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Unique Fixed Point Results for Sequence of Self Mappings in Generalized Fuzzy Metric Spaces

Vishal Gupta[1,*], Manu Verma[1], Naveen Gulati[2]
[1]*Maharishi Markandeshwar University, Mullana, Ambala, Haryana, India*
[2]*Department of Mathematics, S.D. College, Ambala cantt, Haryana, India*

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Abstract

We prove the existence and uniqueness of common fixed point for a sequence of mapping, considering each pair of mapping satisfies the contractive condition on fuzzy metric space. We also prove same for generalized fuzzy metric spaces. We give the lemma to prove the main result in generalized fuzzy metric space. In this paper we extend the theorem of Gajic [5] in fuzzy metric space as well as in generalized fuzzy metric space for sequence of mappings. We give example to explain our main result.

Keywords: fixed point, fuzzy metric space, generalized fuzzy metric space

1 Introduction

Zadeh's [22] fuzzy set commencement has evolved the concept of fixed point theory in different directions. In 1992, concept of D-metric space was introduced by Dhage [4]. This D-metric space also known as generalized metric space. In 2006, Mustafa and Sims [15] introduced a new structure of generalized metric space called G-Metric spaces. Sung and Yang [21] introduced the notion of generalized fuzzy metric spaces. Generalized fuzzy metric space is a combination of the concept of fuzzy set introduced by Zadeh [22] in 1965 and the concept of fuzzy metric space introduced by Kramosil and Michalek [13]. George and Veeramani [6, 7] modified the definition of fuzzy metric space by defining the Hausdorff topology on Kramosil's fuzzy metric space. Many authors have also studied fuzzy metric space and generalized fuzzy metric space [1, 2, 3, 9, 10, 11, 12, 14, 16, 17, 18, 20]. Gajic [5] proved a common fixed point theorem for a sequence of self mappings defined on D-metric space given as follows:

Theorem 1.1 [13] Let (X, D) be a complete D-metric space, $f_n : X \to X, n \in N,$ be a sequence of mappings with property that for each $x, y, z \in X$ and any $i, j, k \in N \setminus \Delta, \Delta = \{(\mathrm{n,n,n}) : \mathrm{n} \in N\}$,

$$\mathrm{D}(f_i(x), f_j(y), f_k(z)) \geq q.\mathrm{D}(x, y, z) \text{ for some } q < 1. \tag{1}$$

If there exists $x_0 \in X$ such that $\sup_{y \in X} D(x_0, f_1(x_0), y) = M$, for some $M > 0$, then there exists a unique common fixed point for the family $\{f_n\}$.

2 Preliminaries

Definition 2.1 [19] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm. If $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,

[*] Corresponding author.
Emails: vishal.gmn@gmail.com (V. Gupta), ammanu7@gmail.com (M. Verma), dr.naveengulati31gmail.com (N. Gulati).

(iv) $a*b \le c*d$, whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Definition 2.2 [13] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is fuzzy sets on $X^2 \times [0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$,

(FM-5) $M(x, y, .) : [0,\infty) \to [0,1]$ is left continuous.

The function $M(x, y, t)$ denotes the degree of nearness between x and y w.r.t. t.

Remark 2.3 Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. Let us consider (X, M, T) is a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Lemma 2.4 [8] In fuzzy metric space $(X, M, *)$, $M(x, y, .)$ is non-decreasing for all $x, y \in X$.

Definition 2.5 [8] Let $(X, M, *)$ be a fuzzy metric space, then

a) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$;

b) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ or if for each $\varepsilon > 0$, there exist $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$, for each $n, m \ge n_0$;

c) a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.6 Let $X = \{1/n : n \in N\} \cup \{0\}$ and T be the continuous t-norm defined by $T(a,b) = ab$ $(T(a,b) = \min\{a,b\})$ respectively, for all $a, b \in [0,1]$. For each $t > 0$ and $x, y \in X$, define (X, M, T) by

$$M(x, y, t) = \begin{Bmatrix} \dfrac{t}{t+|x+y|}, & t > 0 \\ 0, & t = 0 \end{Bmatrix},$$

clearly (X, M, T) is a complete fuzzy metric space.

Definition 2.7 [21] A 3-tuple $(X, G, *)$ is said to be a G-fuzzy metric space (denoted by GF-space) if X is an arbitrary nonempty set, $*$ is a continuous t-norm and G is a fuzzy set on $X^3 \times [0,\infty)$ satisfying the following conditions for each $t, s > 0$:

$(GF-1)$ $G(x, x, y, t) > 0$ for all $x, y \in X$ with $x \ne y$,

$(GF-2)$ $G(x, x, y, t) \ge G(x, y, z, t)$ for all $x, y, z \in X$ with $y \ne z$,

$(GF-3)$ $G(x, y, z, t) = 1$ if and only if $x = y = z$,

$(GF-4)$ $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,

$(GF-5)$ $G(x, a, a, t) * G(a, y, z, t) \le G(x, y, z, t+s)$ (triangle inequality),

$(GF-6)$ $G(x, y, z, .) : (0,\infty) \to [0,1]$ is continuous.

Definition 2.8 [21] Let $(X, G, *)$ be a GF-space, then

(1) a sequence $\{x_n\}$ in X is said to be convergent to x (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} G(x_n, x_n, x, t) = 1$ for all $t > 0$,

(2) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} G(x_n, x_n, x_m, t) = 1$,

(3) a *GF*-space is said to be complete if every Cauchy sequence in X is convergent.

Remark 2.9 [12] Let $(X,G,*)$ be a *GF*-space with the following condition:

$(GF-7)$ $\lim_{t\to\infty} G(x,y,z,t)=1$ for all $x,y\in X$.

Lemma 2.10 [21] Let $(X,G,*)$ be a *GF*-space. Then $G(x,y,z,t)$ is non-decreasing with respect to t for all $x,y,z\in X$.

Lemma 2.11 [21] Let $(X,G,*)$ be a *GF*-space. Then G is a continuous function on $X^3\times(0,\infty)$.

3 Main Results

Here we give the introductory results for sequence of mappings to prove our result. The first result assumes that each pair of self mappings satisfies the same contraction condition and sequence $\{f_n\}$ has a common fixed point. The second result assumes that each f_n satisfies the same contraction condition and sequence $\{f_n\}$ tends to point wise to a limit function f, then f has a fixed point z, which is the limit of each of the fixed points z_n of f_n. The third result assumes that each $\{f_n\}$ converges uniformly to a function f which satisfies the particular contraction condition and each f_n has a fixed point z_n, then a sequence $\{z_n\}$ converges to z, where z is a fixed point of f. In this paper we prove a fixed point result of first type of sequence of mappings in fuzzy metric space and generalized fuzzy metric space. To prove our main result in fuzzy metric space we apply the following lemma.

Lemma 3.1 [14] Let $\{x_n\}$ be a sequence in fuzzy metric space $(X,M,*)$ with the condition (*FM*-6), if there exists a number $q\in(0,1)$ such that $M(x_{n+2},x_{n+1},qt)\geq M(x_{n+1},x_n,t)$ for all $t>0$ and $n=1,2,3,\ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 3.2 [14] If for all $x,y\in X$, $t>0$ and for a number $q\in(0,1)$, $M(x,y,qt)\geq M(x,y,t)$, then $x=y$.

Theorem 3.3 Let $(X,M,*)$ be a complete fuzzy metric space and $f_n: X\to X, n\in N$, be a sequence of mappings with property that for each $x,y,z\in X$ and any $i,j\in N\setminus\Delta, \Delta=\{(n,n):n\in N\}$,

$$M(f_i(x),f_j(y),qt)\geq M(x,y,t) \text{ for some } q<1. \quad (2)$$

Then there exists a unique common fixed point for the family $\{f_n\}$.

Proof: For $x\in X$, define a sequence $\{x_n\}=\{f_n(x_{n-1})\}$, $n\in N$, first we prove that $\{x_n\}$ is a Cauchy sequence. For any $n,p\in N$ and using (2) we have

$$M(x_n,x_{n+p},qt)=M(f_n(x_{n-1}),f_{n+p}(x_{n+p-1}),qt)\geq M(x_{n-1},x_{n+p-1},t).$$

Using Lemma (3.1), we observe that $\{x_n\}$ is a Cauchy sequence. Since $(X,M,*)$ being complete fuzzy metric space, there exists $z\in Z$ such that $z=\lim_{n\to\infty} x_n$.

Now we prove that z is the unique fixed point of the sequence $\{f_n\}$.

For fixed $k\in N$ and for any $m\in N, m>k$,

$$M(x_m,f_k(z),qt)=M(f_m(x_{m-1}),f_k(z),qt)\geq M(x_{m-1},z,t).$$

Since M is continuous, it follows that $M(z,f_k(z),qt)\geq M(z,z,t)$. Consequently $z=f_k(z)$. Hence z is a fixed point of the sequence $\{f_n\}$.

Uniqueness: We suppose that for some $y\in X$, $f_k(y)=y$ for all $k\in N$, as $q<1$ and $M(z,y,qt)$ $=M(f_k(z),f_k(y),qt)\geq M(z,y,t)$. Using Lemma (3.2), we have $z=y$. So the uniqueness is proved and hence the proof is completed.

Corollary 3.4 Let $(X,M,*)$ be a complete fuzzy metric space and $f_n : X \to X, n \in N,$ be a sequence of mappings with property that $M(f_i^m(x), f_j^m(y), qt) \geq M(x,y,t)$ for some $m \in N,$ each $x,y,z \in X$ and any $i,j \in N \setminus \Delta, \Delta = \{(n,n) : n \in N\}$ for some $q<1.$ Then there exists a unique common fixed point for the family $\{f_n\}$.

Proof: Theorem 3.3 implies that there exists a unique common fixed point for the sequence $\{f_k^m\}$. But by uniqueness, fixed point for f_k^m is a fixed point for f_k. Hence the proof is completed.

Example 3.5 Let the sequence $\{f_n\},$ where $f_n(x) = x/(1+nx^2)$ is defined in fuzzy metric space with $(X,M,*),$ $X=[0,1],$ converges point wise to limit $f(x)=0$. Also $x=0$ is a unique common fixed point for the family $\{f_n\}$.

To prove our main result in G-fuzzy metric space, first we prove lemma in G-fuzzy metric space.

Lemma 3.6 Let $\{x_n\}$ be a sequence in a G-fuzzy metric space $(X,G,*)$ with the condition $(GF-7)$. If there exists a number $q \in (0,1)$ such that

$$G(x_{n+1}, x_{n+1}, x_{n+2}, qt) \geq G(x_n, x_n, x_{n+1}, t) \text{ for all } t>0 \text{ and } n=1,2,3,..., \tag{3}$$

then $\{x_n\}$ is a Cauchy sequence in X.

Proof: For $t>0$ and $q \in (0,1),$ we have $G(x_2,x_2,x_3,qt) \geq G(x_1,x_1,x_2,t) \geq G\left(x_0,x_0,x_1, t/q\right),$ this implies

$$G(x_2,x_2,x_3,t) \geq G\left(x_0,x_0,x_1, t/q^2\right).$$

By simple induction with the condition (3), we have

$$G(x_{n+1},x_{n+1},x_{n+2},qt) \geq G\left(x_0,x_0,x_1, t/q^n\right) \tag{4}$$

for all $t>0$ and $n=1,2,3....$ Thus by (4) and $(GF-5)$, for any positive integer p and real number $t>0,$ we have

$$G(x_{n+1},x_{n+1},x_{n+p},t) \geq G\left(x_{n+1},x_{n+1},x_{n+2}, t/p\right) * \cdots p-times \cdots * G\left(x_{n+p-1},x_{n+p-1},x_{n+p}, t/p\right)$$

$$\geq G\left(x_1,x_1,x_2, t/pq^{n-1}\right) * \cdots p-times \cdots * G\left(x_1,x_1,x_2, t/pq^{n+p-2}\right).$$

Therefore, by $(GF-7)$, we have

$$\lim_{n\to\infty} G(x_{n+1},x_{n+1},x_{n+p},t) \geq 1*1*\cdots*1 \geq 1,$$

this implies that $\{x_n\}$ is a Cauchy sequence in X. This completes the proof.

Lemma 3.7 [12] If for all $x,y \in X,$ $t>0$ and for a number $q \in (0,1)$, $G(x,x,y,qt) \geq G(x,x,y,t),$ then $x=y.$

Theorem 3.8 Let $(X,G,*)$ be a complete G-fuzzy metric space, $f_n : X \to X, n \in N,$ be a sequence of mappings with the property that for each $x,y,z \in X$ and any $i,j,k \in N \setminus \Delta, \Delta = \{(\mathrm{n,n,n}) : \mathrm{n} \in N\},$

$$G(f_i(x), f_j(y), f_k(z), qt) \geq G(x,y,z,t) \text{ for some } q<1. \tag{5}$$

Then there exists a unique common fixed point for the family $\{f_n\}$.

Proof: For $x \in X,$ define a sequence $\{x_n\} = \{f_n(x_{n-1})\},$ $n \in N$, we prove that $\{x_n\}$ is a G-Cauchy sequence.
For any $n,p \in N$ and using (5) we have

$$G(x_n,x_n,x_{n+p},qt) = G(f_n(x_{n-1}), f_n(x_{n-1}), f_{n+p}(x_{n+p-1}), qt) \geq G(x_{n-1},x_{n-1},x_{n+p-1},t).$$

Now using Lemma (3.6) we observe that $\{x_n\}$ is a G-Cauchy sequence. Since $(X,G,*)$ being complete fuzzy metric space, there exists $z \in Z$ such that $z = \lim_{n\to\infty} x_n$.

We prove that z is the unique fixed point of the sequence $\{f_n\}$.

Fixed $k \in N$ for any $m \in N, m>k$,

$$G(x_m, f_k(z), f_k(z), qt) = G(f_m(x_{m-1}), f_k(z), f_k(z), qt) \geq G(x_{m-1}, z, z, t).$$

Since G is continuous, it follows that $G(z, f_k(z), f_k(z), qt) \geq G(z, z, z, t)$. Consequently $z = f_k(z)$. Hence z is a fixed point of the sequence $\{f_n\}$.

Uniqueness: We suppose for some $y \in X$, $f_k(y) = y$, for all $k \in N$, as $q < 1$ and

$$G(z, z, y, qt) = G(f_k(z), f_k(z), f_k(y), qt) \geq G(z, z, y, t).$$

Using Lemma (3.7), we have $z = y$. So, uniqueness is proved and the proof is completed.

Corollary 3.9 Let $(X, G, *)$ be a complete G-fuzzy metric space. $f_n : X \to X, n \in N$, be sequence of mappings with property that for some $m \in N$, each $x, y, z \in X$ and any $i, j, k \in N \setminus \Delta, \Delta = \{(\text{n}, \text{n}, \text{n}) : \text{n} \in N\}$,

$$G(f_i^m(x), f_j^m(y), f_k^m(z), qt) \geq G(x, y, z, t) \text{ for some } q < 1. \quad (6)$$

Then there exists a unique common fixed point for the family $\{f_n\}$.

Proof: Here, Theorem 3.8 implies that there exists the unique common fixed point for the sequence $\{f_k^m\}$, but the fixed point for f_k^m by uniqueness is a fixed point for f_k. This completes the proof.

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