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Managing Water Supply Risk through an Option Contract in Uncertain Environment

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Abstract

Managing water supply risk in uncertain environment is becoming more and more important to water users with the rapid development of social economy. However, since mature water market has not been established in China, water price is highly influenced by human objective uncertainties. In consideration of human uncertainties and jumps in water price, a model of water option pricing based on the theory of uncertain jump process should be a more appropriate choice. In this paper, uncertain jump process is defined more generally and integral and differential with respect to this process are proposed. Then by supposing water price is driven by an uncertain jump process whose right-continuous pure jump part is determined by an uncertain renewal reward process, an exotic water option is designed which allows the holders to decide how much water to buy at the expiration date at an agreed price. A pricing formula for this water option contract is derived by using the operational law of uncertain variables. Finally, a numerical example is described.
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1 Introduction

Water supply risk is becoming critical with the development of economy and society and the emergence of environmental and pollution problems. The South-North Water Transfer Project is a mega-project in China which aims to channel 44.8 billion cubic meters of water from the Yangze River in southern China to reduce the risk of water scarcity in the industrialized north China [7]. However, the project can not solve the water scarcity problems completely, so additional means must be taken to manage water supply risk.

Market-based policies can help water users to reduce risk exposure and improve water allocation efficiency in drought years. Javier [13] showed that farmers' economic vulnerability across irrigation seasons can be reduced through the exchange of water in annual spot market. Frederick [9] argued that water market may increase the value of water by transfering it from low-value agricultural users to hydropower or urban users. But risks that may arise from water scarcity and the variation in water prices in drought years still exist. Therefore more sophisticated financial instruments such as futures and options are needed.

Several exports have studies option contracts for water. Bjornlund [1] founded that the economic basis and market mechanisms for introducing sophisticated water derivative instruments are emerging in Australia. Truong [32] proposed a two factor model and used the model to analyze real options and other derivatives in water markets. Villinski [33] proposed a more exotic option for water which can be exercised in any seven of fifteen years. He presented the methodology for valuing this multi-exercise option by stochastic dynamic programming and got valuation results for Texas. Tomkins [31] proposed a bilateral option contract model for water and compared the actual contracts prices in California with model-predicted prices. The model considered the possibility of conveyance losses and random delivery, and characterized the optimal option contracts and welfare gains. Other research results on water supply risk management include Cui [8], Hansen [12] and Gómez [11].

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Indeterminacy must be taken into consideration when options in water market are dealt with. A mass of data are needed when models based on probability or statistics are constructed. However, due to the lack of real-world data and the complexity of environment, when making decisions, people have to consult with domain experts. In this case, information and knowledge can not be described well by random variables since human beings usually overweight unlikely events. In order to model this type of human uncertainties, Liu [25] suggested to deal with it with uncertainty theory. Based on normality, duality, subadditivity and product axioms, uncertainty theory was well developed in both theory aspect and practice aspect, see Liu [15, 18, 22].

To our knowledge, there are no studies in which uncertainty theory is applied to water resource management. This paper seeks to construct a model of water option pricing by using uncertainty theory. We will design an exotic option contract for mitigating water supply risk in uncertain environment. This contract allows a potential user to decide how much water to buy at the expiration date at an agreed price. In Section 2, we introduce previous works on uncertainty theory and uncertain option pricing models for stocks. The fundamental results of uncertainty theory, such as uncertain variable and uncertain differential equation, are introduced at the same time. In Section 3, price dynamics in water market are described based on uncertain jump process. Uncertain integral with respect to uncertain jump process is also proposed. In Section 4, we design an uncertain water option contract, pricing formula and numerical example for this option being derived. Finally, a conclusion is drawn in Section 5.

2 Uncertainty Theory and Uncertain Stock Models

Uncertainty theory is an axiomatic mathematical system founded by Liu [15] in 2007, and attracts many researchers to conduct studies on uncertain finance [26], uncertain programming [18], uncertain risk analysis [20], uncertain statistics [22], uncertain set [21], uncertain logic [23] and other application fields. To start with, some useful concepts about uncertain variables are introduced.

2.1 Uncertainty Theory

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, where Γ is a nonempty set, $\mathcal{L}$ is a σ-algebra defined on Γ and $\mathcal{M}$ is an uncertain measure which was defined by Liu [15] as a set function satisfying the following axioms:

Axiom 1. *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ *for the* Γ.

Axiom 2. *(Duality Axiom)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any* $\Lambda \in \mathcal{L}$.

Axiom 3. *(Subadditivity Axiom) For every countable sequence* $\{\Lambda_i\} \subset \mathcal{L}$*, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4. *(Product Axiom) Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \ldots$*. The product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\},$$

where $\Lambda_k \in \mathcal{L}_k$ *for* $k = 1, 2, \ldots$*, respectively.*

Following the concept of uncertainty space, uncertain variable ξ is defined as a measurable function from an uncertainty space to the set of real numbers. For describing an uncertain variable ξ, uncertainty distribution $\Phi(x) = \mathcal{M}\{\xi \le x\}$ ($x \in \Re, \Re$ is the set of real numbers) is proposed, and if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$, $\Phi(x)$ is said to be regular.

Definition 2.1. (Liu [19]) *The uncertain variables* $\xi_1, \xi_2, \ldots, \xi_n$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{2.1}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ *of real numbers.*

Note that the product axiom is the main difference between uncertainty theory and probability theory which implies that uncertain variable and random variable obey different operational laws.

A sequence or other collection of uncertain variables is said to be independent and identically distributed (i.i.d.) if each uncertain variable has the same uncertainty distribution as the others and all are mutually independent.

Theorem 2.1. (Liu [22], Operational Law) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$, and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.2}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \left(\min_{1 \le i \le m} \Phi_i(x_i) \wedge \min_{m+1 \le i \le n} (1 - \Phi_i(x_i)) \right). \tag{2.3}$$

Definition 2.2. (Liu [15]) *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\} \mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\} \mathrm{d}x. \tag{2.4}$$

provided that at least one of the two integrals is finite.

For exploring more details and recent developments of uncertainty theory, readers can consult the book by Liu [24].

2.2 Uncertain Stock Models

Wiener process was used as the stock price process by Bachelier as early as 1900. Since Wiener process allows negative value which is violating the reality, geometric Brownian Motion with positive drift was proposed to describe the stock price by Samuelson [30]. In 1973, Black and Scholes [2] and Merton [28] developed the famous analytic option pricing formulas based on assuming the underlying stock price followed geometric Brownian Motion. Since then, geometric Brownian Motion has become the basic component part for option pricing and many studies have derived various option pricing formulas under different assumptions about the market.

Uncertain process was first introduced by Liu [16] to model the uncertain dynamics systems. And then, Liu [17] designed a canonical Liu process as the basic building block for uncertain calculus [17], uncertain differential equation [16] and uncertain finance [26].

Definition 2.3. (Liu [17]) *An uncertain process C_t($t \in T$, T is a time set) is said to be a canonical Liu process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with distribution

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t} \right) \right)^{-1}, \quad x \in \Re. \tag{2.5}$$

Definition 2.4. (Liu [17]) *Let X_t be an uncertain process and let C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|. \tag{2.6}$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}) \tag{2.7}$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Definition 2.5. (Liu [16]) *Suppose C_t is a canonical Liu process, and f and g are some given functions. Then*

$$\mathrm{d}Z_t = f(t, Z_t)\mathrm{d}t + g(t, Z_t)\mathrm{d}C_t \tag{2.8}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfy (2.8) identically in t.

A European call option gives the contract holder the right to buy a stock at an expiration time s for a strike price K. Liu [17] proposed the stock price Y_t followed a geometric canonical Liu process with log-drift e and log-diffusion σ. Let X_t denote the bond price, r be the risk-free interest rate and C_t be a canonical Liu process, Liu's stock model is given by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t, \\ \mathrm{d}Z_t = eZ_t\mathrm{d}t + \sigma Z_t\mathrm{d}C_t. \end{cases} \tag{2.9}$$

Then the European call option has price

$$f_c = \exp(-rs)E\left[(Z_s - K)^+\right]. \tag{2.10}$$

Liu [17] provided pricing formula for European call option as follows,

$$f_c = \exp(-rs)Z_0 \int_{K/Y_0}^{+\infty} \left(1 + \exp\left(\frac{\pi(\ln y - es)}{\sqrt{3}\sigma s}\right)\right)\mathrm{d}y. \tag{2.11}$$

Based on Liu's framework, many other uncertain financial models were developed. Chen [3] derived American option pricing formulas for uncertain stock model. Yao [35] proved a no-arbitrage theorem for Liu's stock model. Chen, Liu and Ralescu [5] presented some option pricing formula for an uncertain stock model with periodic dividends. Peng and Yao [29] proposed an uncertain stock model with mean-reverting process. Liu, Chen and Ralescu [27] proposed an uncertain currency model and derived a currency option pricing formula. Chen and Gao [4] presented an uncertain interest rate model and got the price of zero-coupon bond.

3 Water Price Process

In order to set up a reasonable water option pricing model, it is necessary to study the behaviour of water price dynamics. Villinski [33] estimated parameters for mean reversion and geometric Brownian Motion with data from short-term water market in Texas. Truong [32] used mean reverting process to describe dynamics of the log of short-run price to long-run price ratio and the log of long-run price to long-run mean ratio. Through analyzing water price series in northern Victoria Australia, Cui [8] found that jumps in water price are the most important feature. However, since mature water market has not been established in China, water price can not be decided through the market mechanism. This lead to the lack of adequate water price data. For those reasons, uncertain process with jumps is a more appropriate choice for water price in China.

3.1 Uncertain Jump Process

Liu [16] introduced an uncertain renewal process which has uncertain events occurrence times. Yao [34] defined uncertain integral with respect to uncertain renewal process and proposed a concept of uncertain differential equation with jumps. After that, Yu [36] proposed an uncertain stock model with positive constant jumps. Ji and Zhou [14] developed an uncertain stock model with positive and negative jumps, but the jump sizes were still constant. However, a more general uncertain jump process should be proposed to model the stock price, see [10].

In this section we consider uncertain process of the following form

$$Z_t = Z_0 + \int_0^t \mu_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}C_t + J_t, \tag{3.1}$$

where C_t is a canonical Liu process, μ_t and σ_t are two integrable uncertain processes. J_t is a right-continuous pure jump uncertain process which has only finite jumps on each finite interval and is constant between jumps.

Definition 3.1. *A process Z_t described above is called an uncertain jump process.* $Z_t^c = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_t$ *is called the continuous part of this process.*

Definition 3.2. *Let Z_t be an uncertain jump process defined by (3.1), X_t be an uncertain process. Then the uncertain integral of X_t with respect to Z_t is defined to be*

$$\int_0^t X_s \mathrm{d}Z_s = \int_0^t X_s \mu_s \mathrm{d}s + \int_0^t X_s \sigma_s \mathrm{d}C_s + \sum_{0<s\le t} X_s \Delta J_s, \tag{3.2}$$

where ΔJ_s denotes the jump size at time s. In differential notation:

$$X_t \mathrm{d}Z_t = X_t \mu_t \mathrm{d}t + X_t \sigma_t \mathrm{d}C_t + X_t \mathrm{d}J_t = X_t \mathrm{d}Z_t^c + X_t \mathrm{d}J_t. \tag{3.3}$$

Theorem 3.1. (Chen and Ralescu [6]) *Suppose C_t is a canonical Liu process, and μ_t and σ_t are two uncertain processes. Let Z_t be an uncertain process satisfies the following uncertain differential equation*

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t. \tag{3.4}$$

Assume $G(t,x)$ is a continuously differentiable function. Then the uncertain process $G(t,Z_t)$ satisfies

$$G(t,Z_t) = G(0,Z_0) + \int_0^t \left(\frac{\partial G}{\partial s}(s,Z_s) + \frac{\partial G}{\partial x}(s,Z_s)\cdot \mu_s \right) \mathrm{d}s + \int_0^t \frac{\partial G}{\partial x}(s,Z_s)\cdot \sigma_s \mathrm{d}C_s. \tag{3.5}$$

In differential form:

$$\mathrm{d}G(t,Z_t) = \left(\frac{\partial G}{\partial t}(t,Z_t) + \frac{\partial G}{\partial x}(t,Z_t)\cdot \mu_t \right) \mathrm{d}t + \frac{\partial G}{\partial x}(t,Z_t)\cdot \sigma_t \mathrm{d}C_t. \tag{3.6}$$

Theorem 3.2. *Let Z_t be an uncertain jump process and $G(t,x)$ be a continuously differentiable function. Then*

$$\begin{aligned} G(t,Z_t) \;=\; & G(0,Z_0) + \int_0^t \left(\frac{\partial G}{\partial s}(s,Z_s) + \frac{\partial G}{\partial x}(s,Z_s)\cdot \mu_s \right) \mathrm{d}s + \int_0^t \frac{\partial G}{\partial x}(s,Z_s)\cdot \sigma_t \mathrm{d}C_t \\ & + \sum_{0<s\le t} \left[G(s,Z_s) - G(s,Z_{s-}) \right]. \end{aligned} \tag{3.7}$$

Proof: Let $Y_t = G(t,Z_t)$ and $T_i, i=1,\ldots,N_t$ denote the jump times of Z. Let u,v be in the same (T_i,T_{i+1}). In $[u,v]$ Z_t evolves according to

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t.$$

By applying Theorem 3.1, we obtain

$$Z_v - Z_u = \int_u^v \left(\frac{\partial G}{\partial s}(s,Z_s) + \frac{\partial G}{\partial x}(s,Z_s)\cdot \mu_s \right) \mathrm{d}s + \int_u^v \frac{\partial G}{\partial x}(s,Z_s)\cdot \sigma_s \mathrm{d}C_s.$$

Let $u \downarrow T_i$ and $v \uparrow T_{i+1}$. We have

$$Z_{T_{i+1}} - Z_{T_i} = \int_{T_i}^{T_{i+1}^-} \left(\frac{\partial G}{\partial s}(s,Z_s) + \frac{\partial G}{\partial x}(s,Z_s)\cdot \mu_s \right) \mathrm{d}s + \int_{T_i}^{T_{i+1}^-} \frac{\partial G}{\partial x}(s,Z_s)\cdot \sigma_s \mathrm{d}C_s.$$

If there is a jump at T_{i+1}, then the resulting change in Y is $G(t,Z_{T_{i+1}}) - G(t,Z_{T_{i+1}^-})$. Summing over these two contributions, we obtain

$$\begin{aligned} G(t,Z_t) \;=\; & G(0,Z_0) + \int_0^t \left(\frac{\partial G}{\partial s}(s,Z_s) + \frac{\partial G}{\partial x}(s,Z_s)\cdot \mu_s \right) \mathrm{d}s + \int_0^t \frac{\partial G}{\partial x}(s,Z_s)\cdot \sigma_t \mathrm{d}C_t \\ & + \sum_{0<s\le t} \left[G(s,Z_s) - G(s,Z_{s-}) \right]. \end{aligned}$$

This completes the proof.

Following the same method, we give the two-dimensional version of Theorem 3.2.

Theorem 3.3. *Let $Z_t^{(1)}$ and $Z_t^{(2)}$ be two jump processes and $G(t, x_1, x_2)$ be a continuously differentiable function. Then*

$$\begin{aligned} G(t, Z_t^{(1)}, Z_t^{(2)}) &= G(0, Z_0^{(1)}, Z_0^{(2)}) \\ &+ \int_0^t \left(\frac{\partial G}{\partial s}(s, Z_s^{(1)}, Z_s^{(2)}) + \frac{\partial G}{\partial x_1}(s, Z_s^{(1)}, Z_s^{(2)}) \cdot \mu_s^{(1)} + \frac{\partial G}{\partial x_2}(s, Z_s^{(1)}, Z_s^{(2)}) \cdot \mu_s^{(2)} \right) \mathrm{d}s \\ &+ \int_0^t \frac{\partial G}{\partial x_1}(s, Z_s^{(1)}, Z_s^{(2)}) \cdot \sigma_t^{(1)} \mathrm{d}C_t^{(1)} + \int_0^t \frac{\partial G}{\partial x_2}(s, Z_s^{(1)}, Z_s^{(2)}) \cdot \sigma_t^{(2)} \mathrm{d}C_t^{(2)} \\ &+ \sum_{0<s\le t} \left[G(s, Z_s^{(1)}, Z_s^{(2)}) - G(s, Z_{s^-}^{(1)}, Z_{s^-}^{(2)}) \right]. \end{aligned} \tag{3.8}$$

3.2 Uncertain Water Price Process with Jumps

In this section, we suppose that the water price follows an uncertain jump process whose right-continuous pure jump part is determined by an uncertain renewal reward process. $R_t = \sum_{i=1}^{N_t} Y_i$ is an uncertain renewal reward process, where $Y_1, Y_2, \ldots$ are independent and identically distributed ($i.i.d$) uncertain renewals, $X_1, X_2, \ldots$ are $i.i.d$ uncertain interarrival times. Assume $X_1, Y_1, X_2, Y_2, \ldots$ are independent uncertain variables, then the underlying water price model is given by

$$\mathrm{d}Z_t = eZ_t\mathrm{d}t + \sigma Z_t \mathrm{d}C_t + Z_{t-}\mathrm{d}R_t. \tag{3.9}$$

Let $Y_i > -1, i = 1, 2, \ldots$. This assumption guarantees that the water price can jump down but it cannot jump from positive to negative or to zero. The solution to (3.9) is

$$Z_t = Z_0 \exp\{et + \sigma C_t\} \prod_{i=1}^{N_t} (1 + Y_i). \tag{3.10}$$

To check (3.10) satisfies the uncertain differential equation (3.9), define the continuous part

$$Z_t^c = Z_0 \exp\{et + \sigma C_t\} \tag{3.11}$$

and the pure jump part

$$J_t = \prod_{i=1}^{N_t} (1 + Z_i), \tag{3.12}$$

then $Z_t = Z_t^c J_t$. By using Theorem 3.1, we have

$$\mathrm{d}Z_t^c = eZ_t^c \mathrm{d}t + \sigma Z_t^c \mathrm{d}C_t. \tag{3.13}$$

And the jump size of J_t at time t is

$$\Delta J_t = J_t - J_{t-} = J_{t-}(1 + Y_i) - J_{t-} = J_{t-}\Delta R_t. \tag{3.14}$$

By using Theorem 3.3 and the above equations, we obtain

$$\begin{aligned} Z_t &= Z_t^c J_t \\ &= Z_0^c J_0 + \int_0^t eZ_s^c J_s \mathrm{d}s + \int_0^t \sigma Z_s^c J_s \mathrm{d}C_s + \sum_{0<s\le t} \left[Z_s^c J_s - Z_{s-}^c J_{s-} \right] \\ &= Z_0^c J_0 + \int_0^t eZ_s^c J_s \mathrm{d}s + \int_0^t \sigma Z_s^c J_s \mathrm{d}C_s + \sum_{0<s\le t} Z_{s-}^c J_{s-} \Delta R_s \\ &= Z_0 + \int_0^t eZ_s \mathrm{d}s + \int_0^t \sigma Z_s \mathrm{d}C_s + \sum_{0<s\le t} Z_{s-} \Delta R_s. \end{aligned} \tag{3.15}$$

This verifies that (3.10) is the solution to (3.9).

By the operation of uncertain variables, the uncertainty distribution of water prices Z_t can be obtained.

Theorem 3.4. *Assume C_t has distribution $\Phi_t(x)$, $Y_1, Y_2, \ldots$ and $X_1, X_2, \ldots$ have distributions $F(x)$ and $H(x)$ respectively. Then Z_t has an uncertainty distribution*

$$\Psi_t(x) = \sup_{x_1 x_2 = x/Z_0} \left\{ \Phi_t\left(\frac{\ln x_1 - et}{\sigma}\right) \wedge \left(\max_{k\geq 0}\left(1 - H\left(\frac{t}{k+1}\right)\right) \wedge F\left(\exp\left(\frac{x_2}{k}\right) - 1\right)\right)\right\}. \tag{3.16}$$

Proof: Because Y_i has distribution $F(x)$, $\ln(1+Y_i)$ has distribution $F(\exp(x)-1)$. Then $\prod_{i=1}^{N_t}(1+Y_i)$ has an uncertainty distribution

$$\begin{aligned}
\Upsilon_t(x) &= \mathcal{M}\left\{\prod_{i=1}^{N_t}(1+Y_i) \leq x\right\} \\
&= \mathcal{M}\left\{\sum_{i=1}^{N_t}\ln(1+Y_i) \leq \ln x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_t = k) \cap \sum_{i=1}^{k}\ln(1+Y_i) \leq x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_t = k) \cap \left(\ln(1+Y_1) \leq \frac{x}{k}\right)\right\} \\
&= \max_{k\geq 0}\mathcal{M}\left\{(N_t \leq k) \cap \left(\ln(1+Y_1) \leq \frac{x}{k}\right)\right\} \\
&= \max_{k\geq 0}\mathcal{M}\left\{(N_t \leq k)\right\} \wedge \mathcal{M}\left\{\left(\ln(1+Y_1) \leq \frac{x}{k}\right)\right\} \\
&= \max_{k\geq 0}\left(1 - H\left(\frac{t}{k+1}\right)\right) \wedge F\left(\exp\left(\frac{x}{k}\right) - 1\right).
\end{aligned} \tag{3.17}$$

On the other hand, $\exp\{et+\sigma C_t\}$ has uncertainty distribution $\Phi\left(\frac{\ln x - et}{\sigma}\right)$. By the operational law of uncertain variables, Z_t has an uncertainty distribution

$$\begin{aligned}
\Psi_t(x) &= \mathcal{M}\{Z_t \leq x\} \\
&= \mathcal{M}\left\{Z_0 \exp\{et + \sigma C_t\}\prod_{i=1}^{N_t}(1+Y_i) \leq x\right\} \\
&= \sup_{x_1 x_2 = x/Z_0} \mathcal{M}\left\{(\exp\{et+\sigma C_t\} \leq x_1) \cap \left(\prod_{i=1}^{N_t}(1+Y_i) \leq x_2\right)\right\} \\
&= \sup_{x_1 x_2 = x/Z_0} \mathcal{M}\left\{(\exp\{et+\sigma C_t\} \leq x_1)\right\} \wedge \mathcal{M}\left\{\left(\prod_{i=1}^{N_t}(1+Y_i) \leq x_2\right)\right\} \\
&= \sup_{x_1 x_2 = x/Z_0} \left\{ \Phi_t\left(\frac{\ln x_1 - et}{\sigma}\right) \wedge \left(\max_{k\geq 0}\left(1 - H\left(\frac{t}{k+1}\right)\right) \wedge F\left(\exp\left(\frac{x_2}{k}\right) - 1\right)\right)\right\}.
\end{aligned}$$

This completes the proof.

Note that by Theorem 3.4, an accurate expression of water price at a future time is obtained. The water option pricing formula in the next sector is based on this expression.

4 Option Contract for Water in Uncertain Environment

Option contracts are frequently-used risk management tools. Because there are great uncertainties in supply and demand in water market, water option contracts are more complicated than plain vanilla option models for stocks. In this section, we suppose that a reservoir's surface water is the main source of an urban water supply company. The company need a call contract to hedge water supply risk during drought periods with low accumulated inflows. On the other hand, a separate water supply system, such as the South-North Water Transfer Project, is in charge of offering an additional amount of water. The additional water is no more than the maximum permissible if the contract is exercised. In the following part, we design an exotic option contract for water which allows a potential holder to decide how much water to buy at the expiration date at an agreed price if accumulated inflows fall below a predetermined threshold in a drought period. However, the amount of water provided by this option contracts can not exceed a predetermined maximum. Suppose water price follows the uncertain jump process, then the payoff function of this water option contract is

$$V_s = \begin{cases} 0, & \text{if } A_s \geq U, \\ (Z_s - K)^+(U - A_s), & \text{if } U > A_s > L, \\ (Z_s - K)^+(U - L), & \text{if } A_s \leq \ L, \end{cases} \tag{4.1}$$

where A_s denotes accumulated inflows in time s which has an uncertainty distribution function $\Theta_s(x)$, U and L are the upper and lower thresholds of the contract, s is the expiration time, K is the strike price, and Z_s is the water price. This formula can be equivalent to a succinct expression as follows,

$$V_s = (Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right). \tag{4.2}$$

Then the price of this water option contract is

$$f = \exp(-rs)E\left[(Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right)\right]. \tag{4.3}$$

Because $\min\left(U - L, (U - A_s)^+\right)$ is an uncertain variable with distribution

$$Q(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 - \Theta_s(U - x), & \text{if } 0 < x \leq U - L \\ 1, & \text{if } x > U - L, \end{cases} \tag{4.4}$$

$(Z_s - K)^+$ has a distribution

$$V(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \Psi_s(x + K), & \text{if } x > 0, \end{cases} \tag{4.5}$$

by the operational law $(Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right)$ has a distribution

$$W(x) = \sup_{x_1 x_2 = x} Q(x_1) \wedge V(x_2). \tag{4.6}$$

By the definition of expected value

$$\begin{aligned} & E\left[(Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right)\right] \\ &= \int_0^{+\infty} \mathcal{M}\left\{(Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right) \geq x\right\} \mathrm{d}x \\ &= \int_0^{+\infty} \left(1 - \mathcal{M}\left\{(Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right) < x\right\}\right) \mathrm{d}x \\ &= \int_0^{+\infty} \left(1 - \sup_{x_1 x_2 = x} Q(x_1) \wedge V(x_2)\right) \mathrm{d}x, \end{aligned} \tag{4.7}$$

where $Q(x)$ and $V(x)$ are given by (4.4) and (4.5). Hence the price of the water option is

$$\begin{aligned} f &= \exp(-rs)E\left[(Z_s - K)^+ \min\left(U - L, (U - A_s)^+\right)\right] \\ &= \exp(-rs) \int_0^{+\infty} \left(1 - \sup_{x_1 x_2 = x} Q(x_1) \wedge V(x_2)\right) \mathrm{d}x. \end{aligned}$$

The proof is completed.

Example 4.1. *Suppose that the jump size $Y_1, Y_2, \ldots$ follow normal uncertainty distribution with expected value 0, variance 1 and is no more than 0.2, the interarrival times follow lognormal uncertainty distributions as follows*

$$H(x) = \left(1 + \exp\left(\frac{-\pi \ln x}{\sqrt{3}}\right)\right)^{-1}$$

the accumulated flow takes values in $\{2, 3.5, 5, 6.5, 8, 9.5\}$ *with an empirical uncertainty distribution*

$$\Theta_s(x) = \begin{cases} 0, & \text{if } x < 2, \\ 0.1 + \frac{1}{15}(x-2), & \text{if } 2 \le x < 3.5, \\ 0.2 + \frac{1}{15}(x-3.5), & \text{if } 3.5 \le x < 5, \\ 0.3 + \frac{1}{15}(x-5), & \text{if } 5 \le x < 6.5, \\ 0.4 + \frac{1}{15}(x-6.5), & \text{if } 6.5 \le x < 8, \\ 0.5 + \frac{1}{15}(x-8), & \text{if } 8 \le x < 9.5, \\ 1, & \text{if } x \ge 9.5. \end{cases} \tag{4.8}$$

Assume the upper threshold $U = 3.5$, the lower threshold $L = 2.0$, the interest rate $r = 0.10$, the log-drift $e = 0.03$, the log-diffusion $\sigma = 0.25$, the initial price $Z_0 = 2.0$, the strike price $K = 2.2$, and the expiration time $s = 0.6$, then the water option price is about 0.3531.

As the actual water market has not been established, we can only obtain numerical results under hypothetical conditions in this example. However, a scientific and appropriate pricing method of water options is critical in water options market. Water users can use this water options as a powerful tool to secure their water risk exposure and profit from favourable water price movement.

5 Conclusions

Water option contracts can help water users manage water supply risk and make water allocation more efficient. This paper proposed an uncertain water price model when the price is not determined by the market but by governments and experts. We designed an option contract with this water price model and the pricing formulas were derived. In the future, more realistic water price model and more exotic water option should be further studied.

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