Fuzzy Hukuhara Delta Differential and Applications to Fuzzy Dynamic Equations on Time Scales

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Abstract

Using the concept of Hukuhara difference, in this paper we introduce a class of new derivatives called Hukuhara delta derivative and a class of new integrals called Hukuhara delta integral for fuzzy set-valued functions on time scales. Moreover, some corresponding properties of Hukuhara delta derivative and Hukuhara delta integral are discussed. Furthermore, sufficient conditions are established for the existence and uniqueness of solution to the fuzzy dynamic equations on time scales with the help of Banach contraction principle.

Keywords: fuzzy differential equations, ∆₇₆-differential, ∆₇₆-integral, time scales

1 Introduction

1.1 Dynamic Equations on Time Scales

The theory of time scales was introduced by Stefan Hilger [9, 10], which attracted the attention of many researchers due to the ability of this theory to model many real world problems as the dynamical systems include both continuous and discrete nature simultaneously. For fundamental results in the theory of time scales, we refer Agarwal et al. [1, 2], Bohner et al. [4, 5, 3], Guseinov [7], Lakshmikantham et al. [15]. Recently, Hashmi et al. [8] studied the existence and uniqueness of the solution for the dynamic systems on time scales with uncertain parameters.

1.2 Fuzzy Set Theory and Its Applications

The rise and development of new fields such as general system theory, robotics, artificial intelligence and language theory, force us to be engaged in specifying imprecise notions. When a real world problem is transformed into a deterministic model, we cannot usually be sure that the model is perfect. Using fuzzy set theory, we can model the meaning of vague notions and also certain kinds of human reasoning. Fuzzy set theory and its applications have been developed by Kaleva [12, 13], Lakshmikantham et al. [14], Liu et al. [16], Murty et al. [18, 19, 20, 21, 22], Puri et al. [23, 24], You [27].

1.3 Motivation

The differential and integral calculus for multivalued functions on time scales using Hukuhara difference was introduced and developed by Hong [11]. Later, Lupulescu [17] studied the differentiability and integrability for the interval-valued functions on time scales using generalized Hukuhara difference (gH-difference). Further, Fard et al. [6] studied the calculus of fuzzy-number-valued functions on time scales using gH-difference. Furthermore, Vasavi et al. [25] studied the fundamental properties of fuzzy set-valued functions on time scales using generalized delta derivative (Δ₇₆-derivative) with Hukuhara difference. Recently, Vasavi et al. [26] studied the properties of second type Hukuhara delta derivative (Δ₇₆₇₆-derivative) for fuzzy set-valued functions

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on time scales and established the existence and uniqueness of solutions for the fuzzy dynamic equations on time scales. The main aim of this paper is to fill the gap in the literature of fuzzy time scale calculus [6, 25, 26]. In this context, we introduce fuzzy Hukuhara delta derivative and integral for fuzzy set-valued functions on time scales. The main aim of this paper is to fill the gap in the literature of fuzzy time scale calculus [6, 25, 26].

1.4 Structure of the Paper

The paper is organized as follows. In Section 2, some basic definitions and results related to fuzzy calculus as well as time scale calculus are presented. In Section 3, we introduce and study the properties of new class of derivatives called Hukuhara difference. In Section 4, we introduce Hukuhara delta integral (\(\Delta_H\)-integral) and studied some properties. In Section 5, we establish sufficient conditions for the existence and uniqueness of solutions for the fuzzy dynamic equations on time scales.

2 Preliminaries

Firstly, we recall some results related to fuzzy calculus. Let \(P_k(\mathbb{R}^n)\) denotes the family of all nonempty compact convex subsets of \(\mathbb{R}^n\). Define the addition and scalar multiplication in \(P_k(\mathbb{R}^n)\) as usual. Then \(P_k(\mathbb{R}^n)\) is a commutative semigroup [13] under addition, which satisfies the cancellation law. Moreover, if \(\alpha, \beta \in \mathbb{R}\) and \(A, B \in P_k(\mathbb{R}^n)\), then

\[
\alpha(A + B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha \beta)A, \quad 1.A = A,
\]

and if \(\alpha, \beta \geq 0\) then \((\alpha + \beta)A = \alpha A + \beta A\). Let \(A\) and \(B\) be two nonempty bounded subsets of \(\mathbb{R}^n\). The distance between \(A\) and \(B\) is defined by the Hausdorff metric

\[
d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} ||a - b||, \sup_{b \in B} \inf_{a \in A} ||a - b|| \right\}
\]

where \(||.||\) denotes the Euclidean norm in \(\mathbb{R}^n\) (or) equivalently, we define the Hausdorff distance between \(A\) and \(B\) as

\[
d_H(A, B) = \max \left\{ \sup_{a \in A} \sup_{b \in B} d(a, b), \sup_{b \in B} \sup_{a \in A} d(b, A) \right\},
\]

where \(d(a, b) = \inf_{x \in A} ||a - b||\), \(d(b, A) = \inf_{x \in A} ||a - b||\). Then \((P_k(\mathbb{R}^n), d_H)\) becomes a complete and separable metric space [13]. Define

\[
\mathbb{E}^n = \{ u : \mathbb{R}^n \to [0, 1] / u \text{satisfies (i)-(iv) below} \},
\]

where

(i) \(u\) is normal, i.e., there exists an \(x_0 \in \mathbb{R}^n\) such that \(u(x_0) = 1\),

(ii) \(u\) is fuzzy convex,

(iii) \(u\) is upper semicontinuous,

(iv) the closure of \(\{ x \in \mathbb{R}^n / u(x) > 0 \}\), denoted by \(u^0\), is compact.

For \(0 < \alpha \leq 1\), denote \([u]^\alpha = \{ x \in \mathbb{R}^n / u(x) \geq \alpha \}\), then from (i)-(iv) it follows that the \(\alpha\)-level set \([u]^\alpha \in P_k(\mathbb{R}^n)\) for all \(0 < \alpha \leq 1\).

If \(g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is a function then according to Zadeh’s extension principle we can extend \(g : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^n\) by

\[
g(u, v)(z) = \sup_{z = g(x, y)} \min\{u(x), v(y)\}.
\]

It is well known that \([g(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)\), for all \(u, v \in \mathbb{E}^n\), \(0 < \alpha \leq 1\) and \(g\) is continuous. For addition and scalar multiplication, we have

\[
[u + v]^\alpha = [u]^\alpha + [v]^\alpha, [ku]^\alpha = k[u]^\alpha, \text{where } u, v \in \mathbb{E}^n, k \in \mathbb{R}, 0 \leq \alpha \leq 1.
\]
Theorem 2.1. ([13]) If \( u \in \mathbb{E}^n \), then

(i) \( [u]^\alpha \in \mathbb{P}_k(\mathbb{R}^n) \) for all \( 0 \leq \alpha \leq 1 \),

(ii) \( [u]^\alpha_2 \subset [u]^\alpha_1 \) for all \( 0 \leq \alpha_1 \leq \alpha_2 \leq 1 \),

(iii) If \( \{\alpha_k\} \subset [0,1] \) is a nondecreasing sequence converging to \( \alpha > 0 \), then

\[
[u]^\alpha = \bigcap_{k \geq 1} [u]^\alpha_k.
\]

Conversely, if \( \{A^n/0 \leq \alpha \leq 1\} \) is a family of subsets of \( \mathbb{R}^n \) satisfying(i)-(iii),then there exists \( u \in \mathbb{E}^n \) such that

\[
[u]^\alpha = A^n, \quad \text{for} \quad 0 < \alpha \leq 1, \quad \text{and}
\]

\[
[u]^0 = \text{cl} \left\{ \bigcup_{0<\alpha \leq 1} A^n \right\} \subset A^0, \quad \text{where cl denotes the closure of the set.}
\]

Theorem 2.2. ([13]) Let \( \{A_k\} \) be a sequence in \( \mathbb{P}_K(\mathbb{R}^n) \) converging to \( A \) and \( d(A_k,A) \to 0 \) as \( k \to \infty \) then

\[
A = \bigcap_{k \geq 1} \text{cl} \left\{ \bigcup_{m \geq k} A_m \right\}.
\]

Define \( D: \mathbb{E}^n \times \mathbb{E}^n \to [0, \infty) \) by

\[
D(u,v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha,[v]^\alpha),
\]

where \( d_H \) is the Hausdorff metric defined in \( \mathbb{P}_K(\mathbb{R}^n) \). Then \( (\mathbb{E}^n,D) \) is a complete metric space [13].

Let \( x, y \in \mathbb{E}^n \). If there exists a \( z \in \mathbb{E}^n \) such that \( x = y + z \) then we call \( z \) the \( H \)- difference of \( x \) and \( y \) denoted by \( x \odot_H y \). For any \( A,B,C,D \in \mathbb{E}^n \) and \( \lambda \in \mathbb{R} \),

(i) \( D(A,B) = 0 \Leftrightarrow A = B \),

(ii) \( D(\lambda A,\lambda B) = |\lambda| D(A,B) \),

(iii) \( D(A \odot_H C,B \odot_H C) = D(A,B) \),

(iv) \( D(A \odot_H B,C \odot_H D) \leq D(A,C) + D(B,D) \).

Here we assume that the Hukuhara differences appearing in the above formulae exist.

Definition 2.1. ([13]) A mapping \( F: T \to \mathbb{E}^n \) is Hukuhara differentiable at \( t_0 \in T \) if there exists a \( F'(t_0) \in \mathbb{E}^n \) such that the limits

\[
\lim_{h \to 0^+} \frac{F(t_0 + h) \odot_H F(t_0)}{h}, \quad \lim_{h \to 0^+} \frac{F(t_0) \odot_H F(t_0 - h)}{h}
\]

exist in the topology of \( \mathbb{E}^n \) and equal to \( F'(t_0) \). Here the limit is taken in the metric space \( (\mathbb{E}^n,D) \). At the end points of \( T \) we consider only the one-sided derivatives.

Remark 2.1. ([13]) If \( F \) is differentiable then the multivalued mapping \( F_\alpha \) is Hukuhara differentiable for all \( \alpha \in [0,1] \) and

\[
[F_\alpha(t)]' = [F'(t)]^\alpha.
\]

Here \( [F_\alpha]' \) denotes the Hukuhara derivative of \( F_\alpha \).

Definition 2.2. ([13]) A mapping \( F: T \to \mathbb{E}^n \) is strongly measurable if for all \( \alpha \in [0,1] \) the set-valued mapping \( F_\alpha\colon T \to P_k(\mathbb{R}^n) \) defined by \( F_\alpha(t) = [F(t)]^\alpha \) is (Lebesgue) measurable, when \( P_k(\mathbb{R}^n) \) is endowed with the topology generated by the Hausdorff metric \( d_H \).
Remark 2.2. ([13]) If \( F : T \to \mathbb{E}^n \) is continuous, then it is measurable with respect to the metric \( D \) and \( F \) is said to be integrably bounded if there exists an integrable function \( h \) such that \( \|x\| \leq h(t) \) for all \( x \in F_0(t) \).

Definition 2.3. ([13]) let \( F : T \to \mathbb{E}^n \). The integral of \( F \) over \( T \), denoted \( \int_T F(t)dt \) or \( \int_d^b F(t)dt \), is defined levelwise by the equation

\[
\left[ \int_T F(t)dt \right] = \int_T F_\alpha(t)dt = \left\{ \int_T f(t)dt/f : T \to \mathbb{R}^n \right\}
\]

where \( f \) is a measurable selection for \( F_\alpha \), for all \( 0 < \alpha \leq 1 \).

A strongly measurable and integrably bounded mapping \( F : T \to \mathbb{E}^n \) is said to be integrable over \( T \) if \( \int_T F(t)dt \in \mathbb{E}^n \).

Remark 2.3. ([13]) If \( \{\alpha_k\} \) is a nonincreasing sequence converging to zero for all \( u \in \mathbb{E}^n \), then

\[
\lim_{k \to \infty} d_H([u]^0, [u]^{\alpha_k}) = 0.
\]

Remark 2.4. ([13]) If \( F : T \to \mathbb{E}^n \) is continuous, then it is integrable.

Theorem 2.3. ([14]) Let \( F, G : T \to \mathbb{E}^n \) be integrable. Then

(i) \( \int F + G = \int F + \int G \),

(ii) \( \int \lambda F = \lambda \int F \), where \( \lambda \in \mathbb{R} \),

(iii) \( \int_a^b F = \int_a^c F + \int_c^b F \), where \( c \in \mathbb{R} \),

(iv) \( D(F,G) \) is integrable,

(v) \( D(\int F, \int G) \leq \int D(F,G) \).

Now, we present some basic notations, definitions and results of time scales. Let \( T \) be a time scale, i.e. an arbitrary nonempty closed subset of real numbers. Since a time scale \( T \) is not connected, we need the concept of jump operators.

Definition 2.4. ([4]) The forward jump operator \( \sigma : T \to T \), the backward jump operator \( \rho : T \to T \), and the graininess \( \mu : T \to \mathbb{R}^+ \) are defined by

\[
\sigma(t) = \inf \{s \in T : s > t\}, \quad \rho(t) = \sup \{s \in T : s < t\}, \quad \mu(t) = \sigma(t) - t \text{ for } t \in T,
\]

respectively. If \( \sigma(t) = t \), \( t \) is called right-dense (otherwise: right-scattered), and if \( \rho(t) = t \), then \( t \) is called left-dense (otherwise: left-scattered). If \( T \) has a left-scattered maximum \( m \), then \( T^k = T - \{m\} \). Otherwise \( T^k = T \).

If \( f : T \to \mathbb{R} \) is a function, then the function \( f^\sigma : T \to \mathbb{R} \) defined by

\[
f^\sigma(t) = f(\sigma(t)), \text{ for all } t \in T.
\]

Definition 2.5. ([4]) Let \( f : T \to \mathbb{R} \) be a function and \( t \in T^k \). Then \( f^\Delta(t) \) be the number(provided it exists) with the property that given any \( \epsilon > 0 \), there is a neighbourhood \( U \) of \( t \) (i.e., \( U = (t - \delta, t + \delta) \cap T \) for some \( \delta > 0 \)) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|, \text{ for all } s \in U.
\]

In this case, \( f^\Delta(t) \) called the delta (or Hilger) derivative of \( f \) at \( t \). Moreover, \( f \) said to be delta (or Hilger) differentiable on \( T \) if \( f^\Delta(t) \) exists for all \( t \in T^k \). The function \( f^\Delta : T^k \to \mathbb{R} \) is then called the delta derivative of \( f \) on \( T^k \).

Definition 2.6. ([4]) A function \( f : T \to \mathbb{R} \) is called regulated provided its right-sided limits exist(finite) at all right dense points in \( T \) and its left-sided limits exist(finite) at all left-dense points in \( T \) and \( F \) is said to be rd-continuous if it is continuous at all right-dense points in \( T \) and its left-sided limits exists(finite) at all left-dense points in \( T \).
The set of all rd-continuous functions \( F : \mathbb{T} \to \mathbb{R} \) is denoted by \( C_{rd}(\mathbb{T}) \). The set of functions \( F : \mathbb{T} \to \mathbb{R} \) that are differentiable and whose derivatives are rd-continuous is denoted by \( C^1_{rd}(\mathbb{T}) \).

**Lemma 2.1.** (4) Assume \( F : \mathbb{T} \to \mathbb{R} \).

(i) If \( F \) is continuous, then \( F \) is rd-continuous.

(ii) If \( F \) is rd-continuous, then \( F \) is regulated.

(iii) The jump operator \( \sigma \) is rd-continuous.

(iv) If \( F \) is regulated or rd-continuous, then so is \( F^\sigma \).

**Definition 2.7.** (4) A continuous function \( f : \mathbb{T} \to \mathbb{R} \) is called pre-differentiable with region of differentiation \( D \), provided \( D \subset \mathbb{T}^k, \mathbb{T}^k / D \) is countable and contains no right-scattered elements of \( \mathbb{T} \), and \( F \) is differentiable at each \( t \in D \). If \( F \) is regulated, and \( f^\Delta(t) = F(t) \) for all \( t \in D \), then the indefinite integral of a regulated function \( F \) is

\[
\int F(t) \Delta(t) = f(t) + C,
\]

where \( C \) is an arbitrary constant and \( f \) is a pre-antiderivative of \( F \). The cauchy integral is given by

\[
\int_r^s F(t) \Delta(t) = f(s) - f(r), \quad \text{for all } r, s \in \mathbb{T}.
\]

**Lemma 2.2.** (4) If \( a, b, c \in \mathbb{T}, a \in \mathbb{R}, \) and \( F, G \in C_{rd} \), then

(i) \( \int_a^b [F(t) + G(t)] \Delta t = \int_a^b F(t) \Delta t + \int_a^b G(t) \Delta t \),

(ii) \( \int_a^b (\alpha F(t)) \Delta t = \alpha \int_a^b F(t) \Delta t \),

(iii) \( \int_a^b F(t) \Delta t = - \int_b^a F(t) \Delta t \),

(iv) \( \int_a^b F(t) \Delta t = \int_a^c F(t) \Delta t + \int_c^b F(t) \Delta t \),

(v) \( \int_a^a F(t) \Delta t = 0 \).

### 3 Differentiation of Fuzzy Set-valued Functions on Time Scales

In this section we define and study the properties of Hukuhara delta derivative (\( \Delta_H \)-derivative) for fuzzy set-valued functions on time scales. To facilitate the discussion below, we introduce some notation: For \( t \in \mathbb{T} \), the neighbourhood \( t \) of \( \mathbb{T} \) is denoted by \( U_\mathbb{T} = U_\mathbb{T}(t, \delta) \) (i.e. \( U_\mathbb{T}(t, \delta) = (t - \delta, t + \delta) \cap \mathbb{T} \) for some \( \delta > 0 \)). In the present section we work in \((\mathbb{E}^n, D)\).

**Definition 3.1.** A fuzzy set-valued function \( F : \mathbb{T} \to \mathbb{E}^n \) has a \( \mathbb{T} \)-limit \( A \in \mathbb{E}^n \) at \( t_0 \in \mathbb{T} \) if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( D(F(t) \ominus_H A, \{0\}) \leq \epsilon \) for all \( t \in U_\mathbb{T}(t_0, \delta) \). If \( F \) has a \( \mathbb{T} \)-limit \( A \in \mathbb{E}^n \) at \( t_0 \in \mathbb{T} \), then it is unique and is denoted by \( \mathbb{T} - \lim_{t \to t_0} F(t) \).

**Remark 3.1.** From the above definition we have,

\[
\mathbb{T} - \lim_{t \to t_0} F(t) = A \in \mathbb{E}^n \iff \mathbb{T} - \lim_{t \to t_0} (F(t) \ominus_H A) = \{0\},
\]

where the limits are in the metric \( D \).

**Definition 3.2.** A fuzzy set-valued mapping \( F : \mathbb{T} \to \mathbb{E}^n \) is continuous at \( t_0 \in \mathbb{T} \), if \( \mathbb{T} - \lim_{t \to t_0} F(t) \in \mathbb{E}^n \) exists and \( \mathbb{T} - \lim_{t \to t_0} F(t) = F(t_0), \) i.e.

\[
\mathbb{T} - \lim_{t \to t_0} (F(t) \ominus_H F(t_0)) = \{0\}.
\]
Remark 3.2. It follows from the above definition that if \( F : \mathbb{T} \to \mathbb{E}^n \) is continuous at \( t_0 \in \mathbb{T} \), then for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
D(F(t) \ominus_H F(t_0), \{0\}) \leq \epsilon, \quad \text{for all } t \in U_T.
\]

Definition 3.3. Assume \( F : \mathbb{T} \to \mathbb{E}^n \) is a fuzzy set-valued function and \( t \in \mathbb{T}^k \). Let \( \Delta_H F(t) \) be an element of \( \mathbb{E}^n \) (provided it exists) with the property that given any \( \epsilon > 0 \), there is a neighbourhood \( U_T \) of \( t \) for any \( \delta > 0 \) such that

\[
D[(F(t + h) \ominus_H F(\sigma(t))), \Delta_H F(t)(h - \mu(t))] \leq \epsilon(h - \mu(t)),
\]

\[
D[(F(\sigma(t)) \ominus_H F(t - h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon(h + \mu(t))
\]

for all \( t - h, t + h \in U_T \) with \( 0 < h < \delta \) where \( \mu(t) = \sigma(t) - t \). We call \( \Delta_H F(t) \), the \( \Delta_H \)-derivative of \( F \) at \( t \). We say that \( F \) is \( \Delta_H \)-differentiable at \( t \) if its \( \Delta_H \)-derivative exists at \( t \). Moreover, we say that \( F \) is \( \Delta_H \)-differentiable on \( \mathbb{T}^k \) if its \( \Delta_H \)-derivative exists at each \( t \in \mathbb{T}^k \). The fuzzy set-valued function \( \Delta_H F : \mathbb{T}^k \to \mathbb{E}^n \) is called the \( \Delta_H \)-derivative of \( F \) on \( \mathbb{T}^k \) or equivalently,

\[
\lim_{h \to 0^+} \frac{F(t + h) \ominus_H F(\sigma(t))}{h - \mu(t)}, \lim_{h \to 0^+} \frac{F(\sigma(t)) \ominus_H F(t - h)}{h + \mu(t)}
\]

exists and equal to \( \Delta_H F(t) \).

Lemma 3.1. If the \( \Delta_H \)-derivative of \( F \) at \( t \) exists, then it is unique.

Proof. Let \( 1_{\Delta_H} F(t) \) and \( 2_{\Delta_H} F(t) \) be the \( \Delta_H \)-derivatives of \( F \) at \( t \). Then

\[
D[1_{\Delta_H} F(t)(h - \mu(t)), F(t + h) \ominus_H F(\sigma(t))] \leq \epsilon(h - \mu(t)),
\]

\[
D[2_{\Delta_H} F(t)(h - \mu(t)), F(t + h) \ominus_H F(\sigma(t))] \leq \epsilon(h - \mu(t)).
\]

Consider

\[
D[1_{\Delta_H} F(t), 2_{\Delta_H} F(t)] = \frac{1}{h - \mu(t)} D[1_{\Delta_H} F(t)(h - \mu(t)), 2_{\Delta_H} F(t)(h - \mu(t))]
\]

\[
\leq \frac{1}{h - \mu(t)} [D[1_{\Delta_H} F(t)(h - \mu(t)), F(t + h) \ominus_H F(\sigma(t))] + D[F(t + h) \ominus_H F(\sigma(t)), 2_{\Delta_H} F(t)(h - \mu(t))]
\]

\[
\leq \epsilon + \epsilon = 2\epsilon
\]

for all \( |h - \mu(t)| \neq 0 \). Since \( \epsilon > 0 \) is arbitrary, then \( D[1_{\Delta_H} F(t), 2_{\Delta_H} F(t)] = 0 \). This implies that \( 1_{\Delta_H} F(t) = 2_{\Delta_H} F(t) \). Hence \( \Delta_H \)-derivative is well defined.

\[\Box\]

Theorem 3.1. Let \( F : \mathbb{T} \to \mathbb{E}^n \) be a fuzzy set-valued function and \( t \in \mathbb{T}^k \). Then we have the following:

(i) If \( F \) is \( \Delta_H \)-differentiable at \( t \), then \( F \) is continuous at \( t \).

(ii) If \( F \) is continuous at \( t \) and \( t \) is right-scattered, then \( F \) is \( \Delta_H \)-differentiable at \( t \) with

\[
\Delta_H F(t) = \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}.
\]

(iii) If \( t \) is right-dense, then \( F \) is \( \Delta_H \)-differentiable at \( t \) iff the limits

\[
\lim_{h \to 0^+} \frac{F(t + h) \ominus_H F(t)}{h}, \lim_{h \to 0^+} \frac{F(t) \ominus_H F(t - h)}{h}
\]

exists as a finite number and satisfy the equations

\[
\lim_{h \to 0^+} \frac{F(t + h) \ominus_H F(t)}{h} = \lim_{h \to 0^+} \frac{F(t) \ominus_H F(t - h)}{h} = \Delta_H F(t).
\]
Moreover, for all $\epsilon > 0$, there exists a neighbourhood $U_T$ of $t$ such that

$$D[(F(t+h) \ominus_H F(t), \Delta_H F(t)(h - \mu(t))] \leq \epsilon_1(h - \mu(t)),$$

$$D[(F((\sigma(t)) \ominus_H F(t-h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon_1(h + \mu(t))$$

for all $0 < h < \delta$ with $t-h, t+h \in U_T$. Therefore, for all $t-h, t+h \in U_T \cap (t - \epsilon_1, t + \epsilon_1)$ and $0 < h < \epsilon_1$, we have

$$D[F(t+h), F(t)] = D[F(t+h) \ominus_H F(\sigma(t)), F(t) \ominus_H F(\sigma(t))]$$

$$\leq D[F(t+h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))]$$

$$+ D[\Delta_H F(t)(h - \mu(t)), \theta] + D[\theta, \Delta_H F(t)(\mu(t))]$$

$$+ D[F(t) \ominus_H F(\sigma(t)), \Delta_H F(t)(\mu(t))]$$

$$+ (h - \mu(t))D[\Delta_H F(t), \theta] + \mu(t)D[\Delta_H F(t), \theta]$$

$$\leq \epsilon_1(h - \mu(t)) + \epsilon_1\mu(t) + hD[\Delta_H F(t), \theta]$$

$$< \epsilon_1(1 + ||\Delta_H F(t)||) = \epsilon.$$
for all $0 < h < \delta$ with $t - h, t + h \in U_T$. Hence
\[
\Delta_H F(t) = \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)}.
\]

(iii) Assume that $F$ is $\Delta_H$-differentiable at $t$ and $t$ is right-dense. Since $F$ is $\Delta_H$-differentiable at $t$, for any given $\epsilon > 0$, there exists a neighbourhood $U_T$ of $t$ such that
\[
D[(F(t + h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \epsilon(h - \mu(t)),
\]
\[
D[(F(\sigma(t)) \ominus_H F(t - h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon(h + \mu(t))
\]
for all $0 < h < \delta$ with $t - h, t + h \in U_T$. Since $\sigma(t) = t$, i.e. $\mu(t) = 0$, we have
\[
D[(F(t + h) \ominus_H F(t), h \Delta_H F(t)] \leq \epsilon h,
\]
\[
D[(F(t) \ominus_H F(t - h), h \Delta_H F(t)] \leq \epsilon h
\]
for all $0 < h < \delta$ with $t - h, t + h \in U_T$. This yields
\[
D \left[ \frac{F(t + h) \ominus_H F(t)}{h}, \Delta_H F(t) \right] \leq \epsilon,
\]
\[
D \left[ \frac{F(t) \ominus_H F(t - h)}{h}, \Delta_H F(t) \right] \leq \epsilon
\]
for all $0 < h < \delta$ with $t - h, t + h \in U_T$. As $\epsilon$ is arbitrary, we have
\[
\Delta_H F(t) = \lim_{h \to 0} \frac{F(t + h) \ominus_H F(t)}{h} = \lim_{h \to 0} \frac{F(t) \ominus_H F(t - h)}{h}.
\]

Conversely, suppose that $t$ is right-dense. For all $0 < h < \delta$ with $t - h, t + h \in U_T$, there is a neighbourhood $U_T$ of $t$ such that
\[
D \left[ \frac{F(t + h) \ominus_H F(t)}{h}, \Delta_H F(t) \right] \leq \epsilon,
\]
\[
D \left[ \frac{F(t) \ominus_H F(t - h)}{h}, \Delta_H F(t) \right] \leq \epsilon.
\]

From these inequalities, $F$ is $\Delta_H$-differentiable at $t$.

(iv) If $\sigma(t) = t$, then $\mu(t) = 0$ and we have that
\[
F(\sigma(t)) = F(t) = F(t) + \mu(t) \Delta_H F(t).
\]

On the other hand if $\sigma(t) > t$, then by (ii)
\[
F(\sigma(t)) = F(t) + \mu(t) \frac{F(\sigma(t)) \ominus_H F(t)}{\mu(t)} = F(t) + \mu(t) \Delta_H F(t).
\]

Example 3.1 We consider the two cases $T = \mathbb{R}$ and $T = \mathbb{Z}$.

(i) If $T = \mathbb{R}$, then from Theorem 3.1 (iii) $F : \mathbb{R} \to \mathbb{E}^n$ is $\Delta_H$-differentiable at $t \in \mathbb{R}$ iff
\[
\Delta_H F(t) = \lim_{h \to 0} \frac{F(t + h) \ominus_H F(t)}{h} = \lim_{h \to 0} \frac{F(t) \ominus_H F(t - h)}{h} = F'(t).
\]
(ii) If $T = Z$, then from Theorem 3.2, it directly follows that if $F : Z \rightarrow \mathbb{E}^n$ is $\Delta_H$-differentiable at $t \in Z$ and

$$\Delta_H F(t) = \frac{F(\sigma(t) \ominus_H F(t))}{\mu(t)} = F(t + 1) \ominus_H F(t) = \Delta F(t),$$

where $\Delta$ is the forward Hukuhara difference operator.

**Theorem 3.2.** Let $F : T \rightarrow \mathbb{E}^n$ be the fuzzy set-valued function and denote $[F(t)]^\alpha = F_\alpha(t)$, for each $\alpha \in [0, 1]$. If $F$ is $\Delta_H$-differentiable, then $F_\alpha$ is also $\Delta_H$-differentiable and

$$\Delta_H [F(t)]^\alpha = \Delta_H F_\alpha(t).$$

**Proof.** If $F$ is $\Delta_H$-differentiable at $t \in T^k$ and $t$ is right-scattered, then for any $\alpha \in [0, 1]$, we get

$$[F(t + h) \ominus_H F(\sigma(t))]^\alpha = [F_\alpha(t + h) \ominus_H F_\alpha(\sigma(t))],$$

dividing by $h - \mu(t) > 0$ and taking the limit $h \rightarrow 0^+$, we have

$$\lim_{h \rightarrow 0^+} \frac{1}{h - \mu(t)} [F_\alpha(t + h) \ominus_H F_\alpha(\sigma(t))] = \Delta_H F_\alpha(t).$$

Similarly,

$$\lim_{h \rightarrow 0^+} \frac{1}{h + \mu(t)} [F_\alpha(\sigma(t)) \ominus_H F_\alpha(t - h)] = \Delta_H F_\alpha(t).$$

If $F$ is $\Delta_H$-differentiable at $t \in T^k$ and $t$ is right-dense, then for $\alpha \in [0, 1]$, we get

$$[F(t + h) \ominus_H F(t)]^\alpha = [F_\alpha(t + h) \ominus_H F_\alpha(t)],$$

dividing by $h > 0$ and taking the limit $h \rightarrow 0^+$, we have

$$\lim_{h \rightarrow 0^+} \frac{1}{h} [F_\alpha(t + h) \ominus_H F_\alpha(t)] = \Delta_H F_\alpha(t).$$

Similarly, $\lim_{h \rightarrow 0^+} [F_\alpha(t) \ominus_H F_\alpha(t - h)]/h = \Delta_H F_\alpha(t).$ \qed

**Remark 3.1.** From the above Theorem 3.2, it directly follows that if $F$ is $\Delta_H$-differentiable then the multi-valued mapping $F_\alpha$ is $\Delta_H$-differentiable for all $\alpha \in [0, 1]$, but the converse result doesn’t hold. Since the existence of $H$-differences $[x]^\alpha \ominus_H [y]^\alpha$, $\alpha \in [0, 1]$, does not imply the existence of $H$-difference $x \ominus_H y$.

However, for the converse result we have the following:

**Theorem 3.3.** Let $F : T \rightarrow \mathbb{E}^n$ satisfy the assumptions:

(i) for each $t \in T$ there exists a $\beta > 0$ such that the $H$-differences $F(t + h) \ominus_H F(\sigma(t))$ and $F(\sigma(t)) \ominus_H F(t - h)$ exists for all $0 < h < \beta$ and for all $t - h, t + h \in U_T$;

(ii) the set-valued mappings $F_\alpha$, $\alpha \in [0, 1]$, are uniformly $\Delta_H$-differentiable with derivative $\Delta_H F_\alpha$, i.e., for each $t \in T$ and $\epsilon > 0$ there exists a $\delta > 0$ such that

$$D \left\{ \frac{F_\alpha(t + h) \ominus_H F_\alpha(\sigma(t))}{h - \mu(t)}, \Delta_H F_\alpha(t) \right\} < \epsilon\right.$$

and

$$D \left\{ \frac{F_\alpha(\sigma(t)) \ominus_H F_\alpha(t - h)}{h + \mu(t)}, \Delta_H F_\alpha(t) \right\} < \epsilon\right.$$\] for all $0 < h < \delta$, $t - h, t + h \in U_T$, $\alpha \in [0, 1]$.

Then $F$ is $\Delta_H$-differentiable and the derivative is given by $\Delta_H F_\alpha(t) = \Delta_H [F(t)]^\alpha$. 

Proof. Consider the family \( \{ \Delta_H F_\alpha(t), \alpha \in [0, 1] \} \). By definition \( \Delta_H F_\alpha(t) \) is a compact, convex and non-empty subset of \( \mathbb{R}^n \). If \( \alpha_1 \leq \alpha_2 \) then by assumption (i),
\[
F_{\alpha_1}(t + h) \ominus_H F_{\alpha_1}(\sigma(t)) \supset F_{\alpha_2}(t + h) \ominus_H F_{\alpha_2}(\sigma(t))
\]
for \( 0 < h < \beta \) and for all \( t - h, t + h \in U_T \) and consequently \( \Delta_H F_{\alpha_1}(t) \supset \Delta_H F_{\alpha_2}(t) \).

Let \( \alpha > 0 \) and \( \{ \alpha_k \} \) be a nondecreasing sequence converging to \( \alpha \). For \( \epsilon > 0 \) choose \( h > 0 \) such that the equation (2) holds true. Then consider
\[
D(\Delta_H F_{\alpha}(t), \Delta_H F_{\alpha_k}(t)) 
\]
\[
\leq D \left( \Delta_H F_{\alpha}(t), \frac{F_{\alpha}(t + h) \ominus_H F_{\alpha}(\sigma(t))}{h - \mu(t)} \right) + D \left( \frac{F_{\alpha}(t + h) \ominus_H F_{\alpha}(\sigma(t))}{h - \mu(t)}, \Delta_H F_{\alpha_k}(t) \right) 
\]
\[
< \epsilon + \frac{1}{h - \mu(t)} D[F_{\alpha}(t + h) \ominus_H F_{\alpha}(\sigma(t)), F_{\alpha_k}(t + h) \ominus_H F_{\alpha_k}(\sigma(t))] 
\]
\[
+ \frac{1}{h - \mu(t)} D[F_{\alpha_k}(t + h) \ominus_H F_{\alpha_k}(\sigma(t)), \Delta_H F_{\alpha_k}(t)] 
\]
\[
< 2\epsilon + \frac{1}{h - \mu(t)} D[F_{\alpha}(t + h) \ominus_H F_{\alpha}(\sigma(t)), F_{\alpha_k}(t + h) \ominus_H F_{\alpha_k}(\sigma(t))].
\]

By assumption (i), the rightmost term goes to zero as \( k \to \infty \) and hence
\[
\lim_{k \to \infty} D(\Delta_H F_{\alpha}(t), \Delta_H F_{\alpha_k}(t)) = 0.
\]

From Theorem 2.2, we have
\[
\Delta_H F_{\alpha}(t) = \bigcap_{k \geq 1} \Delta_H F_{\alpha_k}(t).
\]

If \( \alpha = 0 \), we deduce as before
\[
\lim_{k \to \infty} D(\Delta_H F_0(t), \Delta_H F_{\alpha_k}(t)) = 0,
\]

where \( \{ \alpha_k \} \) be a nondecreasing sequence converging to zero, and consequently
\[
\Delta_H F_0(t) = \text{cl} \left( \bigcup_{k \geq 1} \Delta_H F_{\alpha_k}(t) \right).
\]

Then from Theorem 2.1 it follows that there is an element \( u \in \mathbb{E}^n \) such that
\[
[u]^\alpha = \Delta_H F_\alpha(t), \quad \alpha \in [0, 1].
\]

Furthermore, let \( t \in \mathbb{T}, \epsilon > 0, \delta > 0 \) and \( t - h, t + h \in U_T \) be as in assumption (ii). Then
\[
D \left( \frac{F_{\alpha}(t + h) \ominus_H F_{\alpha}(\sigma(t))}{h - \mu(t)}, u_\alpha \right) = D \left( \frac{F_{\alpha}(t + h) \ominus_H F_{\alpha}(\sigma(t))}{h - \mu(t)}, \Delta_H F_{\alpha}(t) \right) < \epsilon
\]
for \( 0 < h < \delta \) and \( t - h, t + h \in U_T \).

Similarly
\[
D \left( \frac{F_{\alpha}(\sigma(t)) \ominus_H F_{\alpha}(t - h)}{h - \mu(t)}, u_\alpha \right) < \epsilon.
\]

Hence \( F \) is \( \Delta_H \)-differentiable.

**Theorem 3.4.** Let \( F : \mathbb{T} \to \mathbb{E}^1 \) be \( \Delta_H \)-differentiable on \( \mathbb{T} \). Denote \( F_\alpha(t) = [f_\alpha(t), g_\alpha(t)], \alpha \in [0, 1] \). Then \( f_\alpha \) and \( g_\alpha \) are \( \Delta_H \)-differentiable on \( \mathbb{T} \) and
\[
[\Delta H F(t)]^\alpha = [\Delta H f_\alpha(t), \Delta H g_\alpha(t)].
\]
Proof. Given $F$ is \( \Delta_H \)-differentiable at $t \in \mathbb{T}^k$ and $t$ is right-scattered, then for any $\alpha \in [0, 1]$, consider
\[
[F(t+h) \ominus_H F(\sigma(t))]^\alpha = [f_\alpha(t+h) \ominus_H f_\alpha(\sigma(t)), g_\alpha(t+h) \ominus_H g_\alpha(\sigma(t))]
\]
and dividing by $h - \mu(t)$ and taking limit as $h \to 0^+$, we get
\[
\lim_{h \to 0^+} \frac{1}{\mu(t)} [F(t+h) \ominus_H F(\sigma(t))]^\alpha = 1 \lim_{h \to 0^+} \frac{f_\alpha(t+h) \ominus_H f_\alpha(\sigma(t))}{h} \lim_{h \to 0^+} \frac{g_\alpha(t+h) \ominus_H g_\alpha(\sigma(t))}{h} = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].
\]
Similarly,
\[
\lim_{h \to 0^+} \frac{1}{\mu(t)} [F(\sigma(t)) \ominus_H F(t-h)]^\alpha = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].
\]
If $F$ is \( \Delta_H \)-differentiable at $t \in \mathbb{T}^k$ and $t$ is right-dense, then for any $\alpha \in [0, 1]$, consider
\[
[F(t+h) \ominus_H F(t)]^\alpha = [f_\alpha(t+h) \ominus_H f_\alpha(t), g_\alpha(t+h) \ominus_H g_\alpha(t)]
\]
and dividing by $h > 0$ and taking limit as $h \to 0^+$, we get
\[
\lim_{h \to 0^+} \frac{1}{h} [F(t+h) \ominus_H F(t)]^\alpha = 1 \lim_{h \to 0^+} \frac{f_\alpha(t+h) \ominus_H f_\alpha(t)}{h} \lim_{h \to 0^+} \frac{g_\alpha(t+h) \ominus_H g_\alpha(t)}{h} = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].
\]
Similarly,
\[
\lim_{h \to 0^+} \frac{1}{h} [F(\sigma(t)) \ominus_H F(t-h)]^\alpha = [\Delta_H f_\alpha(t), \Delta_H g_\alpha(t)].
\]

\[\square\]

**Remark 3.4** Let $F : \mathbb{T} \to \mathbb{E}^n$ be a fuzzy set-valued function. If $F$ is \( \Delta_H \)-differentiable on \( \mathbb{T}^k \), then there exists $\delta > 0$ such that for $\alpha \in [0, 1]$, we have
\[
diam[F(t_0-h)]^\alpha \leq diam[F(\sigma(t))]^\alpha \leq diam[F(t_0+h)]^\alpha, \quad \text{for } 0 < h < \delta,
\]
for each $\alpha \in [0, 1]$, the real function $t \to diam[F(t)]^\alpha$ is nondecreasing on \( \mathbb{T}^k \).

**Theorem 3.5.** Let $F : \mathbb{T} \to \mathbb{E}^n$ be \( \Delta_H \)-differentiable and nondecreasing on \( \mathbb{T}^k \). If $t_1, t_2 \in \mathbb{T}^k$ with $t_1 \leq t_2$, then there exists a $C \in \mathbb{E}^n$ such that $F(t_2) \ominus_H F(t_1) = C$.

Proof. For each $w \in [t_1, t_2]$ there exists a $\delta(w) > 0$ such that the $H$-differences $F(w+h) \ominus_H F(\sigma(w))$ and $F(\sigma(w)) \ominus_H F(w-h)$ exist for all $0 < h < \delta(w)$. Then we can find a finite sequence $t_1 = w_1 < w_2 < w_3 < \cdots < w_n = t_2$ such that the family $\{I_{w_i} = (w_i - \delta(w_i), w_i + \delta(w_i))/i = 1, 2, \ldots, n\}$, covers $[t_1, t_2]$ and $I_{w_i} \cap I_{w_{i+1}} \neq \emptyset$. Select a $s_i \in I_{w_i} \cap I_{w_{i+1}}$, $i = 1, 2, \ldots, n-1$, such that $w_i < s_i < w_{i+1}$. Then
\[
F(w_{i+1}) = F(s_i) + k_1 = F(w_i) + k_2 + k_1 = F(w_i) + B_i, \quad i = 1, 2, \ldots, n-1,
\]
for some $k_1, k_2, B_i \in \mathbb{E}^n$. Hence,
\[
F(t_2) = F(t_1) + \sum_{i=1}^{n-1} B_i = F(t_1) + C,
\]
which implies that $F(t_2) \ominus_H F(t_1) = C$. \[\square\]

In the following theorem we obtain the \( \Delta_H \)-derivatives of sums and scalar products of \( \Delta_H \)-differentiable functions on time scales in \((\mathbb{E}^n, D)\).
Theorem 3.6. Let \(F, G : \mathbb{T} \to \mathbb{E}^n\) be \(\Delta_H\)-differentiable at \(t \in \mathbb{T}^k\). Then,

(i) the sum \(F + G : \mathbb{T} \to \mathbb{E}^n\) is \(\Delta_H\)-differentiable at \(t \in \mathbb{T}^k\) with
\[
\Delta_H(F + G)(t) = \Delta_H F(t) + \Delta_H G(t),
\]

(ii) for any constant \(\lambda, \lambda F : \mathbb{T} \to \mathbb{E}^n\) is \(\Delta_H\)-differentiable at \(t\) with
\[
\Delta_H(\lambda F)(t) = \lambda \Delta_H F(t),
\]

(iii) the product \(FG : \mathbb{T} \to \mathbb{E}^n\) is \(\Delta_H\)-differentiable at \(t \in \mathbb{T}^k\) with
\[
\Delta_H(FG)(t) = F(\sigma(t))\Delta_H G(t) + G(t)\Delta_H F(t) = F(t)\Delta_H G(t) + G(\sigma(t))\Delta_H F(t).
\]

Proof. Let \(F\) and \(G\) be \(\Delta_H\)-differentiable at \(t \in \mathbb{T}^k\).

(i) Let \(\epsilon > 0\), then there exists neighbourhoods \(U_1\), and \(U_2\) of \(t\) with
\[
D[(F(t + h) \ominus H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \frac{\epsilon}{2}(h - \mu(t)),
\]
\[
D[(\sigma(t)) \ominus H F(t - h), \Delta_H F(t)(h + \mu(t))] \leq \frac{\epsilon}{2}(h + \mu(t))
\]
for \(0 < h < \delta\) with \(t - h, t + h \in U_1\) and
\[
D[(G(t + h) \ominus H G(\sigma(t)), \Delta_H G(t)(h - \mu(t))] \leq \frac{\epsilon}{2}(h - \mu(t)),
\]
\[
D[(\sigma(t)) \ominus H G(t - h), \Delta_H G(t)(h + \mu(t))] \leq \frac{\epsilon}{2}(h + \mu(t))
\]
for \(0 < h < \delta\) with \(t - h, t + h \in U_2\). Let \(U = U_1 \cap U_2\), then for \(0 < h < \delta\) with \(t - h, t + h \in U\), we have
\[
D[(F + G)(t + h) \ominus H (F + G)(\sigma(t)), (\Delta_H F(t) + \Delta_H G(t))(h - \mu(t))] \leq \frac{\epsilon}{2}(h - \mu(t)) + \frac{\epsilon}{2}(h + \mu(t)) = \epsilon(h - \mu(t)).
\]
Therefore, \(F + G\) is \(\Delta_H\)-differentiable at \(t\) and
\[
\Delta_H(F + G)(t) = \Delta_H F(t) + \Delta_H G(t).
\]

(ii) For \(\lambda = 0\), the result is trivial. We assume that \(\lambda > 0\). Since \(F\) is \(\Delta_H\)-differentiable at \(t \in \mathbb{T}^k\) there exists a neighbourhood \(U_T\) of \(t\) such that, for given \(\epsilon > 0\), we have
\[
D[(F(t + h) \ominus H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \frac{\epsilon}{\lambda}(h - \mu(t)),
\]
\[
D[(\sigma(t)) \ominus H F(t - h), \Delta_H F(t)(h + \mu(t))] \leq \frac{\epsilon}{\lambda}(h + \mu(t))
\]
for \(0 < h < \delta\) with \(t - h, t + h \in U_T\) and
\[
D[\lambda F(t + h) \ominus H \lambda F(\sigma(t)), \lambda \Delta_H F(t)(h - \mu(t))] = \lambda D[(F(t + h) \ominus H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \frac{\epsilon}{\lambda}(h - \mu(t)) = \epsilon(h - \mu(t)).
\]
Similarly, we get
\[
D[\lambda F(t - h) \ominus H \lambda F(\sigma(t)), \lambda \Delta_H F(t)(h + \mu(t))] \leq \epsilon(h + \mu(t))
\]
for \(0 < h < \delta\) with \(t - h, t + h \in U_T\). Therefore \(\lambda F\) is \(\Delta_H\)-differentiable at \(t\) and \(\Delta_H(\lambda F)(t) = \lambda \Delta_H F(t)\).
(iii) Let \( \epsilon \in (0, 1) \) and define \( \epsilon^* = \epsilon [1 + ||F(\sigma(t))|| + ||G(t)|| + ||\Delta_H F(t)||]^{-1} \), then \( \epsilon^* \in (0, 1) \). Thus, there exist neighbourhoods \( U_1, U_2, U_3 \) of \( t \) such that

\[
D[(F(t + h) \ominus_H F(\sigma(t)), \Delta_H F(t)(h - \mu(t))] \leq \epsilon^*(h - \mu(t)),
\]

\[
D[(F((\sigma(t)) \ominus_H F(t - h), \Delta_H F(t)(h + \mu(t))] \leq \epsilon^*(h + \mu(t))
\]

for \( 0 < h < \delta \) with \( t - h, t + h \in U_1 \) and

\[
D[(G(t + h) \ominus_H G(\sigma(t)), \Delta_H G(t)(h - \mu(t))] \leq \epsilon^*(h - \mu(t))
\]

\[
D[(G((\sigma(t)) \ominus_H G(t - h), \Delta_H G(t)(\mu(t) + h)] \leq \epsilon^*(h + \mu(t))
\]

for \( 0 < h < \delta \) with \( t - h, t + h \in U_2 \). Since \( G \) is \( \Delta_H \)-differentiable at \( t \), it follows from Theorem 3.1 that \( G \) is continuous at \( t \). Hence

\[
D[G(t + h), G(t)] \leq \epsilon^*, D[G(t), G(t - h)] \leq \epsilon^*,
\]

\[
D(G(t + h)) \leq D(G(t)) + 1, D(G(t - h)) \leq D(G(t)) + 1
\]

for \( 0 < h < \delta \) with \( t - h, t + h \in U_3 \). Let \( U_T = U_1 \cap U_2 \cap U_3 \) and \( t - h, t + h \in U_T \), then we have

\[
D[(FG)(t + h) \ominus_H (FG)(\sigma(t)), (F(\sigma(t)))\Delta_H G(t) + G(t)\Delta_H F(t)(h - \mu(t))] \leq D[(G(t + h) \ominus_H G(\sigma(t)))F(\sigma(t)), \Delta_H G(t)(h - \mu(t))F(\sigma(t))]
\]

\[
+ D[(F(t + h) \ominus_H F(\sigma(t)))G(t + h), \Delta_H F(t)(h - \mu(t))G(t + h)]
\]

\[
+ D[\theta, \Delta_H F(t)(h - \mu(t))(G(t) - G(t + h))]
\]

\[
\leq D[(G(t + h) \ominus_H G(\sigma(t))), \Delta_H G(t)(h - \mu(t))](F(\sigma(t)))
\]

\[
+ D[(F(t + h) \ominus_H F(\sigma(t))), \Delta_H F(t)(h - \mu(t))](G(t + h))
\]

\[
+ D[G(t + h), G(t)]\|\Delta_H F(t)\|(h - \mu(t))
\]

\[
\leq \epsilon^*(h - \mu(t))\|F(\sigma(t))\| + \epsilon^*(h - \mu(t))\|G(t)\| + 1
\]

\[
+ \epsilon^*(h - \mu(t))\|\Delta_H F(t)\|(h - \mu(t)) = \epsilon(h - \mu(t)).
\]

Similarly, we get

\[
D[(FG)(\sigma(t)) \ominus_H (FG)(t - h), (F(\sigma(t)))\Delta_H G(t) + G(t)\Delta_H F(t)(\mu(t) + h)] \leq \epsilon(h + \mu(t)).
\]

Thus \( \Delta_H (FG)(t) = F(\sigma(t))\Delta_H G(t) + G(t)\Delta_H F(t) \) holds at \( t \). The other product rule in (iii) of this theorem follows from this last equation by interchanging the functions \( F \) and \( G \).

\[\Box\]

4 Integration of Fuzzy Set-valued Functions on Time Scales

In this section we introduced \( \Delta_H \)-integral for fuzzy set-valued functions and also studied their properties.

**Definition 4.1.** Let \( I \subset \mathbb{T} \). A function \( f : I \to \mathbb{R} \) is called a \( \Delta \)-measurable sector of the fuzzy set-valued function \( F : I \to \mathbb{E}^n \) if \( f(t) \in F(t) \) for all \( t \in I \).

**Definition 4.2.** A fuzzy set-valued function \( F : \mathbb{T} \to \mathbb{E}^n \) is said to be regulated provided its regulated \( \Delta \)-measurable sectors exist. A fuzzy set-valued function \( F : \mathbb{T} \to \mathbb{E}^n \) is said to be rd-continuous provided its rd-continuous \( \Delta \)-measurable sectors exist.

In this paper, the set of rd-continuous fuzzy set-valued functions \( F : I \subset \mathbb{T} \to \mathbb{E}^n \) is denoted by

\[
C_{frd} = C_{frd}(I) = C_{frd}(I, \mathbb{E}^n).
\]

The set of fuzzy set-valued functions \( F : I \subset \mathbb{T} \to \mathbb{E}^n \) which are \( \Delta_H \)-differentiable and whose \( \Delta_H \)-derivative is rd-continuous is denoted by

\[
C'_{frd} = C'_{frd}(I) = C'_{frd}(I, \mathbb{E}^n).
\]
Definition 4.3. A continuous fuzzy set-valued function $F : \mathbb{T} \to E^n$ is said to be pre-differentiable with (region of differentiation) $D$, provided $D \subseteq \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and $F$ is $\Delta_H$-differentiable at each $t \in D$.

Remark 4.1. It is evident that $F$ is regulated provided $F$ is rd-continuous, i.e. there exists a fuzzy set-valued function $f$ which is pre-differentiable with region of differentiation $D$ such that $\Delta_H f(t) = F(t)$ for all $t \in D$.

Definition 4.4. A fuzzy set-valued function $F : \mathbb{T} \to E^n$ is said to be $\Delta_H$-integrable on $I \subset \mathbb{T}$ if $F$ has a rd-continuous $\Delta$-measurable sector on $I$. In this case, we define the $\Delta_H$-integral of $F$ on $I$, denoted by $\int_I F(s) \Delta s$, and defined levelwise by the equation

$$\left[ \int_I F(s) \Delta s \right] = \int_I F_\alpha(s) \Delta s = \left\{ \int_I f(s) \Delta s : f \in S_{F_\alpha}(I) \right\},$$

where $S_{F_\alpha}(I)$, the set of all $\Delta_H$-integrable sectors of $F_\alpha$ on $I$.

Theorem 4.1. Let $t_0, T \in \mathbb{T}$ and $F, G : [t_0, T]_\mathbb{T} \to E^n$ be $\Delta_H$-integrable and have rd-continuous $\Delta$-measurable sectors, then we have

(i) $\int_{t_0}^T [F(s) + G(s)] \Delta s = \int_{t_0}^T F(s) \Delta s + \int_{t_0}^T G(s) \Delta s$.

(ii) $\int_{t_0}^T \lambda F(s) \Delta s = \lambda \int_{t_0}^T F(s) \Delta s, \quad \lambda \in \mathbb{R}^+.$

(iii) $\int_{t_0}^T F(s) \Delta s = \int_{t_0}^T F(s) \Delta s + \int_{t_0}^T F(s) \Delta s$.

(iv) $\int_{t_0}^T (F(s) + G(s)) \Delta s = \int_{t_0}^T D(F(s), \Delta(s)) \Delta s.$

(vi) If $f \in S_F([t_0, T]_\mathbb{T})$, then $D(F(\cdot), \theta) : [t_0, T]_\mathbb{T} \to \mathbb{R}^+$ is $\Delta_H$-integrable and

$$D \left( \int_{t_0}^T F(s) \Delta s, \theta \right) \leq \int_{t_0}^T D(F(s), \theta) \Delta s.$$
On the other hand, let \( z = \int_{t_0}^{t} g_1(s) \Delta s + \int_{t}^{T} g_2(s) \Delta s \), where \( g_1 \) is a \( \Delta \)-measurable sector for \( F_\alpha \) in \([t_0, t]\) and \( g_2 \) is a \( \Delta \)-measurable sector for \( F_\alpha \) in \([t, T]\). Then \( f \) defined by

\[
f(t) = \begin{cases} 
g_1(t), & \text{if } t \in [t_0, t] 
g_2(t), & \text{if } t \in (t, T] 
\end{cases}
\]

is a \( \Delta \)-measurable sector for \( F_\alpha \) in \([t_0, T]\) and

\[
\int_{t_0}^{T} f(s) \Delta s = \int_{t_0}^{t} g_1(s) \Delta s + \int_{t}^{T} g_2(s) \Delta s = z.
\]

Thus,

\[
\int_{t_0}^{T} F_\alpha(s) \Delta s + \int_{t}^{T} F_\alpha(s) \Delta s \subset \int_{t_0}^{T} F_\alpha(s) \Delta s.
\]

(iv) Let \( \alpha \in [0, 1] \) and \( f \) be a \( \Delta \)-measurable sector for \( F_\alpha \). From Lemma 2.2 (v), \( \int_{t_0}^{t} f(s) \Delta s = \{0\} \), then we have

\[
\int_{t_0}^{t} F_\alpha(s) \Delta s = \int_{t_0}^{t} f(s) \Delta s = 0.
\]

(vi) It is sufficient to prove (vi) because (v) is a particular case when \( G(t) = \theta \) in (v). For given \( f \in C_{frd} \), we have the inequalities

\[
d(f_\alpha(s), G_\alpha(s)) \leq d(f_\alpha(s), g_\alpha(s)) \leq d(f_\alpha(s), f_\alpha(\tau)) + d(f_\alpha(\tau), g_\alpha(s))
\]

for each \( g_\alpha(s) \in G_\alpha(s) \), which implies

\[
d(f_\alpha(s), G_\alpha(s)) \cap H d(f_\alpha(s), f_\alpha(\tau)) \leq \inf d(f_\alpha(\tau), g_\alpha(s)) = d(f_\alpha(\tau), G_\alpha(s)).
\]

Therefore,

\[
d(f_\alpha(s), G_\alpha(s)) \cap H d(f_\alpha(s), G_\alpha(s)) \leq d(f_\alpha(s), f_\alpha(\tau)).
\]

The same inequality holds with \( s \) and \( \tau \) are interchanged and rd-continuity of \( d_H(f_\alpha(\cdot), G_\alpha(s)) \) at \( s \in [t_0, T]\) follows for each \( f_\alpha \in C_{frd} \). Thus,

\[
D(F(t), G(t)) = \sup_{k \geq 1} d_H(F_{\alpha_k}(t), G_{\alpha_k}(t))
\]

is rd-continuous, which yields that the integral \( \int_{t_0}^{T} D(F(s), G(s)) \Delta s \) is well defined. Hence for any \( u \in \int_{t_0}^{T} F(s) \Delta s \), there exists a \( \Delta \)-measurable sector \( f \in S_F([t_0, T]\), and for any \( v \in \int_{t_0}^{T} G(s) \Delta s \), there exists a \( \Delta \)-measurable sector \( g \in S_G([t_0, T]\), such that

\[
u = \int_{t_0}^{T} f(s) \Delta s, \quad u = \int_{t_0}^{T} g(s) \Delta s.
\]

From [13], we have

\[
d_H \left( \int F_\alpha, \int G_\alpha \right) \leq \int d_H(F_\alpha, G_\alpha),
\]

and consequently

\[
D(u, v) = D \left( \int F, \int G \right) \leq \sup_{\alpha \in [0, 1]} \int d_H(F_\alpha, G_\alpha)
\]

\[
\leq \int \sup_{\alpha \in [0, 1]} d_H(F_\alpha, G_\alpha) = \int D(F, G).
\]

\[\square\]
Theorem 4.2. Let $I_T = I \cap \mathbb{T}$, where $I \subset \mathbb{R}$ be an interval and $F : I_T \rightarrow \mathbb{E}^n$ is rd-continuous. If $t_0 \in \mathbb{T}$, then $f$ defined by

$$f(t) = X_0 + \int_{t_0}^{t} F(s) \Delta s,$$

for $t \in I_T$ and $X_0 \in \mathbb{E}^n$ is $\Delta_H$-differentiable and $\Delta_H f(t) = F(t)$ a.e. on $I_T$.

Proof. Let $t \in D$, since $f$ is $\Delta_H$-differentiable, then $\Delta_H f(t)$ exists and

$$\Delta_H f(t) = F(t) \text{ a.e. on } D.$$

If $t \in I_T \setminus D$, then $t$ is a right-dense point of $\mathbb{T}$. For any $h \geq 0$ with $t - h, t + h \in I_T$, we have

$$f(t + h) - f(\sigma(t)) = \int_{t_0}^{t+h} F(s) \Delta s - \int_{t_0}^{\sigma(t)} F(s) \Delta s = \int_{\sigma(t)}^{t+h} F(s) \Delta s.$$

Let $\epsilon > 0$ be arbitrary, by the rd-continuity of $F$ we have

$$D \left( \frac{f(t + h) - f(\sigma(t))}{h - \mu(t)}, F(t) \right) = \frac{1}{h - \mu(t)} \left[ D(F(t + h) - f(\sigma(t)), (h - \mu(t))F(t)) \right]$$

$$= \frac{1}{h - \mu(t)} \left( \int_{\sigma(t)}^{t+h} F(s) \Delta s, \int_{\sigma(t)}^{t+h} F(t) \Delta s \right)$$

$$\leq \frac{1}{h - \mu(t)} \int_{\sigma(t)}^{t+h} D(F(s), F(t)) \Delta s < \epsilon$$

for all $t - h, t + h \in I_T$. The integral on the right hand side tends to zero as $h \to 0$, as $t$ is right dense. Hence

$$\lim_{h \to 0} \frac{f(t + h) - f(\sigma(t))}{h - \mu(t)} = F(t)$$

and similarly

$$\lim_{h \to 0} \frac{f(\sigma(t)) - f(t - h)}{h - \mu(t)} = F(t).$$

Remark 4.2 A fuzzy set-valued function $f : \mathbb{T} \rightarrow \mathbb{E}^n$ is called a $\Delta_H$-antiderivative of $F : \mathbb{T} \rightarrow \mathbb{E}^n$ provided $\Delta_H f(t) = F(t)$ for all $t \in \mathbb{T}^k$. From Theorem 4.2 every rd-continuous fuzzy valued function $F$ has a $\Delta_H$-antiderivative $f$. Thus, for $t_0 \in \mathbb{T}$,

$$f(t) = f(t_0) + \int_{t_0}^{t} F(s) \Delta s, \text{ for } t \in \mathbb{T}^k.$$

5 Fuzzy Differential Equations on Time Scales

In this section we consider a fuzzy initial value problem on time scales

$$y^\Delta = F(t, y), \quad y(t_0) = y_0,$$

where $F : \mathbb{T} \times \mathbb{E}^n \rightarrow \mathbb{E}^n$ is a given function and for $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{E}^n$, is called an initial value problem. A function $y : \mathbb{T} \rightarrow \mathbb{R}$ is called a solution of this IVP if

$$y^\Delta(t) = F(t, y(t))$$

is satisfied for all $t \in \mathbb{T}^k$ with $y(t_0) = y_0$. From Theorem 4.2 it immediately follows:
Lemma 5.1 A mapping \( y : \mathbb{T} \to E^n \) is called a solution to the IVP if and only if it is rd-continuous and satisfies the integral equation
\[
y(t) = y_0 + \int_{t_0}^t F(s, y(s)) \Delta s, \quad \text{for all } t \in \mathbb{T}^k.
\] (4)

If \( F \) is Lipschitz continuous then the problem has a unique solution on \( \mathbb{T} \). Further, the solution depends continuously on the initial value.

**Theorem 5.1.** Let \( F : \mathbb{T} \times E^n \to E^n \) be rd-continuous and assume that there exists a \( K > 0 \) such that
\[
D(F(t, x), F(t, y)) \leq KD(x, y)
\]
for all \( t, x, y \in E^n \). Then the problem has a unique solution on \( \mathbb{T} \).

*Proof.* Let \( C_{frd}(J, E^n) \) be the set of all rd-continuous fuzzy mappings from \( J \) to \( E^n \), where \( J \) is an interval in \( \mathbb{T} \). Define a metric \( H \) on \( C_{frd}(J, E^n) \) by
\[
H(\xi, \psi) = \sup_{t \in J} D(\xi(t), \psi(t))
\]
for \( \xi, \psi \in C_{frd}(J, E^n) \). Since \( (E^n, D) \) is a complete metric space, in a similar way it is easy to show that \( C_{frd}(J, E^n), H) \) is complete. Let \((t_1, y_1) \in \mathbb{T} \times E^n \) be arbitrary and \( \eta > 0 \) be such that \( \eta k < 1 \). Now we will show that the IVP
\[
y(t) = F(t, y(t)), \quad y(t_1) = y_1 \tag{5}
\]
has a unique solution on the interval \( I = [t_1, t_1 + \eta] \), by using Banach contraction principle. For \( \xi \in C_{rd}(I, E^n) \), define \( G \xi \) on \( I \) by the equation
\[
G \xi(t) = y_1 + \int_{t_1}^t F(s, \xi(s)) \Delta s.
\]

Now, we will show \( G \xi \in C_{frd}(I, E^n) \) by using rd-continuity of \( F \) and (v) of Theorem 4.1 Let \( t_0, t \in I \) and assume that \( t_0 < t \).

Consider
\[
D(G \xi(t), G \xi(t_0)) = D \left( \int_{t_1}^t F(s, \xi(s)) \Delta s, \int_{t_1}^{t_0} F(s, \xi(s)) \Delta s \right)
\]
\[
= D \left( \int_{t_1}^t F(s, \xi(s)) \Delta s, \int_{t_1}^{t_0} F(s, \xi(s)) \Delta s \right)
\]
\[
= \int_{t_0}^t D(F(s, \xi(s)), F(s, \xi(s))) \Delta s
\]
\[
= \int_{t_0}^t \|F(s, \xi(s))\| \Delta s
\]
\[
\leq M(t - t_0).
\]

Hence \( G \xi \in C_{frd}(I, E^n) \). Consider
\[
H(G \xi, G \psi) = \sup_{t \in I} D \left( \int_{t_1}^t F(s, \xi(s)) \Delta s, \int_{t_1}^t F(s, \psi(s)) \Delta s \right)
\]
\[
\leq \int_{t_1}^{t_1 + \eta} D(F(s, \xi(s)), F(s, \psi(s))) \Delta s
\]
\[
\leq \int_{t_1}^{t_1 + \eta} kD(\xi(s), \psi(s)) \Delta s
\]
\[
\leq \eta kH(\xi, \psi)
\]
for \( \xi, \psi \in C_{frd}(I, E^n) \), by using the Lipschitz continuity of \( F \). Hence by using Banach contraction mapping theorem \( G \) has a unique fixed point, which is the desired solution to (5). Proceeding in this way, problem (5) has unique solution on each interval \( I \). Combining all these solutions together gives the solution to (3). Hence the problem (3) has unique solution on \( \mathbb{T} \).
References