the\ set.$$

Theorem 2.2. ([13]) *Let* $\{A_k\}$ *be a sequence in* $\mathbb{P}_K(\mathbb{R}^n)$ *converging to* A *and* $d(A_k, A) \to 0$ *as* $k \to \infty$ *then*

$$A = \bigcap_{k \geq 1} cl \left\{ \bigcup_{m \geq k} A_m \right\}.$$

Define $D : \mathbb{E}^n \times \mathbb{E}^n \to [0, \infty)$ by

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha),$$

where d_H is the Hausdorff metric defined in $\mathbb{P}_K(\mathbb{R}^n)$. Then $(\mathbb{E}^n, D)$ is a complete metric space [13].

Let $x, y \in \mathbb{E}^n$. If there exists a $z \in \mathbb{E}^n$ such that $x = y + z$ then we call z the H- difference of x and y denoted by $x \ominus_H y$. For any $A, B, C, D \in \mathbb{E}^n$ and $\lambda \in \mathbb{R}$,

(i) $D(A, B) = 0 \Leftrightarrow A = B$,

(ii) $D(\lambda A, \lambda B) = |\lambda| D(A, B)$,

(iii) $D(A \ominus_H C, B \ominus_H C) = D(A, B)$,

(iv) $D(A \ominus_H B, C \ominus_H D) \leq D(A, C) + D(B, D)$.

Here we assume that the Hukuhara differences appearing in the above formulae exist.

Definition 2.1. ([13]) *A mapping* $F : T \to \mathbb{E}^n$ *is Hukuhara differentiable at* $t_0 \in T$ *if there exists a* $F'(t_0) \in \mathbb{E}^n$ *such that the limits*

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus_H F(t_0)}{h}, \ \lim_{h \to 0^+} \frac{F(t_0) \ominus_H F(t_0 - h)}{h}$$

exist in the topology of $\mathbb{E}^n$ *and equal to* $F'(t_0)$. *Here the limit is taken in the metric space* $(\mathbb{E}^n, D)$. *At the end points of* T *we consider only the one-sided derivatives.*

Remark 2.1. ([13]) *If* F *is differentiable then the multivalued mapping* F_α *is Hukuhara differentiable for all* $\alpha \in [0,1]$ *and*

$$[F_\alpha(t)]' = [F'(t)]^\alpha.$$

Here $[F_\alpha]'$ *denotes the Hukuhara derivative of* F_α.

Definition 2.2. ([13]) *A mapping* $F : T \to \mathbb{E}^n$ *is strongly measurable if for all* $\alpha \in [0,1]$ *the set-valued mapping* $F_\alpha : T \to P_k(\mathbb{R}^n)$ *defined by* $F_\alpha(t) = [F(t)]^\alpha$ *is (Lebesgue) measurable, when* $P_k(\mathbb{R}^n)$ *is endowed with the topology generated by the Hausdorff metric* d_H.

Remark 2.2. ([13]) *If $F : T \to \mathbb{E}^n$ is continuous, then it is measurable with respect to the metric D and F is said to be integrably bounded if there exists an integrable function h such that $\|x\| \leq h(t)$ for all $x \in F_0(t)$.*

Definition 2.3. ([13]) *let $F : T \to \mathbb{E}^n$. The integral of F over T, denoted $\int_T F(t)dt$ or $\int_a^b F(t)dt$, is defined levelwise by the equation*

$$\left[\int_T F(t)dt\right] = \int_T F_\alpha(t)dt = \left\{\int_T f(t)dt / f : T \to \mathbb{R}^n\right\}$$

where f is a measurable selection for F_α, for all $0 < \alpha \leq 1$.

A strongly measurable and integrably bounded mapping $F : T \to \mathbb{E}^n$ is said to be integrable over T if $\int_T F(t)dt \in \mathbb{E}^n$.

Remark 2.3. ([13]) *If $\{\alpha_k\}$ is a nonincresing sequence converging to zero for all $u \in \mathbb{E}^n$, then*

$$\lim_{k\to\infty} d_H([u]^0, [u]^{\alpha_k}) = 0.$$

Remark 2.4. ([14]) *If $F : T \to \mathbb{E}^n$ is continuous, then it is integrable.*

Theorem 2.3. ([14]) *Let $F, G : T \to \mathbb{E}^n$ be integrable. Then*

(i) $\int F + G = \int F + \int G$,

(ii) $\int \lambda F = \lambda \int F$, *where* $\lambda \in \mathbb{R}$,

(iii) $\int_a^b F = \int_a^c F + \int_c^b F$, *where* $c \in \mathbb{R}$,

(iv) $D(F, G)$ *is integrable,*

(v) $D(\int F, \int G) \leq \int D(F, G)$.

Now, we present some basic notations, definitions and results of time scales. Let $\mathbb{T}$ be a time scale, i.e. an arbitrary nonempty closed subset of real numbers. Since a time scale $\mathbb{T}$ is not connected, we need the concept of jump operators.

Definition 2.4. ([4]) *The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\},\ \mu(t) = \sigma(t) - t \text{ for } t \in \mathbb{T},$$

respectively. If $\sigma(t) = t$, t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered). If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ defined by

$$f^\sigma(t) = f(\sigma(t)), \text{ for all } t \in \mathbb{T}.$$

Definition 2.5. ([4]) *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^k$. Then $f^\Delta(t)$ be the number(provided it exists) with the property that given any $\epsilon > 0$, there is a neighbourhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

In this case, $f^\Delta(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta (or Hilger) differentiable on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^k$.

Definition 2.6. ([4]) *A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist(finite) at all right dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at all left-dense points in $\mathbb{T}$ and F is said to be rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exists(finite) at all left-dense points in $\mathbb{T}$.*

The set of all rd-continuous functions $F : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T})$. The set of functions $F : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivatives are rd-continuous is denoted by $C^1_{rd}(\mathbb{T})$.

Lemma 2.1. ([4]) *Assume $F : \mathbb{T} \to \mathbb{R}$.*

(i) If F is continuous, then F is rd-continuous.

(ii) If F is rd-continuous, then F is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If F is regulated or rd-continuous, then so is F^σ.

Definition 2.7. ([4]) *A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with region of differentiation D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k/D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and F is differentiable at each $t \in D$. If F is regulated, and $f^\Delta(t) = F(t)$ for all $t \in D$, then the indefinite integral of a regulated function F is*

$$\int F(t)\Delta(t) = f(t) + C,$$

where C is an arbitrary constant and f is a pre-antiderivative of F. The cauchy integral is given by

$$\int_r^s F(t)\Delta(t) = f(s) - f(r), \quad \text{for all } r, s \in \mathbb{T}.$$

Lemma 2.2. ([4]) *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $F, G \in C_{rd}$, then*

(i) $\int_a^b [F(t) + G(t)]\Delta t = \int_a^b F(t)\Delta t + \int_a^b G(t)\Delta t,$

(ii) $\int_a^b (\alpha F(t))\Delta t = \alpha \int_a^b F(t)\Delta t,$

(iii) $\int_a^b F(t)\Delta t = -\int_b^a F(t)\Delta t,$

(iv) $\int_a^b F(t)\Delta t = \int_a^c F(t)\Delta t + \int_c^b F(t)\Delta t,$

(v) $\int_a^a F(t)\Delta t = 0.$

3 Differentiation of Fuzzy Set-valued Functions on Time Scales

In this section we define and study the properties of Hukuhara delta derivative (Δ_H-derivative) for fuzzy set-valued functions on time scales. To facilitate the discussion below, we introduce some notation: For $t \in \mathbb{T}$, the neighbourhood t of $\mathbb{T}$ is denoted by $U_\mathbb{T} = U_\mathbb{T}(t, \delta)$ (i.e. $U_\mathbb{T}(t, \delta) = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$). In the present section we work in $(\mathbb{E}^n, D)$.

Definition 3.1. *A fuzzy set-valued function $F : \mathbb{T} \to \mathbb{E}^n$ has a $\mathbb{T}$-limit $A \in \mathbb{E}^n$ at $t_0 \in \mathbb{T}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $D(F(t) \ominus_H A, \{0\}) \leq \epsilon$ for all $t \in U_\mathbb{T}(t_0, \delta)$. If F has a $\mathbb{T}$-limit $A \in \mathbb{E}^n$ at $t_0 \in \mathbb{T}$, then it is unique and is denoted by $\mathbb{T} - \lim_{t \to t_0} F(t)$.*

Remark 3.1. From the above definition we have,

$$\mathbb{T} - \lim_{t \to t_0} F(t) = A \in \mathbb{E}^n \Longleftrightarrow \mathbb{T} - \lim_{t \to t_0} (F(t) \ominus_H A) = \{0\},$$

where the limits are in the metric D.

Definition 3.2. *A fuzzy set-valued mapping $F : \mathbb{T} \to \mathbb{E}^n$ is continuous at $t_0 \in \mathbb{T}$, if $\mathbb{T} - \lim_{t \to t_0} F(t) \in \mathbb{E}^n$ exists and $\mathbb{T} - \lim_{t \to t_0} F(t) = F(t_0)$, i.e.*

$$\mathbb{T} - \lim_{t \to t_0} (F(t) \ominus_H F(t_0)) = \{0\}.$$

Remark 3.2. It follows from the above definition that if $F : \mathbb{T} \to \mathbb{E}^n$ is continuous at $t_0 \in \mathbb{T}$, then for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$D(F(t) \ominus_H F(t_0), \{0\}) \leq \epsilon, \text{ for all } t \in U_{\mathbb{T}}.$$

Definition 3.3. *Assume $F : \mathbb{T} \to \mathbb{E}^n$ is a fuzzy set-valued function and $t \in \mathbb{T}^k$. Let $\Delta_H F(t)$ be an element of $\mathbb{E}^n$ (provided it exists) with the property that given any $\epsilon > 0$, there is a neighbourhood $U_{\mathbb{T}}$ of t for some $\delta > 0$ such that*

$$\begin{aligned} &D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \epsilon(h - \mu(t)), \\ &D[(F(\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon(h + \mu(t)) \end{aligned} \tag{1}$$

for all $t - h, t + h \in U_{\mathbb{T}}$ with $0 < h < \delta$ where $\mu(t) = \sigma(t) - t$. We call $\Delta_H F(t)$, the Δ_H-derivative of F at t. We say that F is Δ_H-differentiable at t if its Δ_H-derivative exists at t. Moreover, we say that F is Δ_H-differentiable on $\mathbb{T}^k$ if its Δ_H-derivative exists at each $t \in \mathbb{T}^k$. The fuzzy set-valued function $\Delta_H F : \mathbb{T}^k \to \mathbb{E}^n$ is called the Δ_H-derivative of F on $\mathbb{T}^k$ or equivalently,

$$\lim_{h \to 0^+} \frac{F(t+h) \ominus_H F(\sigma(t))}{h - \mu(t)}, \lim_{h \to 0^+} \frac{F(\sigma(t)) \ominus_H F(t-h)}{h + \mu(t)}$$

exists and equal to $\Delta_H F(t)$.

Lemma 3.1. *If the Δ_H-derivative of F at t exists, then it is unique.*

Proof. Let $1_{\Delta_H} F(t)$ and $2_{\Delta_H} F(t)$ be the Δ_H-derivatives of F at t. Then

$$D[1_{\Delta_H} F(t)(h - \mu(t)), F(t+h) \ominus_H F(\sigma(t))] \leq \epsilon(h - \mu(t)),$$

$$D[2_{\Delta_H} F(t)(h - \mu(t)), F(t+h) \ominus_H F(\sigma(t))] \leq \epsilon(h - \mu(t)).$$

Consider

$$\begin{aligned} D[1_{\Delta_H} F(t), 2_{\Delta_H} F(t)] &= \frac{1}{h - \mu(t)} \left[D[1_{\Delta_H} F(t)(h - \mu(t)), 2_{\Delta_H} F(t)(h - \mu(t)]\right] \\ &\leq \frac{1}{h - \mu(t)} \left[D[1_{\Delta_H} F(t)(h - \mu(t)), F(t+h) \ominus_H F(\sigma(t))]\right. \\ &\quad \left. + D[F(t+h) \ominus_H F(\sigma(t)), 2_{\Delta_H} F(t)(h - \mu(t)]\right] \\ &\leq \epsilon + \epsilon = 2\epsilon \end{aligned}$$

for all $|h - \mu(t)| \neq 0$. Since $\epsilon > 0$ is arbitrary, then $D[1_{\Delta_H} F(t), 2_{\Delta_H} F(t)] = 0$. This implies that $1_{\Delta_H} F(t) = 2_{\Delta_H} F(t)$. Hence Δ_H-derivative is well defined. □

Theorem 3.1. *Let $F : \mathbb{T} \to \mathbb{E}^n$ be a fuzzy set-valued function and $t \in \mathbb{T}^k$. Then we have the following:*

(i) If F is Δ_H-differentiable at t, then F is continuous at t.

(ii) If F is continuous at t and t is right-scattered, then F is Δ_H-differentiable at t with

$$\Delta_H F(t) = \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}.$$

(iii) If t is right-dense, then F is Δ_H-differentiable at t iff the limits

$$\lim_{h \to 0^+} \frac{F(t+h) \ominus_H F(t)}{h}, \quad \lim_{h \to 0^+} \frac{F(t) \ominus_H F(t-h)}{h}$$

exists as a finite number and satisfy the equations

$$\lim_{h \to 0^+} \frac{F(t+h) \ominus_H F(t)}{h} = \lim_{h \to 0^+} \frac{F(t) \ominus_H F(t-h)}{h} = \Delta_H F(t).$$

(iv) If F is Δ_H-differentiable at t, then

$$F(\sigma(t)) = F(t) + \mu(t)\Delta_H F(t).$$

Proof. (i) Assume F is Δ_H-differentiable at t. Let $\epsilon \in (0,1)$. Define

$$\epsilon_1 = \epsilon[1 + \|\Delta_H F(t)\|]^{-1}.$$

Clearly, $\epsilon_1 \in (0,1)$. Let θ be the zero element of $\mathbb{E}^n$. By the definition of D,

$$D[u, \theta] = \|u\|.$$

For any $u, v \in \mathbb{E}^n$, $\quad D[u,v] \leq \|u - v\|$.

Since F is Δ_H-differentiable there exists a neighbourhood $U_{\mathbb{T}}$ of t such that

$$\begin{aligned} &D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \epsilon_1(h - \mu(t)), \\ &D[(F((\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon_1(h + \mu(t)) \end{aligned}$$

for all $0 < h < \delta$ with $t - h, t + h \in U_{\mathbb{T}}$. Therefore, for all $t-h, t+h \in U_{\mathbb{T}} \cap (t - \epsilon_1, t + \epsilon_1)$ and $0 < h < \epsilon_1$, we have

$$\begin{aligned} D[F(t+h), F(t)] &= D[F(t+h) \ominus_H F(\sigma(t)), F(t) \ominus_H F(\sigma(t))] \\ &\leq D[F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \\ &\qquad + D[\Delta_H F(t)(h - \mu(t)), \theta] + D[\theta, \Delta_H F(t)\mu(t)] \\ &\qquad + D[F(t) \ominus_H F(\sigma(t)), \Delta_H F(t)\mu(t)] \\ &= D[F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \\ &\qquad + D[F(t) \ominus_H F(\sigma(t)), \Delta_H F(t)\mu(t)] \\ &\qquad + (h - \mu(t))D[\Delta_H F(t), \theta] + \mu(t) D[\Delta_H F(t), \theta] \\ &= D[F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \\ &\qquad + D[F(t) \ominus_H F(\sigma(t)), \Delta_H F(t)\mu(t)] + hD[\Delta_H F(t), \theta] \\ &\leq \epsilon_1(h - \mu(t)) + \epsilon_1\mu(t) + hD[\Delta_H F(t), \theta] \\ &< \epsilon_1(1 + ||\Delta_H F(t))\|) = \epsilon. \end{aligned}$$

Similarly, $D[F(t), F(t-h)] \leq \epsilon$. It follows that F is continuous at t.

(ii) Assume that F is continuous at t and t is right-scattered. By the continuity of F, we have

$$\lim_{h \to 0^+} \frac{F(t+h) \ominus_H F(\sigma(t))}{h - \mu(t)} = \frac{F(t) \ominus_H F(\sigma(t))}{-\mu(t)},$$

$$\lim_{h \to 0^+} \frac{F(\sigma(t)) \ominus_H F(t-h)}{h + \mu(t)} = \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}.$$

Moreover,

$$D\left[\frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}, \frac{F(t) \ominus_H F(\sigma(t))}{-\mu(t)}\right] = 0.$$

Hence given $\epsilon > 0$, there exists a neighbourhood $U_{\mathbb{T}}$ of t such that

$$D\left[\frac{F(t+h) \ominus_H F(\sigma(t))}{h - \mu(t)}, \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}\right] \leq \epsilon,$$

$$D\left[\frac{F(\sigma(t)) \ominus_H F(t-h)}{h + \mu(t)}, \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}\right] \leq \epsilon$$

for all $0 < h < \delta$ with $t-h, t+h \in U_{\mathbb{T}}$. It follows that

$$D\left[F(t+h) \ominus_H F(\sigma(t)), \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}(h - \mu(t))\right] \leq \epsilon(h - \mu(t)),$$

$$D\left[F(\sigma(t)) \ominus_H F(t-h), \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}(h+\mu(t))\right] \leq \epsilon(h+\mu(t))$$

for all $0 < h < \delta$ with $t-h, t+h \in U_{\mathbb{T}}$. Hence

$$\Delta_H F(t) = \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}.$$

(iii) Assume that F is Δ_H-differentiable at t and t is right-dense. Since F is Δ_H-differentiable at t, for any given $\epsilon > 0$, there exists a neighbourhood $U_{\mathbb{T}}$ of t such that

$$D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h-\mu(t))] \leq \epsilon(h-\mu(t)),$$

$$D[(F((\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h+\mu(t))] \leq \epsilon(h+\mu(t))$$

for all $0 < h < \delta$ with $t-h, t+h \in U_{\mathbb{T}}$. Since $\sigma(t) = t$, i.e. $\mu(t) = 0$, we have

$$D[(F(t+h) \ominus_H F(t), h\Delta_H F(t)] \leq \epsilon h,$$

$$D[(F(t) \ominus_H F(t-h), h\Delta_H F(t)] \leq \epsilon h$$

for all $0 < h < \delta$ with $t-h, t+h \in U_{\mathbb{T}}$. This yields

$$D\left[\frac{F(t+h) \ominus_H F(t)}{h}, \Delta_H F(t)\right] \leq \epsilon,$$

$$D\left[\frac{F(t) \ominus_H F(t-h)}{h}, \Delta_H F(t)\right] \leq \epsilon$$

for all $0 < h < \delta$ with $t-h, t+h \in U_{\mathbb{T}}$. As ϵ is arbitrary, we have

$$\Delta_H F(t) = \lim_{h\to 0} \frac{F(t+h) \ominus_H F(t)}{h} = \lim_{h\to 0} \frac{F(t) \ominus_H F(t-h)}{h}.$$

Conversely, suppose that t is right-dense. For all $0 < h < \delta$ with $t-h, t+h \in U_{\mathbb{T}}$, there is a neighbourhood $U_{\mathbb{T}}$ of t such that

$$D\left[\frac{F(t+h) \ominus_H F(t)}{h}, \Delta_H F(t)\right] \leq \epsilon,$$

$$D\left[\frac{F(t) \ominus_H F(t-h)}{h}, \Delta_H F(t)\right] \leq \epsilon.$$

From these inequalities, F is Δ_H-differentiable at t.

(iv) If $\sigma(t) = t$, then $\mu(t) = 0$ and we have that

$$F(\sigma(t)) = F(t) = F(t) + \mu(t)\Delta_H F(t).$$

On the other hand if $\sigma(t) > t$, then by (ii)

$$F(\sigma(t)) = F(t) + \mu(t)\frac{F(\sigma(t) \ominus_H F(t))}{\mu(t)} = F(t) + \mu(t)\Delta_H F(t).$$

□

Example 3.1 We consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then from Theorem 3.1 (iii) $F : \mathbb{R} \to \mathbb{E}^n$ is Δ_H-differentiable at $t \in \mathbb{R}$ iff

$$\Delta_H F(t) = \lim_{h\to 0} \frac{F(t+h) \ominus_H F(t)}{h} = \lim_{h\to 0} \frac{F(t) \ominus_H F(t-h)}{h} = F^{'}(t).$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then from Theorem 3.1 (ii) $F : \mathbb{Z} \to \mathbb{E}^n$ is Δ_H-differentiable at $t \in \mathbb{Z}$ and

$$\Delta_H F(t) = \frac{F(\sigma(t) \ominus_H F(t))}{\mu(t)} = F(t+1) \ominus_H F(t) = \Delta F(t),$$

where Δ is the forward Hukuhara difference operator.

Theorem 3.2. *Let $F : \mathbb{T} \to \mathbb{E}^n$ be the fuzzy set-valued function and denote $[F(t)]^\alpha = F_\alpha(t)$, for each $\alpha \in [0,1]$. If F is Δ_H-differentiable, then F_α is also Δ_H-differentiable and*

$$\Delta_H [F(t)]^\alpha = \Delta_H F_\alpha(t).$$

Proof. If F is Δ_H-differentiable at $t \in \mathbb{T}^k$ and t is right-scaterred, then for any $\alpha \in [0,1]$, we get

$$[F(t+h) \ominus_H F(\sigma(t))]^\alpha = [F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))],$$

dividing by $h - \mu(t) > 0$ and taking the limit $h \to 0^+$, we have

$$\lim_{h \to 0^+} \frac{1}{h - \mu(t)} [F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))] = \Delta_H F_\alpha(t).$$

Similarly,

$$\lim_{h \to 0^+} \frac{1}{h + \mu(t)} \big[F_\alpha(\sigma(t)) \ominus_H F_\alpha(t-h)\big] = \Delta_H F_\alpha(t).$$

If F is Δ_H-differentiable at $t \in \mathbb{T}^k$ and t is right-dense, then for $\alpha \in [0,1]$, we get

$$[F(t+h) \ominus_H F(t)]^\alpha = [F_\alpha(t+h) \ominus_H F_\alpha(t)],$$

dividing by $h > 0$ and taking the limit $h \to 0^+$, we have

$$\lim_{h \to 0^+} \frac{1}{h} [F_\alpha(t+h) \ominus_H F_\alpha(t)] = \Delta_H F_\alpha(t).$$

Similarly, $\lim\limits_{h \to 0^+} \big[F_\alpha(t) \ominus_H F_\alpha(t-h)\big]/h = \Delta_H F_\alpha(t)$. □

Remark 3.1. *From the above Theorem 3.2, it directly follows that if F is Δ_H-differentiable then the multivalued mapping F_α is Δ_H-differentiable for all $\alpha \in [0,1]$, but the converse result doesn't hold. Since the existence of H-differences $[x]^\alpha \ominus_H [y]^\alpha$, $\alpha \in [0,1]$, does not imply the existence of H-difference $x \ominus_H y$.*

However, for the converse result we have the following:

Theorem 3.3. *Let $F : \mathbb{T} \to \mathbb{E}^n$ satisfy the assumptions:*

(i) for each $t \in \mathbb{T}$ there exists a $\beta > 0$ such that the H-differences $F(t+h) \ominus_H F(\sigma(t))$ and $F(\sigma(t)) \ominus_H F(t-h)$ exists for all $0 < h < \beta$ and for all $t-h, t+h \in U_{\mathbb{T}}$;

(ii) the set-valued mappings F_α, $\alpha \in [0,1]$, are uniformly Δ_H-differentiable with derivative $\Delta_H F_\alpha$, i.e., for each $t \in \mathbb{T}$ and $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned} &D\left\{\frac{F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))}{h - \mu(t)}, \Delta_H F_\alpha(t)\right\} < \epsilon, \\ &D\left\{\frac{F_\alpha(\sigma(t)) \ominus_H F_\alpha(t-h)}{h + \mu(t)}, \Delta_H F_\alpha(t)\right\} < \epsilon \end{aligned} \tag{2}$$

for all $0 < h < \delta$, $t-h, t+h \in U_T$, $\alpha \in [0,1]$.

Then F is Δ_H-differentiable and the derivative is given by $\Delta_H F_\alpha(t) = \Delta_H [F(t)]^\alpha$.

Proof. Consider the family $\{\Delta_H F_\alpha(t), \alpha \in [0,1]\}$. By definition $\Delta_H F_\alpha(t)$ is a compact, convex and non-empty subset of $\mathbb{R}^n$. If $\alpha_1 \leq \alpha_2$ then by assumption (i),

$$F_{\alpha_1}(t+h) \ominus_H F_{\alpha_1}(\sigma(t)) \supset F_{\alpha_2}(t+h) \ominus_H F_{\alpha_2}(\sigma(t))$$

for $0 < h < \beta$ and for all $t-h, t+h \in U_T$ and consequently $\Delta_H F_{\alpha_1}(t) \supset \Delta_H F_{\alpha_2}(t)$.

Let $\alpha > 0$ and $\{\alpha_k\}$ be a nondecreasing sequence converging to α. For $\epsilon > 0$ choose $h > 0$ such that the equation (2) holds true. Then consider

$$\begin{aligned}
&D(\Delta_H F_\alpha(t), \Delta_H F_{\alpha_k}(t)) \\
&\leq D\left(\Delta_H F_\alpha(t), \frac{F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))}{h-\mu(t)}\right) + D\left(\frac{F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))}{h-\mu(t)}, \Delta_H F_{\alpha_k}(t)\right) \\
&< \epsilon + \frac{1}{h-\mu(t)} D[F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t)), F_{\alpha_k}(t+h) \ominus_H F_{\alpha_k}(\sigma(t))] \\
&\qquad + \frac{1}{h-\mu(t)} D[F_{\alpha_k}(t+h) \ominus_H F_{\alpha_k}(\sigma(t)), \Delta_H F_{\alpha_k}(t))] \\
&< 2\epsilon + \frac{1}{h-\mu(t)} D[F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t)), F_{\alpha_k}(t+h) \ominus_H F_{\alpha_k}(\sigma(t))].
\end{aligned}$$

By assumption (i), the rightmost term goes to zero as $k \to \infty$ and hence

$$\lim_{k\to\infty} D(\Delta_H F_\alpha(t), \Delta_H F_{\alpha_k}(t)) = 0.$$

From Theorem 2.2, we have

$$\Delta_H F_\alpha(t) = \bigcap_{k\geq 1} \Delta_H F_{\alpha_k}(t).$$

If $\alpha = 0$, we deduce as before

$$\lim_{k\to\infty} D(\Delta_H F_0(t), \Delta_H F_{\alpha_k}(t)) = 0,$$

where $\{\alpha_k\}$ be a nondecreasing sequence converging to zero, and consequently

$$\Delta_H F_0(t) = cl\left(\bigcup_{k\geq 1} \Delta_H F_{\alpha_k}(t)\right).$$

Then from Theorem 2.1, it follows that there is an element $u \in \mathbb{E}^n$ such that

$$[u]^\alpha = \Delta_H F_\alpha(t), \quad \alpha \in [0,1].$$

Furthermore, let $t \in \mathbb{T}$, $\epsilon > 0$, $\delta > 0$ and $t-h, t+h \in U_T$ be as in assumption (ii). Then

$$D\left(\frac{F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))}{h-\mu(t)}, u_\alpha\right) = D\left(\frac{F_\alpha(t+h) \ominus_H F_\alpha(\sigma(t))}{h-\mu(t)}, \Delta_H F_\alpha(t)\right) < \epsilon$$

for $0 < h < \delta$ and $t-h, t+h \in U_T$.

Similarly

$$D\left(\frac{F_\alpha(\sigma(t)) \ominus_H F_\alpha(t-h)}{h-\mu(t)}, u_\alpha\right) < \epsilon.$$

Hence F is Δ_H-differentiable. □

Theorem 3.4. *Let $F : \mathbb{T} \to \mathbb{E}^1$ be Δ_H-differentiable on $\mathbb{T}$. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$, $\alpha \in [0,1]$. Then f_α and g_α are Δ_H-differentiable on $\mathbb{T}$ and*

$$[\Delta_H F(t)]^\alpha = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].$$

Proof. Given F is Δ_H-differentiable at $t \in \mathbb{T}^k$ and t is right-scaterred, then for any $\alpha \in [0,1]$, consider

$$[F(t+h) \ominus_H F(\sigma(t))]^\alpha = [f_\alpha(t+h) \ominus_H f_\alpha(\sigma(t)), g_\alpha(t+h) \ominus_H g_\alpha(\sigma(t))]$$

and dividing by $h - \mu(t)$ and taking limit as $h \to 0^+$, we get

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{\mu(t)}[F(t+h) \ominus_H F(\sigma(t))]^\alpha \\
&\qquad = \left[\lim_{h\to 0^+} \frac{f_\alpha(t+h) \ominus_H f_\alpha(\sigma(t))}{\mu(t)}, \lim_{h\to 0^+} \frac{g_\alpha(t+h) \ominus_H g_\alpha(\sigma(t))}{\mu(t)}\right] \\
&\qquad = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].
\end{aligned}$$

Similarly,

$$\lim_{h\to 0^+} \frac{1}{\mu(t)}[F(\sigma(t)) \ominus_H F(t-h)]^\alpha = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].$$

If F is Δ_H-differentiable at $t \in \mathbb{T}^k$ and t is right-dense, then for any $\alpha \in [0,1]$, consider

$$[F(t+h) \ominus_H F(t)]^\alpha = [f_\alpha(t+h) \ominus_H f_\alpha(t), g_\alpha(t+h) \ominus_H g_\alpha(t)]$$

and dividing by $h > 0$ and taking limit as $h \to 0^+$, we get

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}[F(t+h) \ominus_H F(t)]^\alpha \\
&\qquad = \left[\lim_{h\to 0^+} \frac{f_\alpha(t+h) \ominus_H f_\alpha(t)}{h}, \lim_{h\to 0^+} \frac{g_\alpha(t+h) \ominus_H g_\alpha(t)}{h}\right] \\
&\qquad = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].
\end{aligned}$$

Similarly,

$$\lim_{h\to 0^+} \frac{1}{h}[F(\sigma(t)) \ominus_H F(t-h)]^\alpha = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].$$

□

Remark 3.4 Let $F : \mathbb{T} \to \mathbb{E}^n$ be a fuzzy set-valued function. If F is Δ_H-differentiable on $\mathbb{T}^k$, then there exists $\delta > 0$ such that for $\alpha \in [0,1]$, we have

$$diam[F(t_0 - h)]^\alpha \le diam[F(\sigma(t_0))]^\alpha \le diam[F(t_0 + h)]^\alpha, \text{ for } 0 < h < \delta,$$

i.e. for each $\alpha \in [0,1]$, the real function $t \to diam[F(t)]^\alpha$ is nondecreasing on $\mathbb{T}^k$.

Theorem 3.5. *Let $F : \mathbb{T} \to \mathbb{E}^n$ be Δ_H-differentiable and nondecreasing on $\mathbb{T}^k$. If $t_1, t_2 \in \mathbb{T}^k$ with $t_1 \le t_2$, then there exists a $C \in \mathbb{E}^n$ such that $F(t_2) \ominus_H F(t_1) = C$.*

Proof. For each $w \in [t_1, t_2]$ there exists a $\delta(w) > 0$ such that the H-differences $F(w+h) \ominus_H F(\sigma(w))$ and $F(\sigma(w)) \ominus_H F(w-h)$ exist for all $0 < h < \delta(w)$. Then we can find a finite sequence $t_1 = w_1 < w_2 < w_3 < \cdots < w_n = t_2$ such that the family $\{I_{w_i} = (w_i - \delta(w_i), w_i + \delta(w_i))/i = 1, 2, ..., n\}$, covers $[t_1, t_2]$ and $I_{w_i} \cap I_{w_{i+1}} \ne \emptyset$. Select a $s_i \in I_{w_i} \cap I_{w_{i+1}}$, $i = 1, 2, ..., n-1$, such that $w_i < s_i < w_{i+1}$. Then

$$F(w_{i+1}) = F(s_i) + k_1 = F(w_i) + k_2 + k_1 = F(w_i) + B_i, \quad i = 1, 2, ..., n-1,$$

for some $k_1, k_2, B_i \in \mathbb{E}^n$. Hence,

$$F(t_2) = F(t_1) + \sum_{i=1}^{n-1} B_i = F(t_1) + C,$$

which implies that $F(t_2) \ominus_H F(t_1) = C$. □

In the following theorem we obtain the Δ_H-derivatives of sums and scalar products of Δ_H-differentiable functions on time scales in (E^n, D).

Theorem 3.6. *Let $F, G : \mathbb{T} \to \mathbb{E}^n$ be Δ_H-differentiable at $t \in \mathbb{T}^k$. Then,*

(i) the sum $F + G : \mathbb{T} \to \mathbb{E}^n$ is Δ_H-differentiable at $t \in \mathbb{T}^k$ with

$$\Delta_H(F+G)(t) = \Delta_H F(t) + \Delta_H G(t),$$

(ii) for any constant λ, $\lambda F : \mathbb{T} \to \mathbb{E}^n$ is Δ_H-differentiable at t with

$$\Delta_H(\lambda F)(t) = \lambda \Delta_H F(t),$$

(iii) the product $FG : \mathbb{T} \to \mathbb{E}^n$ is Δ_H-differentiable at $t \in \mathbb{T}^k$ with

$$\Delta_H(FG)(t) = F(\sigma(t))\Delta_H G(t) + G(t)\Delta_H F(t) = F(t)\Delta_H G(t) + G(\sigma(t))\Delta_H F(t).$$

Proof. Let F and G be Δ_H-differentiable at $t \in \mathbb{T}^k$.

(i) Let $\epsilon > 0$, then there exists neighbourhoods U_1, and U_2 of t with

$$D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \le \frac{\epsilon}{2}(h - \mu(t)),$$

$$D[(F((\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h + \mu(t))] \le \frac{\epsilon}{2}(h + \mu(t))$$

for $0 < h < \delta$ with $t - h, t + h \in U_1$ and

$$D[(G(t+h) \ominus_H G(\sigma(t)), \Delta_H G(t)(h - \mu(t))] \le \frac{\epsilon}{2}(h - \mu(t)),$$

$$D[(G((\sigma(t)) \ominus_H G(t-h), \Delta_H G(t)(h + \mu(t))] \le \frac{\epsilon}{2}(h + \mu(t))$$

for $0 < h < \delta$ with $t - h, t + h \in U_2$. Let $U = U_1 \cap U_2$, then for $0 < h < \delta$ with $t - h, t + h \in U$, we have

$$\begin{aligned}
&D[(F+G)(t+h) \ominus_H (F+G)(\sigma(t)), (\Delta_H F(t) + \Delta_H G(t))(h - \mu(t))] \\
&= D[F(t+h) \ominus_H F(\sigma(t)) + G(t+h) \ominus_H G(\sigma(t)), \\
&\qquad \Delta_H F(t)(h - \mu(t)) + \Delta_H G(t)(h - \mu(t))] \\
&\le D[F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \\
&\quad + D[G(t+h) \ominus_H G(\sigma(t)), \Delta_H G(t)(h - \mu(t))] \\
&\le \frac{\epsilon}{2}(h - \mu(t)) + \frac{\epsilon}{2}(h - \mu(t)) \\
&= \epsilon(h - \mu(t)).
\end{aligned}$$

Similarly, we get

$$D[(F+G)((\sigma(t)) \ominus_H (F+G)(t-h), (\Delta_H F(t) + \Delta_H G(t))(h + \mu(t))] \le \epsilon(h + \mu(t)).$$

Therefore, $F + G$ is Δ_H-differentiable at t and

$$\Delta_H(F+G)(t) = \Delta_H F(t) + \Delta_H G(t).$$

(ii) For $\lambda = 0$, the result is trivial. We assume that $\lambda > 0$. Since F is Δ_H-differentiable at $t \in \mathbb{T}^k$ there exists a neighbourhood $U_{\mathbb{T}}$ of t such that, for given $\epsilon > 0$, we have

$$D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \le \frac{\epsilon}{\lambda}(h - \mu(t)),$$

$$D[(F((\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h + \mu(t))] \le \frac{\epsilon}{\lambda}(h + \mu(t))$$

for $0 < h < \delta$ with $t - h, t + h \in U_{\mathbb{T}}$ and

$$\begin{aligned}
&D[\lambda(F(t+h) \ominus_H \lambda F(\sigma(t)), \lambda \Delta_H F(t)(h - \mu(t))] \\
&= \lambda D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \\
&\le \lambda \frac{\epsilon}{\lambda}(h - \mu(t)) = \epsilon(h - \mu(t)).
\end{aligned}$$

Similarly, we get

$$D[\lambda(F((\sigma(t))) \ominus_H \lambda F(t-h), \lambda \Delta_H F(t)(h + \mu(t))] \le \epsilon(h + \mu(t))$$

for $0 < h < \delta$ with $t - h, t + h \in U_{\mathbb{T}}$. Therefore λF is Δ_H-differentiable at t and $\Delta_H(\lambda F)(t) = \lambda \Delta_H F(t)$.

(iii) Let $\epsilon \in (0,1)$ and define $\epsilon^* = \epsilon[1 + \|F(\sigma(t))\| + \|G(t)\| + \|\Delta_H F(t)\|]^{-1}$, then $\epsilon^* \in (0,1)$. Thus, there exist neighbourhoods U_1, U_2, U_3 of t such that

$$D[(F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \epsilon^*(h - \mu(t)),$$

$$D[(F((\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon^*(h + \mu(t))$$

for $0 < h < \delta$ with $t-h, t+h \in U_1$ and

$$D[(G(t+h) \ominus_H G(\sigma(t)), \Delta_H G(t)(h - \mu(t))] \leq \epsilon^*(h - \mu(t)),$$

$$D[(G((\sigma(t)) \ominus_H G(t-h), \Delta_H G(t)(\mu(t)) + h] \leq \epsilon^*(h + \mu(t))$$

for $0 < h < \delta$ with $t-h, t+h \in U_2$. Since G is Δ_H-differentiable at t, it follows from Theorem 3.1 that G is continuous at t. Hence

$$D[G(t+h), G(t)] \leq \epsilon^*, D[G(t), G(t-h)] \leq \epsilon^*,$$
$$D(G(t+h)) \leq D(G(t)) + 1, D(G(t-h)) \leq D(G(t)) + 1$$

for $0 < h < \delta$ with $t-h, t+h \in U_3$. Let $U_{\mathbb{T}} = U_1 \cap U_2 \cap U_3$ and $t-h, t+h \in U_{\mathbb{T}}$, then we have

$$\begin{aligned}
&D[(FG)(t+h) \ominus_H (FG)(\sigma(t)), (F(\sigma(t))\Delta_H G(t) + G(t)\Delta_H F(t))(h - \mu(t))] \\
&\leq D[(G(t+h) \ominus_H G(\sigma(t)))F(\sigma(t)), \Delta_H G(t)(h - \mu(t))F(\sigma(t))] \\
&\qquad + D[(F(t+h) \ominus_H F(\sigma(t)))G(t+h), \Delta_H F(t)(h - \mu(t))G(t+h)] \\
&\qquad + D[\theta, \Delta_H F(t)(h - \mu(t))(G(t) - G(t+h))] \\
&\leq D[(G(t+h) \ominus_H G(\sigma(t))), \Delta_H G(t)(h - \mu(t))](F(\sigma(t))) \\
&\qquad + D[(F(t+h) \ominus_H F(\sigma(t))), \Delta_H F(t)(h - \mu(t))]D(G(t+h)) \\
&\qquad + D[G(t+h), G(t)]\|\Delta_H F(t)\|(h - \mu(t)) \\
&\leq \epsilon^*(h - \mu(t))\|F(\sigma(t)\| + \epsilon^*(h - \mu(t))[\|G(t)\| + 1] \\
&\qquad + \epsilon^*(h - \mu(t))\|\Delta_H F(t)\| \\
&= \epsilon^*[1 + \|F(\sigma(t))\| + \|G(t)\| + \|\Delta_H F(t)\|](h - \mu(t)) = \epsilon(h - \mu(t)).
\end{aligned}$$

Similarly, we get

$$D[(FG)(\sigma(t)) \ominus_H (FG)(t-h), (F(\sigma(t))\Delta_H G(t) + G(t)\Delta_H F(t))(\mu(t) + h)] \leq \epsilon(h + \mu(t)).$$

Thus $\Delta_H(FG)(t) = F(\sigma(t))\Delta_H(G)(t) + G(t)\Delta_H(F)(t)$ holds at t. The other product rule in (iii) of this theorem follows from this last equation by interchanging the functions F and G. □

4 Integration of Fuzzy Set-valued Functions on Time Scales

In this section we introduced Δ_H-integral for fuzzy set-valued functions and also studied their properties.

Definition 4.1. *Let $I \subset \mathbb{T}$. A function $f : I \to \mathbb{R}$ is called a Δ-measurable sector of the fuzzy set-valued function $F : I \to E^n$ if $f(t) \in F(t)$ for all $t \in I$.*

Definition 4.2. *A fuzzy set-valued function $F : \mathbb{T} \to E^n$ is said to be regulated provided its regulated Δ-measurable sectors exist. A fuzzy set-valued function $F : \mathbb{T} \to E^n$ is said to be rd-continuous provided its rd-continuous Δ-measurable sectors exist.*

In this paper, the set of rd-continuous fuzzy set-valued functions $F : I \subset \mathbb{T} \to \mathbb{E}^n$ is denoted by

$$C_{frd} = C_{frd}(I) = C_{frd}(I, \mathbb{E}^n).$$

The set of fuzzy set-valued functions $F : I \subset \mathbb{T} \to \mathbb{E}^n$ which are Δ_H-differentiable and whose Δ_H-derivative is rd-continuous is denoted by

$$C'_{frd} = C'_{frd}(I) = C'_{frd}(I, \mathbb{E}^n).$$

Definition 4.3. *A continuous fuzzy set-valued function $F:\mathbb{T}\to E^n$ is said to be pre-differentiable with (region of differentiation) D, provided $D\subset\mathbb{T}^k$, $\mathbb{T}^k\setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and F is Δ_H -differentiable at each $t\in D$.*

Remark 4.1 It is evident that F is regulated provided F is rd-continuous, i.e. there exists a fuzzy set-valued function f which is pre-differentiable with region of differentiation D such that $\Delta_H f(t)=F(t)$ for all $t\in D$.

Definition 4.4. *A fuzzy set-valued function $F:\mathbb{T}\to E^n$ is said to be Δ_H-integrable on $I\subset\mathbb{T}$ if F has a rd-continuous Δ-measurable sector on I. In this case, we define the Δ_H-integral of F on I, denoted by $\int_I F(s)\Delta s$, and defined levelwise by the equation*

$$\left[\int_I F(s)\Delta s\right]=\int_I F_\alpha(s)\Delta s=\left\{\int_I f(s)\Delta s : f\in S_{F_\alpha}(I)\right\},$$

where $S_{F_\alpha}(I)$, the set of all Δ_H-integrable sectors of F_α on I.

Theorem 4.1. *Let $t_0,T\in\mathbb{T}$ and $F,G:[t_0,T]_\mathbb{T}\to E^n$ be Δ_H-integrable and have rd-continuous Δ-measurable sectors, then we have*

(i) $\int_{t_0}^T[F(s)+G(s)]\Delta s=\int_{t_0}^T F(s)\Delta s+\int_{t_0}^T G(s)\Delta s.$

(ii) $\int_{t_0}^T \lambda F(s)\Delta s=\lambda\int_{t_0}^T F(s)\Delta s,\quad \lambda\in\mathbb{R}^+.$

(iii) $\int_{t_0}^T F(s)\Delta s=\int_{t_0}^t F(s)\Delta s+\int_t^T F(s)\Delta s.$

(iv) $\int_{t_0}^{t_0} F(s)\Delta s=\{0\}.$

(v) If $f\in S_F([t_0,T]_\mathbb{T})$, then $D(F(.),\theta):[t_0,T]_\mathbb{T}\to\mathbb{R}^+$ is Δ_H-integrable and

$$D\left(\int_{t_0}^T F(s)\Delta s,\theta\right)\le\int_{t_0}^T D(F(s),\theta)\Delta s.$$

(vi) If $f\in S_F([t_0,T]_\mathbb{T})$ and $g\in S_G([t_0,T]_\mathbb{T})$ implies that $f,g\in C_{frd}([t_0,T]_\mathbb{T})$, respectively, then $D(F(.),G(.)):[t_0,T]_\mathbb{T}\to\mathbb{R}^+$ is Δ_H-integrable and

$$D\left(\int_{t_0}^T F(s)\Delta s,\int_{t_0}^T G(s)\Delta s\right)\le\int_{t_0}^T D\left(F(s),G(s)\right)\Delta s.$$

Proof. Let $t_0,T\in\mathbb{T}$ and $F,G:[t_0,T]_\mathbb{T}\to E^n$ are Δ_H-integrable.
(i) Since F, G are Δ_H-integrable, for $\alpha\in[0,1]$, F_α and G_α have rd-continuous Δ-measurable sectors. Hence for any $u\in\int_{t_0}^T[F(s)+G(s)]_\alpha\Delta s$, there exist $f\in S_F([t_0,T]_\mathbb{T})$, $g\in S_G([t_0,T]_\mathbb{T})$ such that

$$u=\int_{t_0}^T[f(s)+g(s)]\Delta s=\int_{t_0}^T f(s)\Delta s+\int_{t_0}^T g(s)\Delta s\in\int_{t_0}^T F_\alpha(s)\Delta s+\int_{t_0}^T G_\alpha(s)\Delta s,$$

which implies that $\int_{t_0}^T[F(s)+G(s)]_\alpha\Delta s\subset\int_{t_0}^T[F_\alpha(s)]\Delta s+\int_{t_0}^T[G_\alpha(s)]\Delta s$. In a similar way, we can obtain the converse part.
(ii) Let $u\in\int_{t_0}^T[\lambda F(s)]_\alpha\Delta s$, there exist $f\in S_F([t_0,T]_\mathbb{T})$, such that

$$u=\int_{t_0}^T[\lambda f(s)\Delta s]=\lambda\int_{t_0}^T f(s)\Delta s\in\lambda\int_{t_0}^T F_\alpha(s)\Delta s,$$

which implies that $\int_{t_0}^T[\lambda F(s)]_\alpha\Delta s\subset\lambda\int_{t_0}^T[F(s)]_\alpha\Delta s$. In a similar way, we can obtain the converse part.
(iii) Let $\alpha\in[0,1]$ and f be a Δ- measurable sector for F_α. From Lemma 2.2 (iv), $\int_{t_0}^T f(s)\Delta s=\int_{t_0}^t f(s)\Delta s+\int_t^T f(s)\Delta s$, then we have

$$\int_{t_0}^T F_\alpha(s)\Delta s\subset\int_{t_0}^t F_\alpha(s)\Delta s+\int_t^T F_\alpha(s)\Delta s.$$

On the other hand, let $z = \int_{t_0}^{t} g_1(s)\Delta s + \int_{t}^{T} g_2(s)\Delta s$, where g_1 is a Δ-measurable sector for F_α in $[t_0, t]$ and g_2 is a Δ-measurable sector for F_α in $[t, T]$. Then f defined by

$$f(t) = \begin{cases} g_1(t), & \text{if} \quad t \in [t_0, t] \\ g_2(t), & \text{if} \quad t \in (t, T] \end{cases}$$

is a Δ-measurable sector for F_α in $[t_0, T]_{\mathbb{T}}$ and

$$\int_{t_0}^{T} f(s)\Delta s = \int_{t_0}^{t} g_1(s)\Delta s + \int_{t}^{T} g_2(s)\Delta s = z.$$

Thus,

$$\int_{t_0}^{T} F_\alpha(s)\Delta s + \int_{t}^{T} F_\alpha(s)\Delta s \subset \int_{t_0}^{T} F_\alpha(s)\Delta s.$$

(iv) Let $\alpha \in [0, 1]$ and f be a Δ-measurable sector for F_α. From Lemma 2.2 (v), $\int_{t_0}^{t_0} f(s)\Delta s = \{0\}$, then we have

$$\int_{t_0}^{t_0} F_\alpha(s)\Delta s = \int_{t_0}^{t_0} f(s)\Delta s = 0.$$

(vi) It is sufficient to prove (vi) because (v) is a particular case when $G(t) = \theta$ in (v). For given $f \in C_{frd}$, we have the inequalities

$$d(f_\alpha(s), G_\alpha(s)) \leq d(f_\alpha(s), g_\alpha(s)) \leq d(f_\alpha(s), f_\alpha(\tau)) + d(f_\alpha(\tau), g_\alpha(s))$$

for each $g_\alpha(s) \in G_\alpha(s)$, which implies

$$d(f_\alpha(s), G_\alpha(s)) \ominus_H d(f_\alpha(s), f_\alpha(\tau)) \leq \inf d(f_\alpha(\tau), g_\alpha(s)) = d(f_\alpha(\tau), G_\alpha(s)).$$

Therefore,

$$d(f_\alpha(s), G_\alpha(s)) \ominus_H d(f_\alpha(\tau), G_\alpha(s)) \leq d(f_\alpha(s), f_\alpha(\tau)).$$

The same inequality holds with s and τ are interchanged and rd-continuity of $d_H(f_\alpha(.), G_\alpha(s))$ at $s \in [t_0, T]$ follows for each $f_\alpha \in C_{frd}$. Thus,

$$D(F(t), G(t)) = \sup_{k \geq 1} d_H(F_{\alpha_k}(t), G_{\alpha_k}(t))$$

is rd-continuous, which yields that the integral $\int_{t_0}^{T} D[F(s), G(s)]\Delta s$ is well defined. Hence for any $u \in \int_{t_0}^{T} F(s)\Delta s$, there exists a Δ-measurable sector $f \in S_F([t_0, T]_{\mathbb{T}})$, and for any $v \in \int_{t_0}^{T} G(s)\Delta s$, there exists a Δ-measurable sector $g \in S_G([t_0, T]_{\mathbb{T}})$, such that

$$u = \int_{t_0}^{T} f(s)\Delta s, \quad v = \int_{t_0}^{T} g(s)\Delta s.$$

From [13], we have

$$d_H\left(\int F_\alpha, \int G_\alpha\right) \leq \int d_H(F_\alpha, G_\alpha),$$

and consequently

$$\begin{aligned} D(u, v) = D\left(\int F, \int G\right) &\leq \sup_{\alpha \in [0,1]} \int d_H(F_\alpha, G_\alpha) \\ &\leq \int \sup_{\alpha \in [0,1]} d_H(F_\alpha, G_\alpha) = \int D(F, G). \end{aligned}$$

$\square$

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Theorem 4.2. *Let $I_{\mathbb{T}} = I \cap \mathbb{T}$, where $I \subset \mathbb{R}$ be an interval and $F : I_{\mathbb{T}} \to \mathbb{E}^n$ is rd-continuous. If $t_0 \in \mathbb{T}$, then f defined by*

$$f(t) = X_0 + \int_{t_0}^{t} F(s)\Delta s, \quad \textit{for } t \in I_{\mathbb{T}} \textit{and } X_0 \in \mathbb{E}^n$$

is Δ_H-differentiable and $\Delta_H f(t) = F(t)$ a.e. on $I_{\mathbb{T}}$.

Proof. Let $t \in D$, since f is Δ_H- differentiable, then $\Delta_H f(t)$ exists and

$$\Delta_H f(t) = F(t) \quad \text{a.e. on } D.$$

If $t \in I_{T^k} \setminus D$, then t is a right-dense point of $\mathbb{T}$. For any $h \geq 0$ with $t - h, t + h \in I_T$, we have

$$f(t+h) - f(\sigma(t)) = \int_{t_0}^{t+h} F(s)\Delta s - \int_{t_0}^{\sigma(t)} F(s)\Delta s = \int_{\sigma(t)}^{t+h} F(s)\Delta s.$$

Let $\epsilon > 0$ be arbitrary, by the rd-continuity of F we have

$$\begin{aligned} D\left(\frac{f(t+h) - f(\sigma(t))}{h - \mu(t)}, F(t)\right) &= \frac{1}{h - \mu(t)} \left[D(f(t+h) - f(\sigma(t))), (h - \mu(t))F(t))\right] \\ &= \frac{1}{h - \mu(t)} \left(\int_{\sigma(t)}^{t+h} F(s)\Delta s, \int_{\sigma(t)}^{t+h} F(t)\Delta s\right) \\ &\leq \frac{1}{h - \mu(t)} \int_{\sigma(t)}^{t+h} D(F(s), F(t))\Delta s < \epsilon \end{aligned}$$

for all $t - h, t + h \in I_T$. The integral on the right hand side tends to zero as $h \to 0$, as t is right dense. Hence

$$\lim_{h \to 0} \frac{f(t+h) - f(\sigma(t))}{h - \mu(t)} = F(t)$$

and similarly

$$\lim_{h \to 0} \frac{f(\sigma(t)) - f(t-h)}{h - \mu(t)} = F(t).$$

□

Remark 4.2 A fuzzy set-valued function $f : \mathbb{T} \to E^n$ is called a Δ_H-antiderivative of $F : \mathbb{T} \to E^n$ provided $\Delta_H f(t) = F(t)$ for all $t \in \mathbb{T}^k$. From Theorem4.2 every rd-continuous fuzzy valued function F has a Δ_H-antiderivative f. Thus, for $t_0 \in \mathbb{T}$,

$$f(t) = f(t_0) + \int_{t_0}^{t} F(s)\Delta s, \quad \textit{for } t \in \mathbb{T}^k.$$

5 Fuzzy Differential Equations on Time Scales

In this section we consider a fuzzy initial value problem on time scales

$$y^{\Delta} = F(t, y), \qquad y(t_0) = y_0, \tag{3}$$

where $F : \mathbb{T} \times \mathbb{E}^n \to \mathbb{E}^n$ is a given function and for $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{E}^n$, is called an initial value problem. A function $y : \mathbb{T} \to \mathbb{R}$ is called a solution of this IVP if

$$y^{\Delta}(t) = F(t, y(t))$$

is satisfied for all $t \in \mathbb{T}^k$ with $y(t_0) = y_0$. From Theorem4.2 it immediately follows:

Lemma 5.1 A mapping $y : \mathbb{T} \to E^n$ is called a solution to the IVP (3) if and only if it is rd-continuous and satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(s, y(s))\Delta s, \qquad \text{for all } t \in \mathbb{T}^k. \tag{4}$$

If F is Lipschitz continuous then the problem (3) has a unique solution on $\mathbb{T}$. Further, the solution depends continuously on the initial value.

Theorem 5.1. *Let $F : \mathbb{T} \times \mathbb{E}^n \to \mathbb{E}^n$ be rd-continuous and assume that there exists a $K > 0$ such that*

$$D(F(t,x), F(t,y)) \leq KD(x,y)$$

for all $t \in \mathbb{T}$, $x, y \in \mathbb{E}^n$. Then the problem (3) has a unique solution on $\mathbb{T}$.

Proof. Let $C_{frd}(J, \mathbb{E}^n)$ be the set of all rd-continuous fuzzy mappings from J to $\mathbb{E}^n$, where J is an interval in $\mathbb{T}$. Define a metric H on $C_{frd}(J, \mathbb{E}^n)$ by

$$H(\xi, \psi) = \sup_{t \in J} D(\xi(t), \psi(t))$$

for $\xi, \psi \in C_{frd}(J, \mathbb{E}^n)$. Since $(\mathbb{E}^n, D)$ is a complete metric space, in a similar way it is easy to show that $C_{frd}((J, \mathbb{E}^n), H)$ is complete. Let $(t_1, y_1) \in \mathbb{T} \times \mathbb{E}^n$ be arbitrary and $\eta > 0$ be such that $\eta k < 1$. Now we will show that the IVP

$$y^{\Delta}(t) = F(t, y(t)), \qquad y(t_1) = y_1 \tag{5}$$

has a unique solution on the interval $I = [t_1, t_1 + \eta]$, by using Banach contraction principle. For $\xi \in C_{rd}(I, \mathbb{E}^n)$, define $G\xi$ on I by the equation

$$G\xi(t) = y_1 + \int_{t_1}^{t} f(s, \xi(s))\Delta s.$$

Now, we will show $G\xi \in C_{frd}(I, \mathbb{E}^n)$ by using rd-continuity of F and (v)of Theorem 4.1. Let $t_0, t \in I$ and assume that $t_0 < t$.

Consider

$$\begin{aligned}
D\left(G\xi(t), G\xi(t_0)\right) &= D\left(\int_{t_1}^{t} F(s, \xi(s))\Delta s, \int_{t_1}^{t_0} F(s, \xi(s))\Delta s\right) \\
&= D\left(\int_{t_0}^{t} F(s, \xi(s))\Delta s, \theta\right) \\
&\leq \int_{t_0}^{t} D\left(F(s, \xi(s))\Delta s, \theta\right) \\
&= \int_{t_0}^{t} ||(F(s, \xi(s))\|\Delta s \\
&\leq M(t - t_0).
\end{aligned}$$

Hence $G\xi \in C_{frd}(I, \mathbb{E}^n)$. Consider

$$\begin{aligned}
H(G\xi, G\psi) = \sup_{t \in I} D\left(\int_{t_1}^{t} F(s, \xi(s))\Delta s, \int_{t_1}^{t} F(s, \psi(s))\Delta s\right) \\
\leq \int_{t_1}^{t_1+\eta} D\left(F(s, \xi(s)), F(s, \psi(s))\right)\Delta s \\
\leq \int_{t_1}^{t_1+\eta} kD\left(\xi(s), \psi(s)\right)\Delta s \\
\leq \eta k H(\xi, \psi)
\end{aligned}$$

for $\xi, \psi \in C_{frd}(I, \mathbb{E}^n)$, by using the Lipschitz continuity of F. Hence by using Banach contraction mapping theorem G has a unique fixed point, which is the desired solution to (5). Proceeding in this way, problem (5) has unique solution on each interval I. Combining all these solutions together gives the solution to (3). Hence the problem (3) has unique solution on $\mathbb{T}$. □

References

[1] Agarwal, R.P., and M. Bohner, Basic calculus on time scales and some of its applications, *Results in Mathematics*, vol.35, pp.3–22, 1999.

[2] Agarwal, R.P., Bohner, M., O'Regan, D., and A. Peterson, Dynamic equations on time scales: a survey, *Journal of Computational and Applied Mathematics*, vol.141, nos.1-2, pp.1–26, 2002.

[3] Bohner, M., and G.S. Guseinov, Improper integrals on time scales, *Dynamic Systems and Applications*, vol.12, pp.45–65, 2003.

[4] Bohner, M., and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, 2001.

[5] Bohner, M., and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhauser, Boston, 2003.

[6] Fard, O.S., and T.A. Bidgoli, Calculus of fuzzy functions on time scales (I), *Soft Computing, Methodologies and Applications*, vol.19, no.2, pp.293–305, 2015.

[7] Guseinov, G.S., Integration on time scales, *Journal of Mathematical Analysis and Applications*, vol.285, pp.107–127, 2003.

[8] Hashmi, U.A., Lupulescu, V., and G. Rahman, Uncertain dynamic systems on time scales, *Journal of Uncertain Systems*, vol.9, no.1, pp.17–30, 2015.

[9] Hilger, S., *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. Thesis, Universitat Wurzburg, 1988.

[10] Hilger, S., Analysis on measure chains - a unified approach to continuous and discrete calculus, *Results in Mathematics*, vol.18, pp.18–56, 1990.

[11] Hong, S., Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations, *Nonlinear Analysis*, vol.71, pp.3622–3637, 2009.

[12] Kaleva, O., On the convergence of fuzzy sets, *Fuzzy Sets and Systems*, vol.17, pp.53–65, 1985.

[13] Kaleva, O., Fuzzy differential equations, *Fuzzy Sets and Systems*, vol.24, pp.301–317, 1987.

[14] Lakshmikantham, V., and R.N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, 2003.

[15] Lakshmikantham, V., Sivasundaram, S. and B. Kaymakcalan, *Dynamic Equations on Measure Chains*, Kluwer, Dordrecht, 1996.

[16] Liu, B., Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, vol.2, no.1, pp.3–16, 2008.

[17] Lupulescu, V., Hukuhara differentiability of interval-valued functions and interval differential equations on time scales, *Information Sciences*, vol.248, pp.50–67, 2013.

[18] Murthy, M.S.N., and G. Suresh Kumar, Three Point boundary value problems for third order fuzzy differential equations, *Journal of Chungcheong Mathematical Society*, vol.19, no.1, pp.101–110, 2006.

[19] Murthy, M.S.N., and G. Suresh Kumar, Initial and boundary value problems for fuzzy differential equations, *Demonstratio Mathematical*, vol.XL, no.4, pp.827–838, 2007.

[20] Murthy, M.S.N., and G. Suresh Kumar, On controllability and observability of fuzzy dynamical matrix Lyapunov systems, *Advances in Fuzzy Systems*, vol.2008, pp.1–16, 2008.

[21] Murthy, M.S.N., and G. Suresh Kumar, On observability of fuzzy dynamical matrix Lyapunov systems, *Kyungpook Mathematical Journal*, vol.48, pp.473–486, 2008.

[22] Murthy, M.S.N., Suresh Kumar, G., Appa Rao, B.V., and K.A.S.N.V. Prasad, On controllability of fuzzy dynamical matrix Lyapunov systems, *Analele Universitatii de Vest, Timisoara*, vol.LI, no.2, pp.73–87, 2013.

[23] Puri, M.L., and D.A. Ralescu, Differentials of fuzzy functions, *Journal of Mathematical Analysis and Applications*, vol.91, pp.552–558, 1983.

[24] Puri, M.L., and D.A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, vol.114, pp.409–422, 1986.

[25] Vasavi, C.H., Suresh Kumar, G., and M.S.N. Murty, Generalized differentiability and integrability for fuzzy set-valued functions on time scales, *Soft Computing, Methodologies and Applications*, vol.20, pp.1093–1104, 2016.

[26] Vasavi, C.H., Suresh Kumar, G., and M.S.N. Murty, Fuzzy dynamic equations on time scales under second type Hukuhara delta derivative, *International Journal of Chemical Sciences*, vol.14, no.1, pp.49–66, 2016.

[27] You, C., Wang, W., and H. Huae, Existence and uniqueness theorems for fuzzy differential equations, *Journal of Uncertain Systems*, vol.7, no.4, pp.303–315, 2013.