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Optimizing Fuzzy Multi-item Single-period Inventory Problem under Risk-neutral Criterion

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Abstract

On the basis of credibilistic optimization method, the multi-item single-period inventory problem is studied. A new risk-neutral inventory problem with uncertain demand is presented, in which the expected value is adopted in the formulation of profit objective function. We use both discrete and continuous possibility distributions to describe uncertain demands in our inventory problem. To compute expected value objective, we assume uncertain demands follow triangular, trapezoidal and Erlang possibility distributions. Finally, the numerical discussion is given in the cases of discrete distribution and continuous triangular distribution.

Keywords: inventory problem, uncertain demand, risk-neutral criterion, credibilistic optimization

1 Introduction

The financial and decision theoretical aspects are important in inventory management. The multi-item single-period inventory problem is of significance in terms of both theoretical and practical consideration in real life. In this work, we model the single-item/multi-item single-period inventory problem under risk-neutral criterion, and consider a firm producing multiple products in a single period and the buyers can order goods in advance. The key points of the problem are to determine the order quantity as well as the demands in order to maximize the total profit.

In the recent literature, the importance of financial and decision theoretical aspects in inventory management has been evidenced by a variety of optimal methods [9, 16, 19, 27, 24, 28]. Many relevant literature of single-item/multi-item inventory problem have been made in the probabilistic framework, in which the uncertainty of demand is characterized by the randomness. Benjaafar and Elhafsi [2] analyzed the optimal production and inventory control of an assemble-to-order system with m components, one end-product, and n customer classes. Ahmed [1] considered an extension of the classical multi-period, single-item, linear cost inventory problem, where the objective function was a coherent risk measure. Chen et al. [5] proposed a framework for incorporating risk aversion in multi-period inventory models as well as multi-period models that coordinate inventory and pricing strategies. Gotoh and Takano [7] considered the minimization of the conditional value-at-risk (CVaR) in the well-known single-period newsvendor problem, which was originally formulated as the maximization of the expected profit or the minimization of the expected cost. Keren [8] studied a special form of the single-period inventory problem with a known demand and stochastic supply. By assuming random demand, Ozler et al. [18] considered a single-period stochastic inventory (newsvendor) problem with downside risk constraints. See and Sim [20] proposed a robust optimization approach to address a multi-period inventory control problem under ambiguous demands. Taleizadeh et al. [22] optimized multiproduct multiconstraint inventory control systems with stochastic period length and emergency order.

In real life there exist situations where there are no sufficient historical data or historical data are unavailable. Then the demands in the inventory problem have to be given mainly by experts' estimations instead of

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historical data, and thus contain much subjective imprecision rather than randomness. Under this consideration, some researchers addressed fuzzy uncertainty in inventory management. For example, Yao et al. [26] applied a stochastic single-period inventory management approach to analyzing optimal cash management policies with fuzzy cash demand. Shao and Ji [21] studied the multi-product newsboy problem with fuzzy demands under budget constraint [10]. Chang and Yeh [4] investigated the effects of the manufacturer's refund on retailer's unsold products for the two-echelon decentralized and centralized supply chains of a short life and returnable product with trapezoidal fuzzy demand. Yaghin et al. [25] proposed a hybrid bi-objective credibility-based fuzzy optimization model including both quantitative and qualitative objectives to cope with these issues.

Many researchers investigated the single-item/multi-item single-period inventory problem with risk-neutral or risk-averse polices. Borgonovo and Peccati [3] adopted some methods like expected value, quadratic utility, mean-absolute and conditional value at risk (CVaR) to optimize the inventory problems. In this study, we maximize the profit by using the risk-neutral method in the multi-item single-period inventory problem. In addition, the uncertain demands in our problem is expressed by a fuzzy variable with known possibility distribution functions.

The purpose of this work is to discuss the financial management in inventory problems under uncertain demands. We model a multi-item single-period inventory system by using credibilistic optimization methods [13], in which the financial characteristics are described by expectation profit function. In our optimization model, the uncertain demands follow several common possibility distributions. Our purpose is to find the optimal order quantity by maximizing the profit with respect to the expected value. Firstly, by the properties of the demand, we calculate the expected value about the reciprocal of the demand so that the optimal policy can be obtained. Secondly, we deal with the calculation of the expected value when the demand follows a discrete distribution and common continuous distributions. Finally, the numerical discussion is carried out in the situations of that the demand is a discrete fuzzy variable and a triangular fuzzy variable. The reader may refer to the recent works [6, 17, 23] about the practical applications of credibilistic optimization methods.

The main contributions of the paper can be summarized as follows. (i) We propose the credibilistic optimization model for multi-item single-period inventory problems, in which the demands are described by known possibility distribution functions. (ii) We identify the conditions to obtain the optimal order policy for profit objective function. (iii) We discuss the properties of the demand to calculate the expected value about the reciprocal of the demand.

The structure of this paper is organized as follows. Section 2 builds a risk-neural expected value model for the single-period inventory problem. Section 3 deals with the computation about the expected value of the discrete fuzzy demand and the continuous fuzzy demand. Section 4 provides some numerical examples to illustrate the proposed optimization method. Section 5 gives the conclusions of this paper.

2 Model Formulation of Inventory Problem

The problem studied in this paper is the multi-item single-period inventory problem. To describe our problem clearly, we adopt the notations in Table 1.

The firm allows the buyers to order goods in advance, and it can obtain the revenue through the order quantity. Since the firm allows buyers ordering goods in advance, we consider two types of costs. One is the fixed order cost component and the other is the holding cost which are both paid at the beginning in the single-item inventory problem. Now, we describe the two kinds of costs in details.

(1) *Order cost*: We consider a fixed component $a > 0$ per order, so that during the ordering process the ordering cost amounts to a.

(2) *Holding cost*: Since the average amount of inventories is $x/2$ and the holding cost is assumed to be proportional to the quantity on stock and to time, the holding cost amounts to $hxT/2$ with the constant $h > 0$ and $T = x/D$. Its expected value is $mhx^2/2$, where x is the order quantity, D is the fuzzy demand, and

$$m = \mathrm{E}\left[\frac{1}{D}\right] = \int_0^{+\infty} \mathrm{Cr}\left\{\frac{1}{D} \geq r\right\} \mathrm{d}r$$

with Cr the credibility measure defined in [10].

The relationship between the demand and the quantity on stock has been showed in Figure 1, where x/D is the time that goods are sold out and the shaded area represents the quantity on stock.

Table 1: The notations in inventory problem

Notations	Definitions
Single-item:	
a	Fixed order cost per inventoried item
p	Revenue per unit of inventoried item
h	Unit holding cost
D	Fuzzy demand in the inventory problem
x	Order quantity in the inventory problem
Multi-item:	
$\mathbf{a} = [a_1, a_2, \cdots, a_n]$	Unit fixed costs per inventoried item
$\mathbf{p} = [p_1, p_2, \cdots, p_n]$	Unit revenues per inventoried item
$\mathbf{h} = [h_1, h_2, \cdots, h_n]$	Unit holding costs per inventoried item
$\mathbf{D} = [D_1, D_2, \cdots, D_n]$	Fuzzy demand vector in the inventory problem
$\mathbf{x} = [x_1, x_2, \cdots, x_n]$	Order quantity vector in the inventory problem

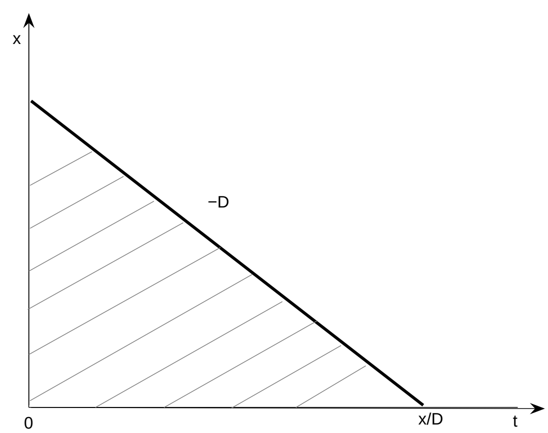

Figure 1: The relationship between the demand and the quantity on stock

If decision makers want to obtain the maximum profit, they often use the expected value of the profit as the objective function, and maximize the total mean profit. In order to determine the optimal policy of replenishment, we have firstly to determine the expected costs associated to each policy.

It is easy to get a single-item inventory model, in which profit objective function is

$$\pi(x, D) = px - a - \frac{hx^2}{2D},$$

where px is the total revenue in the inventory problem, a is the fixed order cost and $hx^2/2D$ is the holding cost. The expected value of fuzzy profit $\pi(x, D)$ is denoted by $\mathrm{E}[\pi(x, D)]$. By the properties of expected value operator [10], one has

$$\begin{aligned} \mathrm{E}[\pi(x, D)] &= \mathrm{E}\left[px - a - \frac{hx^2}{2D}\right] \\ &= px - a - \mathrm{E}\left[\frac{hx^2}{2D}\right] \\ &= px - a - \frac{hx^2}{2}\mathrm{E}\left[\frac{1}{D}\right]. \end{aligned} \tag{1}$$

In this case, the expected value model for the single-item single-period inventory problem reads

$$\begin{cases} \max & \mathrm{E}[\pi(x, D)] \\ \text{s. t.} & x \geq 0. \end{cases} \tag{2}$$

According to Eq. (1), we can obtain the following solution to model (2):

$$x^* = \frac{p}{hm},$$

where $m = \mathrm{E}[1/D]$.

In real life, the multi-item single-period inventory problem is of significance in terms of both theoretical and practical consideration. We next discuss the issue when the inventory has various items in single period. Let $i = 1, 2, \ldots, n$ denote the types of goods (items), and suppose there is no influence between any two items. In this case, the multi-item profit function is expressed as

$$\pi(\mathbf{x}, \mathbf{D}) = \sum_{i=1}^{n} \left(p_i x_i - a_i - \frac{h_i x_i^2}{2D_i} \right).$$

Under the risk-neutral criterion, the expected value model of the multi-item inventory problem is formally built as

$$\begin{cases} \max \quad \mathrm{E}[\pi(\mathbf{x}, \mathbf{D})] \\ \text{s. t.} \quad \mathbf{x} \geq 0. \end{cases} \tag{3}$$

If we denote the demand vector $\mathbf{D} = (D_1, D_2, \cdots, D_n)$, and suppose the demand variables D_i are mutually independent fuzzy variables [15], then the joint possibility distribution $\mu_{\mathbf{D}}$ is represented by

$$\mu_{\mathbf{D}}(t_1, t_2, \cdots, t_n) = \min_{1 \leq i \leq n} \mu_{D_i}(t_i).$$

Let

$$\pi_i(x_i, D_i) = p_i x_i - a_i - \frac{h_i x_i^2}{2D_i}, \quad i = 1, 2, \ldots, n.$$

Then $\pi_i(x_i, D_i), i = 1, 2, \ldots, n$, are also mutually independent. By the independence linearity of expected value operator [14], one has

$$\mathrm{E}[\pi(\mathbf{x}, \mathbf{D})] = \sum_{i=1}^{n} \mathrm{E}[\pi_i(x_i, D_i)].$$

By calculation, we obtain

$$\mathrm{E}[\pi(\mathbf{x}, \mathbf{D})] = \sum_{i=1}^{n} \left(p_i x_i - a_i - \frac{h_i x_i^2}{2} \mathrm{E}\left[\frac{1}{D_i}\right] \right).$$

As a consequence, the equivalent model of model (3) is as follows

$$\begin{cases} \max \quad \displaystyle\sum_{i=1}^{n} \left(p_i x_i - a_i - \frac{m_i h_i x_i^2}{2} \right) \\ \text{s. t.} \quad \mathbf{x} \geq 0, \end{cases} \tag{4}$$

where

$$m_i = \mathrm{E}\left[\frac{1}{D_i}\right], \quad i = 1, 2, \ldots, n.$$

It is evident that model (4) is a convex programming model. By solving the following equations,

$$\frac{\partial}{\partial x_i} \sum_{i=1}^{n} \left(p_i x_i - a_i - \frac{m_i h_i x_i^2}{2} \right) = 0, \quad i = 1, 2, \ldots, n,$$

we obtain

$$\mathbf{x}^* = \left[\frac{p_1}{h_1 m_1}, \frac{p_2}{h_2 m_2}, \ldots, \frac{p_n}{h_n m_n} \right].$$

So far, we have obtained the general solution $\mathbf{x}^*$ to model (4), which depends on the values of $m_i = \mathrm{E}[1/D_i], i = 1, 2, \ldots, n$. In the next section, we discuss the calculation of $m_i, i = 1, 2, \ldots, n$ under common demand distributions.

3 Model Analysis under Common Demand Distributions

In this section, we deal with the calculation of $\mathrm{E}[1/D_i]\,(i=1,2,\ldots,n)$ under common demand distributions. We divide our discussion into two cases. One is the case that the demands $D_i\,(i=1,2,\ldots,n)$ follow discrete distributions, and the second case is that demands follow common continuous distributions.

3.1 Discrete Demand Distributions

Theorem 1. *Suppose demand D in model (2) has the following discrete possibility distribution*

$$D \sim \begin{pmatrix} t_1 & t_2 & \cdots & t_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_n & \cdots \end{pmatrix},$$

where $t_i \geq t_{i+1}$ for any i, and $\mu_i > 0$ with $\max_{1\leq i<+\infty} \mu_i = 1$. Then the expected value $\mathrm{E}[1/D]$ is

$$\mathrm{E}\left[\frac{1}{D}\right] = \sum_{i=1}^{+\infty} \frac{q_i}{t_i}, \tag{5}$$

where the weights q_i are determined by the following formula

$$q_i = \frac{1}{2}(\max_{j\leq i} \mu_j - \max_{j\leq i-1} \mu_j) + \frac{1}{2}(\sup_{j\geq i} \mu_j - \sup_{j\geq i+1} \mu_j) \tag{6}$$

for any i with $\mu_0 = 0$.

Proof. By supposition, the possibility distribution of demand D is as follows

$$\begin{pmatrix} t_1 & t_2 & \cdots & t_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_n & \cdots \end{pmatrix}.$$

Hence, the variable $1/D$ has the following possibility distribution

$$\tfrac{1}{D} \sim \begin{pmatrix} \frac{1}{t_1} & \frac{1}{t_2} & \cdots & \frac{1}{t_n} & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_n & \cdots \end{pmatrix}.$$

Note that for any i, one has

$$\frac{1}{t_i} \leq \frac{1}{t_{i+1}}.$$

Thus, by the definition of expected value operator [10], the expected value $E[1/D]$ has the analytical expression (5). The proof of theorem is complete. □

3.2 Continuous Demand Distributions

Theorem 2. *If demand D in model (2) is a triangular fuzzy variable (r_1, r_2, r_3) with $r_1 > 0$, then the expected value $\mathrm{E}[1/D]$ is*

$$\mathrm{E}\left[\frac{1}{D}\right] = \frac{1}{2(r_3 - r_2)} \ln \frac{r_3}{r_2} + \frac{1}{2(r_2 - r_1)} \ln \frac{r_2}{r_1}.$$

In the case of $r_1 \geq 0$, the expected value $\mathrm{E}[1/D]$ does not exist.

Proof. If we denote $\eta = 1/D$, then the possibility distribution of η is

$$\mu_\eta(t) = \mathrm{Pos}\{\eta = t\} = \mathrm{Pos}\left\{D = \frac{1}{t}\right\} = \mu_D\left(\frac{1}{t}\right).$$

According to the distribution of $D = (r_1, r_2, r_3)$, we have

$$\mu_\eta(t) = \begin{cases} \dfrac{\frac{1}{t} - r_1}{r_2 - r_1}, & r_1 \leq \dfrac{1}{t} < r_2 \\ \dfrac{r_3 - \frac{1}{t}}{r_3 - r_2}, & r_2 \leq \dfrac{1}{t} < r_3 \\ 0, & \text{others.} \end{cases}$$

Thus, the possibility distribution of η is

$$\mu_\eta(t) = \begin{cases} \dfrac{r_3 t - 1}{(r_3 - r_2)t}, & \dfrac{1}{r_3} \le t \le \dfrac{1}{r_2} \\ \dfrac{1 - r_1 t}{(r_2 - r_1)t}, & \dfrac{1}{r_2} \le t \le \dfrac{1}{r_1} \\ 0, & others. \end{cases}$$

For any $r > 0$, the credibility distribution of η is computed by

$$\text{Cr}\{\eta \ge r\} = \text{Cr}\{D \le \tfrac{1}{r}\} = \begin{cases} 1, & 0 < r < \dfrac{1}{r_3} \\ \dfrac{(r_3 - 2r_2)r + 1}{2r(r_3 - r_2)}, & \dfrac{1}{r_3} \le r < \dfrac{1}{r_2} \\ \dfrac{1 - r_1 r}{2r(r_2 - r_1)}, & \dfrac{1}{r_2} \le r < \dfrac{1}{r_1} \\ 0, & r \ge \dfrac{1}{r_1}. \end{cases} \tag{7}$$

As a consequence, the expected value $\text{E}[\eta]$ is computed as follows

$$\begin{aligned} \text{E}[\eta] &= \int_0^{+\infty} \text{Cr}\{\eta \ge r\}\text{dr} - \int_{-\infty}^0 \text{Cr}\{\eta \le r\}\text{dr} \\ &= \int_0^{+\infty} \text{Cr}\{\eta \ge r\}\text{dr} \\ &= \int_0^{\frac{1}{r_3}} \text{dr} + \int_{\frac{1}{r_3}}^{\frac{1}{r_2}} \frac{(r_3 - 2r_2)r + 1}{2r(r_3 - r_2)}\text{dr} + \int_{\frac{1}{r_2}}^{\frac{1}{r_1}} \frac{1 - r_1 r}{2r(r_2 - r_1)}\text{dr} \\ &= \frac{r_3}{2r_2 r_3} + \frac{1}{2(r_3 - r_2)} \ln\frac{r_3}{r_2} + \frac{1}{2(r_2 - r_1)} \ln\frac{r_2}{r_1} - \frac{r_1}{2r_1 r_2} \\ &= \frac{1}{2(r_3 - r_2)} \ln\frac{r_3}{r_2} + \frac{1}{2(r_2 - r_1)} \ln\frac{r_2}{r_1}. \end{aligned}$$

We next to show that when $r_1 = 0$, the expected value $\text{E}[1/D]$ does not exist. In fact, the credibility distribution of η in this case is

$$\text{Cr}\{\eta \ge r\} = \text{Cr}\{D \le \tfrac{1}{r}\} = \begin{cases} 1, & 0 < r < \dfrac{1}{r_3} \\ \dfrac{(r_3 - 2r_2)r + 1}{2r(r_3 - r_2)}, & \dfrac{1}{r_3} \le r < \dfrac{1}{r_2} \\ \dfrac{1}{2r_2 r}, & r \ge \dfrac{1}{r_2}. \end{cases} \tag{8}$$

Thus, the expected value $\text{E}[\eta]$ is computed by

$$\begin{aligned} \text{E}[\eta] &= \int_0^{+\infty} \text{Cr}\{\eta \ge r\}\text{dr} - \int_{-\infty}^0 \text{Cr}\{\eta \le r\}\text{dr} \\ &= \int_0^{+\infty} \text{Cr}\{\eta \ge r\}\text{dr} \\ &= \int_0^{\frac{1}{r_3}} \text{dr} + \int_{\frac{1}{r_3}}^{\frac{1}{r_2}} \frac{(r_3 - 2r_2)r + 1}{2r(r_3 - r_2)}\text{dr} + \lim_{d \to +\infty} \int_{\frac{1}{r_2}}^{d} \frac{1}{2r_2 r}\text{dr} \\ &= \frac{1}{2(r_3 - r_2)} \ln\frac{r_3}{r_2} + \frac{1}{2r_2}(1 + \lim_{d \to +\infty} \ln dr_2), \end{aligned}$$

which implies the expected value $\text{E}[1/D]$ does not exist. The proof of theorem is complete. □

Theorem 3. *If demand D in model (2) is a trapezoidal fuzzy variable (r_1, r_2, r_3, r_4) with $r_1 > 0$, then the expected value* $\mathrm{E}[1/D]$ *is*

$$\mathrm{E}\left[\frac{1}{D}\right] = \frac{1}{2(r_4 - r_3)} \ln \frac{r_4}{r_3} + \frac{1}{2(r_2 - r_1)} \ln \frac{r_2}{r_1}.$$

In the case of $r_1 \geq 0$, the expected value $\mathrm{E}[1/D]$ *does not exist.*

Proof. If we denote $\eta = 1/D$, then the possibility distribution of η is

$$\mu_\eta(t) = \mathrm{Pos}\{\eta = t\} = \mathrm{Pos}\left\{D = \frac{1}{t}\right\} = \mu_D\left(\frac{1}{t}\right).$$

According to the distribution of D, we have

$$\mu_\eta(t) = \begin{cases} \dfrac{r_4 t - 1}{(r_4 - r_3)t}, & \dfrac{1}{r_4} < t \leq \dfrac{1}{r_3} \\ 1, & \dfrac{1}{r_3} < t \leq \dfrac{1}{r_2} \\ \dfrac{1 - r_1 t}{(r_2 - r_1)t}, & \dfrac{1}{r_2} < t \leq \dfrac{1}{r_1} \\ 0, & \text{others.} \end{cases}$$

For any $r > 0$, the credibility distribution of η is

$$\mathrm{Cr}\{\eta \geq r\} = \mathrm{Cr}\{D \leq \tfrac{1}{r}\} = \begin{cases} 1, & 0 < r < \dfrac{1}{r_4} \\ \dfrac{(r_4 - 2r_3)r + 1}{2r(r_4 - r_3)}, & \dfrac{1}{r_4} \leq r < \dfrac{1}{r_3} \\ \dfrac{1}{2}, & \dfrac{1}{r_3} \leq r < \dfrac{1}{r_2} \\ \dfrac{1 - r_1 r}{2r(r_2 - r_1)}, & \dfrac{1}{r_2} \leq r < \dfrac{1}{r_1} \\ 0, & r \geq \dfrac{1}{r_1}. \end{cases} \tag{9}$$

Thus, the expected value $\mathrm{E}[\eta]$ is computed by

$$\begin{aligned} \mathrm{E}[\eta] &= \int_0^{+\infty} \mathrm{Cr}\{\eta \geq r\}\mathrm{dr} - \int_{-\infty}^0 \mathrm{Cr}\{\eta \leq r\}\mathrm{dr} \\ &= \int_0^{+\infty} \mathrm{Cr}\{\eta \geq r\}\mathrm{dr} \\ &= \int_0^{\frac{1}{r_4}} \mathrm{dr} + \int_{\frac{1}{r_4}}^{\frac{1}{r_3}} \frac{(r_4 - 2r_3)r + 1}{2r(r_4 - r_3)}\mathrm{dr} + \int_{\frac{1}{r_3}}^{\frac{1}{r_2}} \frac{1}{2}\mathrm{dr} + \int_{\frac{1}{r_2}}^{\frac{1}{r_1}} \frac{1 - r_1 r}{2r(r_2 - r_1)}\mathrm{dr} \\ &= \frac{1}{2(r_4 - r_3)} \ln \frac{r_4}{r_3} + \frac{1}{2(r_2 - r_1)} \ln \frac{r_2}{r_1}. \end{aligned}$$

We next to show that when $r_1 = 0$, the expected value $\mathrm{E}[1/D]$ does not exist. In fact, the credibility distribution of η in this case is

$$\mathrm{Cr}\{\eta \geq r\} = \mathrm{Cr}\{D \leq \tfrac{1}{r}\} = \begin{cases} 1, & 0 < r < \dfrac{1}{r_4} \\ \dfrac{(r_4 - 2r_3)r + 1}{2r(r_4 - r_3)}, & \dfrac{1}{r_4} \leq r < \dfrac{1}{r_3} \\ \dfrac{1}{2}, & \dfrac{1}{r_3} \leq r < \dfrac{1}{r_2} \\ \dfrac{1}{2r_2 r}, & r \geq \dfrac{1}{r_2}. \end{cases} \tag{10}$$

Furthermore, the expected value $\mathrm{E}[\eta]$ is computed by

$$\begin{aligned}\mathrm{E}[\eta] &= \int_0^{+\infty} \mathrm{Cr}\{\eta \geq r\}\mathrm{dr} - \int_{-\infty}^{0} \mathrm{Cr}\{\eta \leq r\}\mathrm{dr}\\ &= \int_0^{+\infty} \mathrm{Cr}\{\eta \geq r\}\mathrm{dr}\\ &= \int_0^{\frac{1}{r_4}} \mathrm{dr} + \int_{\frac{1}{r_4}}^{\frac{1}{r_3}} \frac{(r_4-2r_3)r+1}{2r(r_4-r_3)}\mathrm{dr} + \int_{\frac{1}{r_3}}^{\frac{1}{r_2}} \frac{1}{2}\mathrm{dr} + \lim_{d\to+\infty}\int_{\frac{1}{r_2}}^{d} \frac{1}{2r_2 r}\mathrm{dr}\\ &= \frac{1}{2(r_4-r_3)}\ln\frac{r_4}{r_3} + \frac{1}{2r_2}(1+\lim_{d\to+\infty}\ln dr_2),\end{aligned}$$

which implies the expected value $\mathrm{E}[1/D]$ does not exist. The proof of theorem is complete. □

Theorem 4. *If demand D in model (2) is an Erlang fuzzy variable $\mathrm{Er}(\lambda, r)$, where $D \in [r_1, +\infty)$ with $r_1 > 0$, r is a positive integer and $\lambda > 0$, then we have*

$$\mathrm{E}\left[\frac{1}{D}\right] = \frac{1}{2}\lambda(\frac{1}{\lambda r})^r\left[\sum_{j=1}^{r}\prod_{i=2}^{j+1}(i-r)(-\lambda)^{j+1}((\frac{1}{r_1})^{j+2-r}\exp(r) - 2(\frac{1}{\lambda r})^{j+2-r}) - (\frac{1}{\lambda r})^{2-r} + (\frac{1}{r_1})^{2-r}\exp(r)\right] + \frac{1}{\lambda r}.$$

In the case of $r_1 \geq 0$, the expected value $\mathrm{E}[1/D]$ does not exist.

Proof. If demand D is an Erlang fuzzy variable $\mathrm{Er}(\lambda, r)$, then the possibility distribution of $\eta = 1/D$ is as follows

$$\mu_\eta(t) = \mathrm{Pos}\{\eta = t\} = \mathrm{Pos}\left\{D = \frac{1}{t}\right\} = \mu_D\left(\frac{1}{t}\right).$$

Since

$$\mu_D(t) = \left(\frac{t}{\lambda r}\right)^r \exp\left(r - \frac{t}{\lambda}\right),$$

one has

$$\mu_\eta(t) = \left(\frac{1}{\lambda r t}\right)^r \exp\left(r - \frac{1}{\lambda t}\right),$$

It follows that the credibility distribution of η is

$$\mathrm{Cr}\{\eta \geq x\} = \mathrm{Cr}\{D \leq \tfrac{1}{x}\} = \begin{cases} 1 - \dfrac{1}{2}\left(\dfrac{1}{\lambda r x}\right)^r \exp\left(r - \dfrac{1}{\lambda x}\right), & 0 < x < \dfrac{1}{\lambda r}\\ \dfrac{1}{2}\left(\dfrac{1}{\lambda r x}\right)^r \exp\left(r - \dfrac{1}{\lambda x}\right), & \dfrac{1}{\lambda r} \leq x \leq \dfrac{1}{r_1}. \end{cases} \tag{11}$$

Thus, the expected value $\mathrm{E}[\eta]$ is computed by

$$\begin{aligned}\mathrm{E}[\eta] &= \int_0^{+\infty} \mathrm{Cr}\{\eta \geq x\}\mathrm{dx} - \int_{-\infty}^{0} \mathrm{Cr}\{\eta \leq x\}\mathrm{dx}\\ &= \int_0^{+\infty} \mathrm{Cr}\{\eta \geq x\}\mathrm{dx}\\ &= \int_0^{\frac{1}{\lambda r}} (1 - \frac{1}{2}(\frac{1}{\lambda r x})^r \exp(r - \frac{1}{\lambda x}))\mathrm{dx} + \int_{\frac{1}{\lambda r}}^{\frac{1}{r_1}} \frac{1}{2}(\frac{1}{\lambda r x})^r \exp(r - \frac{1}{\lambda x})\mathrm{dx}\\ &= \frac{1}{2}\lambda(\frac{1}{\lambda r})^r[-\int_0^{\frac{1}{\lambda r}} x^{-r}\exp(r - \frac{1}{\lambda x})\mathrm{dx} + \int_{\frac{1}{\lambda r}}^{\frac{1}{r_1}} x^{-r}\exp(r - \frac{1}{\lambda x})\mathrm{dx}] + (\frac{1}{\lambda r} - 1)\\ &= \frac{1}{2}\lambda(\frac{1}{\lambda r})^r[\sum_{j=1}^{r}\prod_{i=2}^{j+1}(i-r)(-\lambda)^{j+1}((\frac{1}{r_1})^{j+2-r}\exp(r) - 2(\frac{1}{\lambda r})^{j+2-r}) - (\frac{1}{\lambda r})^{2-r}\\ &\quad + (\frac{1}{r_1})^{2-r}\exp(r)] + \frac{1}{\lambda r}.\end{aligned}$$

The rest of theorem is similar to prove. The proof of theorem is complete. □

4 Numerical Discussion

In this section, we present numerical examples to illustrate the proposed optimization methods for inventory problem.

In our inventory problem, a firm allows the buyers to order goods in advance, and obtain its revenue through the order quantity. There are two types of costs, the fixed order cost and the holding cost. The economics parameters **p**, **a** and **h** in the inventory problem are provided in Table 2, where **p** denotes the revenues per inventoried item, **a** denotes the fixed costs per inventoried item and **h** denotes the holding costs per inventoried item.

Case I: Demand vector **D** follows a discrete distribution. There are 10 items in the inventory problem. The demands D_i $(i = 1, 2, \ldots, 10)$ have the following possibility distributions:

$$D_i \sim \begin{pmatrix} t_1^i & t_2^i & \cdots & t_{10}^i \\ \mu_1^i & \mu_2^i & \cdots & \mu_{10}^i \end{pmatrix}.$$

The values t_j^i in the possibility distributions of demands D_i are collected in Table 3, while the possibilities μ_j^i in the possibility distributions of demands D_i are provided in Table 4.

Table 2: Economics inputs for the inventory problem ($)

Item	1	2	3	4	5	6	7	8	9	10
p	10	11	12.5	13	12	9.5	14	13.5	12.5	15
a	1	2	2.5	1.5	1.8	2.2	2.3	4.1	1.9	2.7
h	0.55	0.6	0.65	0.71	0.53	0.56	0.68	0.81	0.92	0.5

Table 3: The values t_j^i in the distributions of demand D_i

Item	t_1^i	t_2^i	t_3^i	t_4^i	t_5^i	t_6^i	t_7^i	t_8^i	t_9^i	t_{10}^i
1	40	35	30	25	20	18	15	13	10	5
2	40	30	20	10	8	7	6	5	4	3
3	35	30	20	15	14	12	10	9	7	5
4	45	35	25	20	18	15	12	10	8	5
5	50	48	45	40	36	32	30	25	20	15
6	60	55	50	47	43	38	35	30	25	20
7	55	50	44	40	35	32	30	25	21	16
8	68	65	60	57	54	45	42	36	33	25
9	70	65	62	59	55	50	45	40	35	30
10	65	62	54	50	47	43	34	30	28	25

Table 4: The values μ_j^i in the distributions of demand D_i

Item	μ_1^i	μ_2^i	μ_3^i	μ_4^i	μ_5^i	μ_6^i	μ_7^i	μ_8^i	μ_9^i	μ_{10}^i
1	0.2	0.3	1	0.7	0.5	0.4	0.6	0.5	0.2	0.8
2	0.3	1	0.5	0.4	0.2	0.6	0.7	0.8	0.1	0.2
3	0.4	1	0.6	0.3	0.5	0.7	0.8	0.3	0.2	0.1
4	1	0.8	0.4	0.5	0.6	0.3	0.2	0.2	0.1	0.1
5	0.7	0.4	0.6	0.2	0.3	0.5	1	0.4	0.8	0.1
6	0.8	0.7	0.5	0.9	0.3	0.2	0.6	1	0.5	0.4
7	0.4	0.3	0.5	0.8	0.9	0.6	0.2	1	0.7	0.5
8	0.3	0.2	0.4	0.5	0.7	0.9	0.6	0.8	1	0.4
9	0.4	0.3	0.2	0.2	0.5	1	0.6	0.7	0.4	0.8
10	0.4	0.3	0.5	0.8	1	0.7	0.6	0.4	0.5	0.2

Based on the possibility distributions of demands D_i, we compute the expected value $\mathrm{E}[1/D_i]$, where the weights q_i are computed by formula (6), and the computational results are shown in Table 5.

Table 5: The calculation results of the weights

Item	1	2	3	4	5	6	7	8	9	10
q_1	0.1	0.05	0.45	0	0	0	0	0	0	0.4
q_2	0.15	0.45	0	0	0	0	0	0.3	0	0.1
q_3	0.2	0.4	0	0	0	0	0.25	0.05	0.05	0.05
q_4	0.6	0.1	0	0	0.15	0.05	0	0.05	0	0.05
q_5	0.35	0	0	0	0	0	0.25	0	0.35	0.05
q_6	0.4	0	0	0.05	0	0	0	0.3	0.05	0.2
q_7	0.2	0	0.05	0.15	0.05	0	0	0.2	0.1	0.25
q_8	0.15	0	0.05	0.05	0.1	0.1	0	0	0.35	0.2
q_9	0.2	0	0	0	0.05	0.35	0	0	0	0.4
q_{10}	0.2	0	0.05	0.15	0.25	0.05	0.05	0	0.15	0.1

Based on Theorem 1, we compute $\mathrm{E}[1/\mathbf{D}]$ and obtain the optimal order policy

$$\mathbf{x}^* = \left[\frac{p_1}{h_1 m_1}, \frac{p_2}{h_2 m_2}, \cdots, \frac{p_{10}}{h_{10} m_{10}}\right],$$

where $m_i = \mathrm{E}[1/D_i], i = 1, 2, \ldots, 10$. The computational results are reported in Table 6.

Table 6: Optimal order policy under discrete demand distributions

Optimal policy	Item 1	2	3	4	5	6	7	8	9	10
$\mathrm{E}[\frac{1}{\mathbf{D}}]$	0.0989	0.1121	0.0667	0.0429	0.0362	0.0297	0.0383	0.0266	0.0241	0.0243
$\mathbf{x}^*$	184	164	288	427	625	571	538	627	564	1235

According to the data in Tables 2–6, the maximum expected profit $\mathrm{E}[\pi(\mathbf{x}^*, \mathbf{D})]$ to model (4) is \$33623.5.

Case II: Demands $D_i\,(i = 1, 2, \ldots, 10)$ follow triangular distributions (r_1^i, r_2^i, r_3^i) for 10 items in the inventory problem. The possibility distributions of demands D_i are provided in Table 7.

Table 7: The triangular distributions of demands D_i

Item	1	2	3	4	5
(r_1^i, r_2^i, r_3^i)	(10,20,30)	(20,30,40)	(5,15,25)	(15,30,45)	(9,24,39)
Item	6	7	8	9	10
(r_1^i, r_2^i, r_3^i)	(10,15,25)	(8,20,30)	(15,20,35)	(25,35,45)	(16,32,40)

Based on Theorem 2, we compute the expected value $\mathrm{E}[1/\mathbf{D}]$ and obtain the following optimal order policy

$$\mathbf{x}^* = \left[\frac{p_1}{h_1 m_1}, \frac{p_2}{h_2 m_2}, \cdots, \frac{p_{10}}{h_{10} m_{10}}\right],$$

where $m_i = \mathrm{E}[1/D_i], i = 1, 2, \ldots, 10$. The optimal solution $\mathbf{x}^*$ is shown in Table 8.

Table 8: Optimal order policy under triangular demand distributions

Optimal policy	Item 1	2	3	4	5	6	7	8	9	10
$\mathrm{E}[\frac{1}{\mathbf{D}}]$	0.0549	0.0347	0.0805	0.0366	0.0489	0.0661	0.0585	0.0474	0.0294	0.0356
$\mathbf{x}^*$	331	528	239	500	463	257	352	352	462	843

According to the data in Tables 2, 7 and 8, the maximum expected profit $\mathrm{E}[\pi(\mathbf{x}^*, \mathbf{D})]$ to model (4) is \$27329.5.

5 Conclusions

This paper studied the multi-item single-period inventory problem with fuzzy demand. The major conclusions include the following several aspects:

(i) We built a multi-item single-period expected profit model, in which the uncertain demands in the inventory problem are described by possibility distributions. Since the expression of the optimal order policy contains $\mathrm{E}[1/D_i]$ ($i = 1, 2, \ldots, n$), we calculated the expected value about the reciprocal of the demand.

(ii) We addressed the cases of fuzzy demands follow discrete possibility distribution and triangular, trapezoidal and Erlang possibility distributions. The computational results have been summarized in Theorems 1–4, which can help us to obtain the analytic solution to the proposed expected value model.

(iii) Based on the obtained theoretical results, some numerical experiments were conducted to illustrate the proposed methods. The obtained optimal order policies were reported in Tables 6 and 8. The computational results support our arguments.

In our future research, we will introduce risk measure to our inventory problem. In addition, we may address the general case that the marginal possibility distributions of demands D_i are not mutually independent. In this situation, we can adopt the approximation method [11] to evaluate the expected objective $E[\pi(x, D)]$, and discuss the convergence of the approximation method [12].

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