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The Credibility of Liu Process Arrives A Before Arrives B

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Abstract

Brownian motion is very important in stochastic process. The probability of Brownian motion arrives a before arrives b is useful in financial mathematics. In this paper, based on credibility theory, the membership function of the hitting time of general Liu process was deduced. Simultaneously, corresponding to the probability of Brownian motion arrives a before arrives b, the credibilities of some Liu processes arrive a before arrive b were presented.

Keywords: credibility theory, fuzzy process, Liu process, membership function, arrives a before arrives b

1 Introduction

There are many uncertain events and uncertain phenomena in real world. Fuzzy phenomenon is a kind of uncertain phenomena. In order to describe fuzzy phenomenon, fuzzy mathematics is produced. The concept of fuzzy set was initiated by Zadeh [17] via membership function in 1965. In order to measure a fuzzy event, the concept of possibility measure was proposed by Zadeh [18] in 1978. Although possibility measure has been widely used, it has no self-dual property. However a self-dual measure is very important, thus Liu and Liu [7] defined the concept of credibility measure in 2002. Credibility theory was founded by Liu [3] in 2004 and refined by Liu [5] in 2007. A sufficient and necessary condition for credibility measure was given by Li and Liu [2] in 2006. Credibility theory, which is deduced from the normality, monotonicity, self-duality, and maximality axioms, becomes an active branch mathematics for studying the behavior of fuzzy phenomena.

In 2008, Liu [6] gave the introduction of a fuzzy process, a differential formula and a fuzzy integral. Because of their importance and usefulness, the community renamed those three footstones Liu process, Liu formula and Liu integral. Later, many researches surrounding Liu process have been done. For example, Dai proposed reflection principle and Lipschitz continuity of Liu process. Multi-dimensional Liu process, Liu integral and Liu formula were presented by You et al. [12]. At the same time, Qin and Wen[10] discussed complex Liu process, and the properties were proved by You and Wang [15]. Then You et al. [16] discussed existence and uniqueness theorems for homogeneous fuzzy differential equations, Chen and Qin [1] gave a new existence and uniqueness theorem for general fuzzy differential equations. Later, Liu integral were extended to the case of infinite interval and unbounded function by You et al. [14], You and Huo [11], respectively. Liu process has also been applied to stock model and fuzzy finance by Liu [6], Qin and Li [9], Peng [8]. Zhu applied Liu process to control fields.

Considering the importance of Brownian motion and the probability of Brownian motion arrives a before arrives b, this paper aims to discuss the credibilities of some kinds of Liu processes arrives a before arrives b, which will be helpful to the application of Liu process in finance.

This paper is organized as follows: Some theorems and definitions in credibility theory and fuzzy process were introduced in Section 2. In Section 3, the membership function of the hitting time of general Liu process is presented. In Section 4, the credibilities of Liu processes arrive a before arrive b were introduced. At last, a brief conclusion is given.

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2 Preliminary

Let Θ be a nonempty set, and $\mathcal{P}$ the power set of Θ. Each element in $\mathcal{P}(\Theta)$ is called an event. Cr is defined in $\mathcal{P}(\Theta)$, the set function Cr is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

Then the triple $(\Theta, \mathcal{P}, Cr)$ is called a credibility space.

Definition 2.1 (Liu [5]) *A fuzzy variable is a (measurable) function from a credibility space* $(\Theta, \mathcal{P}, \mathrm{Cr})$ *to the set of real numbers.*

Definition 2.2 (Liu [4]) *Let* ξ *be a fuzzy variable defined on the credibility space* $(\Theta, \mathcal{P}, Cr)$. *Then its membership function is derived from the credibility measure by*

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1, x \in R.$$

Definition 2.3 (Liu [4]) *Let* ξ *be a fuzzy variable with membership function* μ. *Then for any set* B *of real numbers, we have*

$$Cr\{\xi \in B\} = \frac{1}{2}(\sup_{x\in B} \mu(x) + 1 - \sup_{x\in B^c} \mu(x)).$$

In [6], a fuzzy process $X(t, \theta)$ is a function of two variables, such that $X(t^*, \theta)$ is a fuzzy variable for each t^*, $X(t, \theta^*)$ is a common function for each θ^*. For tidiness, we use X_t to replace $X(t, \theta)$ in the following sections.

Definition 2.4 (Liu [6]) *A fuzzy process* C_t *is said to be a Liu process if*

(i) $C_0 = 0$,

(ii) C_t *has stationary and independent increments,*

(iii) every increment $C_{t+s} - C_s$ *is a normally distributed fuzzy variable with expected value et and variance* $\sigma^2 t^2$, *whose membership function is*

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

The Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Theorem 2.1 (You and Jiao [13]) *Assume* $\{C_t, t \geq 0\}$ *is a Liu process with drift 0 and diffusion* σ, τ_x *is the hitting time of Liu process* $\{at + bC_t, t \geq 0\}$ *arriving* $x > 0$. *Then the membership function of* τ_x *is as follows,*

$$\mu(t) = 2\left(1 + \exp\left(\frac{\pi|\frac{x-at}{b}|}{\sqrt{6}\sigma t}\right)\right)^{-1}.$$

3 Membership Function of the Hitting Time of General Liu Process

From Definition 2.4, we can deduce the following proposition.

Proposition 3.1 *Let* $\{C_t, t \geq 0\}$ *be a Liu process. Then the membership function of every increment of Liu process* $\{at + bC_t, t \geq 0\}$ *is*

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|\frac{x-at}{b} - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

Proof: Assume the membership function of every increment of Liu process $\{at + bC_t, t \geq 0\}$ is $M(x)$.

It follows from *(iii)* of Definition 2.4 that membership function of every increment $C_{t+s} - C_s$ of Liu process $\{C_t, t \geq 0\}$ is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

It follows from Definition 2.2 and Definition 2.3 that

$$M(x) = 2Cr\{at + bC_t = x\} \wedge 1 = 2Cr\{C_t = \frac{x - at}{b}\} \wedge 1 = 2\left(1 + \exp\left(\frac{\pi|\frac{x-at}{b} - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}.$$

Then the membership function of every increment of Liu process $\{at + bC_t, t \geq 0\}$ is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|\frac{x-at}{b} - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, \ -\infty < x < +\infty.$$

The proof is completed.

For a general Liu process, we can obtain the membership function of the hitting time.

Theorem 3.1 *Assume that $\{C_t, t \geq 0\}$ be a Liu process with expected value et and variance $\sigma^2 t^2$. Let τ_x be the hitting time of Liu process $\{at + bC_t, t \geq 0\}$ arrives $x > 0$. Then the membership function of τ_x is as follows,*

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|\frac{x-at}{b} - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}.$$

Proof: Assume τ_x is the hitting time of Liu process $\{at + bC_t, t \geq 0\}$ arrives $x > 0$, and the membership function of τ_x is $M(x)$.

Assume that $\tau_x^{'}$ is the hitting time of Liu Process $\{C_t, t \geq 0\}$ arrives $x > 0$, then the membership function of $\tau_x^{'}$ is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}.$$

In a similar proof of Proposition 3.1, the conclusion is obtained.

4 The Credibility of Liu Process Arrives A Before Arrives B

In order to describe financial phenomenon in fuzzy environment, it is important to discuss the credibility of Liu process arrives a before arrives b. Thus, we have the following theorem and remarks.

Theorem 4.1 *Assume $\{C_t, t \geq 0\}$ is a Liu process with drift 0 and diffusion σ. Then the credibility of Liu process $\{C_t, t \geq 0\}$ arrives a before arrives b is*

$$Cr\{\tau_a < \tau_b\} = \frac{1}{2}, a > 0, b > 0.$$

Proof: Since the membership function of τ_x corresponding to $\{C_t, t \geq 0\}$ is

$$\mu(x) = 2(1 + \exp(\frac{\pi|x|}{\sqrt{6}\sigma t}))^{-1},$$

$$\tau_a = \inf\{t > 0 | C_t = a\}, \tau_b = \inf\{t > 0 | C_t = b\},$$

the membership function of τ_a and τ_b are as follows,

$$\mu(a) = 2(1 + \exp(\frac{\pi|a|}{\sqrt{6}\sigma t}))^{-1},$$

$$\mu(b) = 2(1 + \exp(\frac{\pi|b|}{\sqrt{6}\sigma t}))^{-1}.$$

Since τ_a and τ_b are independent fuzzy variables, we have

$$Cr\{\tau_a = t^{'}\} = \frac{1}{2}(\sup_{t=t^{'}} \mu_a + 1 - \sup_{t \neq t^{'}} \mu_a) = \frac{1}{1 + e^{\frac{\pi|a|}{\sqrt{6}\sigma t^{'}}}},$$

$$Cr\{\tau_b = t^{'}\} = \frac{1}{1+e^{\frac{\pi|b|}{\sqrt{6}\sigma t^{'}}}}.$$

Therefore,

$$\sup_{t^{'}}\{Cr\{\tau_a = t^{'}\}\} = \frac{1}{2},$$

$$\sup_{t^{'}}\{Cr\{\tau_b = t^{'}\}\} = \frac{1}{2}.$$

Then

$$Cr\{\tau_a < \tau_b\} = Cr\{\bigcup_{t^{'}}(\tau_a = t^{'}, \tau_b > t^{'})\} = \sup_{t^{'}}(Cr\{\tau_a = t^{'}, \tau_b > t^{'}\})$$
$$= \sup_{t^{'}}(Cr\{\tau_a = t^{'}\} \wedge \{\tau_b > t^{'}\}) = \sup_{t^{'}}(Cr\{\tau_a = t^{'}\}) \wedge \sup_{t^{'}}(Cr\{\tau_b = t^{'}\}) = \frac{1}{2}.$$

The proof is completed.

Next, in a similar proof of Theorem 4.1, we will obtain the following remarks.

Remark 4.1 Let $\{C_t, t \geq 0\}$ be a standard Liu process. Then the credibility of Liu process $\{C_t, t \geq 0\}$ arrives a before arrives b is

$$Cr\{\tau_a < \tau_b\} = \frac{1}{2}, a > 0, b > 0.$$

Proof: Because $\{C_t, t \geq 0\}$ is a standard Liu process, we have $\sigma = 1$. In a similar proof of Theorem 4.1, the conclusion is obtained.

Remark 4.2 Let $\{C_t, t \geq 0\}$ be a Liu process with expect value et and variance $\sigma^2 t^2$. Then the credibility of Liu process $\{C_t, t \geq 0\}$ arrives a before arrives b is

$$Cr\{\tau_a < \tau_b\} = \frac{1}{2}, a > 0, b > 0.$$

Proof: Since the membership function of τ_x corresponding to $\{C_t, t \geq 0\}$ is

$$\mu(x) = 2(1+\exp(\frac{\pi|x-et|}{\sqrt{6}\sigma t}))^{-1},$$

we have

$$\mu(a) = 2(1+\exp(\frac{\pi|a-et|}{\sqrt{6}\sigma t}))^{-1},$$

$$\mu(b) = 2(1+\exp(\frac{\pi|b-et|}{\sqrt{6}\sigma t}))^{-1}.$$

Since τ_a and τ_b are independent fuzzy variables,

$$Cr\{\tau_a = t^{'}\} = \frac{1}{2}(\sup_{t=t^{'}}\mu_a + 1 - \sup_{t\neq t^{'}}\mu_a) = (1+\exp(\frac{\pi|a-et^{'}|}{\sqrt{6}\sigma t^{'}}))^{-1},$$

$$Cr\{\tau_b = t^{'}\} = (1+\exp(\frac{\pi|b-et^{'}|}{\sqrt{6}\sigma t^{'}}))^{-1}.$$

Consequently,

$$\sup_{t^{'}}\{Cr\{\tau_a = t^{'}\}\} = \frac{1}{2},$$

$$\sup_{t'}\{Cr\{\tau_b = t'\}\} = \frac{1}{2}.$$

In a similar proof of Theorem 4.1, the conclusion is obtained.

Remark 4.3 Assume $\{C_t, t \geq 0\}$ is a Liu process with drift 0 and diffusion σ. Then the credibility of Liu process $\{ct + dC_t, t \geq 0\}$ arrives a before arrives b is

$$Cr\{\tau_a < \tau_b\} = \frac{1}{2}, a > 0, b > 0.$$

Proof: It follows from Theorem 2.1 that the membership function of τ_x corresponding to $\{ct + dC_t, t \geq 0\}$ is

$$\mu(x) = 2(1 + \exp(\frac{\pi|\frac{x-ct}{d}|}{\sqrt{6}\sigma t}))^{-1},$$

we have

$$\mu(a) = 2(1 + \exp(\frac{\pi|\frac{a-ct}{d}|}{\sqrt{6}\sigma t}))^{-1},$$

$$\mu(b) = 2(1 + \exp(\frac{\pi|\frac{b-ct}{d}|}{\sqrt{6}\sigma t}))^{-1}.$$

Since τ_a and τ_b are independent fuzzy variables,

$$Cr\{\tau_a = t'\} = \frac{1}{2}(\sup_{t=t'}\mu_a + 1 - \sup_{t\neq t'}\mu_a) = (1 + \exp(\frac{\pi|\frac{a-ct'}{d}|}{\sqrt{6}\sigma t'}))^{-1},$$

$$Cr\{\tau_b = t'\} = (1 + \exp(\frac{\pi|\frac{b-ct'}{d}|}{\sqrt{6}\sigma t'}))^{-1}.$$

Then

$$\sup_{t'}\{Cr\{\tau_a = t'\}\} = \frac{1}{2},$$

$$\sup_{t'}\{Cr\{\tau_b = t'\}\} = \frac{1}{2}.$$

In a similar proof of Theorem 4.1, the conclusion is obtained.

Remark 4.4 Let $\{C_t, t \geq 0\}$ be a Liu process with expect value et and variance $\sigma^2 t^2$. Then the credibility of Liu process $\{ct + dC_t, t \geq 0\}$ arrives a before arrives b is

$$Cr\{\tau_a < \tau_b\} = \frac{1}{2}, a > 0, b > 0.$$

Proof: Due to Theorem 3.1, the membership function of τ_x corresponding to $\{ct + dC_t, t \geq 0\}$ is

$$\mu(x) = 2(1 + \exp(\frac{\pi|\frac{x-ct}{d} - et|}{\sqrt{6}\sigma t}))^{-1},$$

we have

$$\mu(a) = 2(1 + \exp(\frac{\pi|\frac{a-ct}{d} - et|}{\sqrt{6}\sigma t}))^{-1},$$

$$\mu(b) = 2(1 + \exp(\frac{\pi|\frac{b-ct}{d} - et|}{\sqrt{6}\sigma t}))^{-1}.$$

Since τ_a and τ_b are independent fuzzy variables,

$$Cr\{\tau_a = t'\} = \frac{1}{2}(\sup_{t=t'}\mu_a + 1 - \sup_{t\neq t'}\mu_a) = (1 + \exp(\frac{\pi|\frac{a-ct'}{d} - et'|}{\sqrt{6}\sigma t'}))^{-1},$$

$$Cr\{\tau_b = t^{'}\} = (1 + \exp(\frac{\pi|\frac{b-ct^{'}}{d} - et^{'}|}{\sqrt{6}\sigma t^{'}}))^{-1}.$$

Then

$$\sup_{t^{'}}\{Cr\{\tau_a = t^{'}\}\} = \frac{1}{2},$$

$$\sup_{t^{'}}\{Cr\{\tau_b = t^{'}\}\} = \frac{1}{2}.$$

In a similar proof of Theorem 4.1, the conclusion is obtained.

5 Conclusions

In this paper, the membership function of the hitting time of general Liu process was proposed. Then the credibilities of four kinds of Liu processes arrives a before arrives b were obtained.

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