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Bridging Credibility Measures and Credibility Distribution Functions on Euclidian Spaces

Xueqin Feng[1], Yankui Liu[2,*]
[1] *College of Economics, Hebei University, Baoding 071002, Hebei, China*
[2] *College of Mathematics & Information Science, Hebei University, Baoding 07100201, Hebei, China*

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Abstract

This paper develops an approach to constructing credibility measures on Euclidian space. Given a credibility measure on the ample field of the n-ary Euclidian space, the value of the credibility distribution function at a point is defined as the credibility measure of the atom containing the point. The fundamental properties of the induced credibility distribution function are studied. After that, we build the credibility measure based on a credibility distribution function. The obtained theoretical results establish a one-to-one correspondence between credibility measures and credibility distribution functions, and have potential applications in various decision-making problems under fuzzy uncertainty, in which the uncertain parameters can be described by creditability distribution functions.

Keywords: credibility measure, atom, credibility distribution function, one-to-one correspondence, decision making

1 Introduction

Since the work of Liu and Liu [16], the theory of credibility measure has been well-developed in the literature. From the theoretical perspectives, Liu et al. [18] developed the infinite-dimensional product possibility theory, which provided the theoretical foundation for the existence of fuzzy process. Based on the convergence modes of fuzzy variables, Liu [23] discussed the convergent results about the use of fuzzy simulation and approximation method. Liu and Gao [24] studied the independence of fuzzy variables as well as the independence of fuzzy events. Li and Liu [12] gave a sufficient and necessary condition for credibility measures. Chen and Liu [4] gave a strong law of large numbers in the theory of credibility measure. Liu and Wang [22] discussed the properties of credibility critical value functions. Wang and Watada [30] discussed some properties of T-independent fuzzy variables. Li and Liu [10] proposed the entropy of credibility distributions for fuzzy variables. Li and Liu [13] established the foundation of credibilistic logic. To overcome the difficulty of computing the variance of fuzzy variable, Wu and Liu [32] defined the spread of variable based on Lebesgue-Stieltjes integral. To gauge the risk resulted from fuzzy uncertainty, Chen et al. [5] defined the absolute deviation and absolute semi-deviation for fuzzy variable by nonlinear fuzzy integrals. Liu and Liu [19] constructed additive interval set functions by virtue of joint credibility distributions of fuzzy vectors. Kamdem et al. [9] gave some properties about the moments and semi-moments of fuzzy variables. Shen and Zhao [28] proposed a credibilistic approach to assumption-based truth maintenance. Tang et al. [29] defined a metric space of fuzzy variables, and proved the completeness of this space under the new distances. Based on the independence of fuzzy interarrival times, Zhao and Liu [36] studied the renewal process and renewal reward process. Li [11] discussed some properties of fuzzy alternating renewal processes. Hong [7] investigated the law of large numbers and the renewal process for T-related fuzzy numbers with respect to the credibility measure. Under expected value of fuzzy variable and continuous Archimedean triangular norms, Wang and Liu [31] developed the renewal process and the renewal reward process for T-independent L-R fuzzy variables in fuzzy decision systems. Hong [8] derived fuzzy Blackwell's theorem based on the expected value of fuzzy variables.

*Corresponding author.
Email: yliu@hbu.edu.cn (Y. Liu).

To further develop the theory of credibility measure, fuzzy possibility theory, an axiomatic approach to dealing with type-2 fuzziness, has been documented in the recent literature [17, 25]. There are four major reduction methods in fuzzy possibility theory to reduce the uncertainty embedded in the secondary possibility distributions. The value-at-risk (VaR) reduction methods were based on the VaRs of regular fuzzy variables in the sense of possibility and credibility measures [1, 2, 3]. The critical value reduction methods were defined by Sugeno fuzzy integrals of regular fuzzy variables [26]. The mean value reduction methods were given by Chouqet fuzzy integral of regular fuzzy variables [27, 35], and the equivalent reduction methods were introduced via classic Lebesgue-Stieltjes integrals of regular fuzzy variables [33, 34]. Furthermore, Liu and Liu [20] and Chen and Shen [6] investigated parametric interval-valued fuzzy variables in fuzzy possibility theory, they described interval-valued secondary possibility distributions by virtue of lambda selection variables. Both reduced fuzzy variables and lambda selection variables have variable parametric possibility distributions, which have been used to build generalized credibility measures in various decision-making problems under fuzzy uncertainty [21]. From the theoretical development mentioned above, we observe the fact that there is a one-to-one correspondence between possibility measures and normalized possibility distribution functions, and a one-to-one correspondence between fuzzy possibility measures and regular fuzzy variable-valued maps [25], which motivates us to bridge credibility measures and credibility distribution functions on Euclidian spaces from a new perspective.

The structure of the paper is as follows. In Section 2, we deal with the properties of the induced credibility distribution function under the condition that a credibility measure is given on the ample field of the n-ary Euclidian space. Section 3 builds the credibility measure via a given credibility distribution function. Finally, the conclusions of the paper are summarized in Section 4.

2 The Induced Credibility Distribution Functions

In this paper, when we use some concepts but not defined, the reader may refer to [14, 15, 21] and the references therein.

Suppose Ξ^n is a nonempty subset of space $\Re^n$, $\mathcal{A}(\Xi^n)$ the ample field on Ξ^n, and Cr is a credibility measure on the ample space $(\Xi^n, \mathcal{A}(\Xi^n))$. Then credibility measure Cr can be characterized by the following four properties [12]:

($C1$) Normality: $\mathrm{Cr}(\Gamma) = 1$;

($C2$) Monotonicity: $\mathrm{Cr}(A) \leq \mathrm{Cr}(B)$ whenever $A, B \in \mathcal{A}(\Xi^n)$ and $A \subseteq B$;

($C3$) Self Duality: $\mathrm{Cr}(A) + \mathrm{Cr}(A^c) = 1$ for all $A \in \mathcal{A}(\Xi^n)$;

($C4$) Arithmetic Law: for any family $\{A_i, i \in I\} \subseteq \mathcal{A}(\Xi^n)$ with $\mathrm{Cr}(A_i) \leq 1/2$, one has

$$\mathrm{Cr}\left(\bigcup_{i\in I} A_i\right) \bigwedge \frac{1}{2} = \sup_{i\in I} \mathrm{Cr}(A_i).$$

Given a credibility measure Cr satisfying properties (C1)–(C4), if we denote

$$\mu_n(x) = \mathrm{Cr}([x]),\ x = (x_1, \ldots, x_n) \in \Xi^n \tag{1}$$

as the credibility distribution function induced by credibility measure Cr at atom $[x] \in \mathcal{A}(\Xi^n)$, then one has the following result:

Proposition 1. *Let* Cr *be a credibility measure on the ample field* $\mathcal{A}(\Xi^n)$*, and* $\mu_n(x)$ *the credibility distribution function induced by* Cr *via Eq. (1). Then for any* $A \in \mathcal{A}(\Xi^n)$*, one has*

$$\mathrm{Cr}(A) = \begin{cases} \sup_{x\in A} \mu_n(x), & \text{if } \sup_{x\in A} \mathrm{Cr}([x]) < 1/2 \\ 1 - \sup_{x\in A^c} \mu_n(x), & \text{if } \sup_{x\in A} \mathrm{Cr}([x]) \geq 1/2. \end{cases} \tag{2}$$

Proof. Since $A = \bigcup_{x\in A}[x]$, when $\sup_{x\in A} \mathrm{Cr}([x]) < 1/2$, by (C4), one has

$$\mathrm{Cr}(A) \wedge \frac{1}{2} = \sup_{x\in A} \mathrm{Cr}([x]) < \frac{1}{2},$$

which implies

$$\mathrm{Cr}(A) = \sup_{x\in A} \mathrm{Cr}([x]) = \sup_{x\in A} \mu_n(x).$$

On the other hand, when $\sup_{x\in A} \mathrm{Cr}([x]) \geq 1/2$, by (C2), one has

$$\mathrm{Cr}(A) \geq \sup_{x\in A} \mathrm{Cr}([x]) \geq \frac{1}{2}.$$

By the self-duality property (C3), we know $\mathrm{Cr}(A^c) \leq 1/2$. By monotonicity property (C2), $\mathrm{Cr}([x]) \leq 1/2$ for any $x \in A^c$. Finally, by arithmetic law (C4), one has

$$\mathrm{Cr}(A^c) = \mathrm{Cr}(A^c) \wedge \frac{1}{2} = \sup_{x\in A^c} \mathrm{Cr}([x]) = \sup_{x\in A^c} \mu_n(x),$$

which implies

$$\mathrm{Cr}(A) = 1 - \mathrm{Cr}(A^c) = 1 - \sup_{x\in A^c} \mu_n(x).$$

The proof of proposition is complete. □

Furthermore, the properties of the induced credibility distribution function $\mu_n(x)$ is summarized in the following theorem.

Theorem 1. *Let* Cr *be a credibility measure on the ample field* $\mathcal{A}(\Xi^n)$. *Then the function* $\mu_n(x)$ *defined by Eq. (1) satisfies the following conditions:*

i) There is some $h \in [1/2, 1]$ *such that* $0 \leq \mu_n(x) \leq h$*;*

ii) $\sup_{x\in\Xi^n} \mu_n(x) = h$*;*

iii) Either $\mu_n(t) < 1/2$ *for all* $t \in \Xi^n$, *or there is some* $t \in \Xi^n$ *with* $\mu_n(t) = h$ *such that*

$$\sup_{x\in\Xi^n\backslash[t]} \mu_n(x) = 1 - h.$$

Proof. We first prove the case $h = 1/2$. In this case, for any $x = (x_1, \ldots, x_n) \in \Xi^n$, $\mathrm{Cr}([x]) \leq 1/2$. Since $\mathrm{Cr}(\Xi^n) = 1$, one has

$$\frac{1}{2} = \mathrm{Cr}(\Xi^n) \wedge \frac{1}{2} = \sup_{x\in\Xi^n} \mathrm{Cr}([x]) = \sup_{x\in\Xi^n} \mu_n(x).$$

If $\mu_n(t) < 1/2$ for all $t \in \Xi^n$, the case is proved. If there is a point $t \in \Xi^n$ with $\mu_n(t) = 1/2$. Then, by the self duality of Cr, one has

$$\mathrm{Cr}(\Xi^n\backslash[t]) = 1 - \mathrm{Cr}([t]) = \frac{1}{2}.$$

As a consequence, one has

$$\frac{1}{2} = \mathrm{Cr}(\Xi^n\backslash[t]) \wedge \frac{1}{2} = \sup_{x\in\Xi^n\backslash[t]} \mathrm{Cr}([x]) = \sup_{x\in\Xi^n\backslash[t]} \mu_n(x).$$

We next prove the case $h > 1/2$. In this case, there is a unique $t \in \Xi^n$ such that $\mu_n(t) = h$, and for any $x \in \Xi^n\backslash[t]$, one has $\mathrm{Cr}([x]) < 1/2$. Then, by the self-duality of Cr, one has

$$\mathrm{Cr}(\Xi^n\backslash[t]) = 1 - \mathrm{Cr}([t]) = 1 - h < \frac{1}{2}.$$

As a consequence, one has

$$1 - h = \mathrm{Cr}(\Xi^n\backslash[t]) \wedge \frac{1}{2} = \sup_{x\in\Xi^n\backslash[t]} \mathrm{Cr}([x]) = \sup_{x\in\Xi^n\backslash[t]} \mu_n(x).$$

It is evident that $\sup_{x\in\Xi^n} \mu_n(x) = h$. The proof of theorem is complete. □

In the next section, we will show that the inverse proposition of Theorem 1 is also valid.

3 Building Credibility Measures

According to Theorem 1, we introduce the following definition about *credibility distribution function*:

Definition 1. *For any function $\mu(x)$ defined on $\Xi^n \subseteq \Re^n$, we call $\mu(x)$ a credibility distribution function if and only if it satisfies conditions i), ii) and iii) in Theorem 1, where the ample field $\mathcal{A}(\Xi^n)$ is replaced by the power set $\mathcal{P}(\Xi^n)$.*

Example 1. The following functions are continuous credibility distribution functions in the sense of Definition 1.

$$\mu_1(x) = \begin{cases} \frac{x-r_1}{2(r_2-r_1)}, & \text{if } r_1 \le x < r_2 \\ \frac{r_3-x}{2(r_3-r_2)}, & \text{if } r_2 \le x \le r_3 \\ 0, & \text{otherwise,} \end{cases} \quad \mu_2(x) = \begin{cases} \frac{x-r_1}{2(r_2-r_1)}, & \text{if } r_1 \le x < r_2 \\ \frac{1}{2}, & \text{if } r_2 \le x < r_3 \\ \frac{r_3-x}{2(r_3-r_2)}, & \text{if } r_3 \le x \le r_4 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mu_3(x) = \frac{1}{2}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}, \quad x \in \Re,$$

where the parameters in the above functions satisfy the conditions $r_1 < r_2 \le r_3 \le r_4$, $a \in \Re$ and $\sigma > 0$.

Example 2. The following is a discrete credibility distribution function in the sense of Definition 1.

$$\mu(k) = \frac{1}{2}\left(1 - \frac{1}{k}\right), k = 2, 3, \ldots.$$

So far, for every credibility measure Cr on the ample space $(\Xi^n, \mathcal{A}(\Xi^n))$, it follows from Theorem 1 that there is a credibility distribution function $\mu_n(x)$ determined by Eq. (1). The following theorem deals with the inverse proposition of Theorem 1:

Theorem 2. *If $\mu = \mu(x)$ is a credibility distribution function defined on $\Xi^n \subseteq \Re^n$ in the sense of Definition 1, then there is a unique credibility measure* Cr *on the ample space $(\Xi^n, \mathcal{P}(\Xi^n))$ such that* $\text{Cr}(\{x\}) = \mu(x)$ *holds for any $x = (x_1, \ldots, x_n) \in \Xi^n$.*

Proof. By Definition 1, $\mu = \mu(x)$ satisfies the following three conditions:

i) There is some $h \in [1/2, 1]$ such that $0 \le \mu(x) \le h$;

ii) $\sup_{x\in\Xi^n} \mu(x) = h$;

iii) Either $\mu(t) < 1/2$ for all $t \in \Xi^n$, or there is some $t \in \Xi^n$ with $\mu(t) = h$ such that

$$\sup_{x\in\Xi^n\setminus\{t\}} \mu(x) = 1 - h.$$

We first prove the validness of theorem in the case of $h = 1/2$.

In this case, for any $A \in \mathcal{P}(\Xi^n)$, we define a set function Cr as follows

$$\text{Cr}(A) = \begin{cases} \sup_{x\in A} \mu(x), & \text{if } \sup_{x\in A} \mu(x) < 1/2 \\ 1 - \sup_{x\in A^c} \mu(x), & \text{if } \sup_{x\in A} \mu(x) = 1/2. \end{cases}$$

Then, one has $\text{Cr}(\{x\}) = \mu(x)$ for any $x = (x_1, \ldots, x_n) \in \Xi^n$. In addition,

$$\text{Cr}(\Xi^n) = 1 - \sup_{x\in\emptyset} \mu(x) = 1.$$

Let $A, B \in \mathcal{P}(\Xi^n)$ and $A \subseteq B$. If $\sup_{x\in B} \mu(x) < 1/2$, then

$$\text{Cr}(A) = \sup_{x\in A} \mu(x) \le \sup_{x\in B} \mu(x) = \text{Cr}(B).$$

If $\sup_{x\in A} \mu(x) < 1/2$ and $\sup_{x\in B} \mu(x) = 1/2$, then

$$\text{Cr}(A) = \sup_{x\in A} \mu(x) < \frac{1}{2} \le 1 - \sup_{x\in B^c} \mu(x) = \text{Cr}(B).$$

If $\sup_{x\in A}\mu(x)=1/2$, then $\sup_{x\in B}\mu(x)=1/2$. Thus,

$$\mathrm{Cr}(A)=1-\sup_{x\in A^c}\mu(x)\leq\ 1-\sup_{x\in B^c}\mu(x)=\mathrm{Cr}(B).$$

We now prove the self-duality of Cr. $A\in\mathcal{P}(\Xi^n)$. If $\sup_{x\in A}\mu(x)<1/2$, then $\sup_{x\in A^c}\mu(x)=1/2$. Thus, one has

$$\mathrm{Cr}(A)=\sup_{x\in A}\mu(x),\mathrm{Cr}(A^c)=1-\sup_{x\in A}\mu(x)=1-\mathrm{Cr}(A).$$

If $\sup_{x\in A}\mu(x)=1/2$, then $\sup_{x\in A^c}\mu(x)\leq 1/2$. When $\sup_{x\in A^c}\mu(x)=1/2$, one has

$$\mathrm{Cr}(A)=1-\sup_{x\in A^c}\mu(x)=\frac{1}{2}=1-\sup_{x\in A}\mu(x)=\mathrm{Cr}(A^c).$$

When $\sup_{x\in A^c}\mu(x)<1/2$, one has

$$\mathrm{Cr}(A)=1-\sup_{x\in A^c}\mu(x)=1-\mathrm{Cr}(A^c).$$

Furthermore, for any family $\{A_i,i\in I\}\subseteq\mathcal{P}(\Xi^n)$ with $\mathrm{Cr}(A_i)\leq 1/2$. If $\sup_{i\in I}\mathrm{Cr}(A_i)=1/2$, then by the monotonicity of Cr, one has

$$\mathrm{Cr}\left(\bigcup_{i\in I}A_i\right)\geq\sup_{i\in I}\mathrm{Cr}(A_i)=\frac{1}{2}.$$

As a consequence, one has

$$\mathrm{Cr}\left(\bigcup_{i\in I}A_i\right)\wedge\frac{1}{2}=\frac{1}{2}=\sup_{i\in I}\mathrm{Cr}(A_i).$$

On the other hand, if $\sup_{i\in I}\mathrm{Cr}(A_i)<1/2$, then

$$\sup_{i\in I}\mathrm{Cr}(A_i)=\sup_{i\in I}\sup_{x\in A_i}\mu(x)=\sup_{x\in\cup_{i\in I}A_i}\mu(x)<\frac{1}{2}.$$

Thus, one has

$$\mathrm{Cr}\left(\bigcup_{i\in I}A_i\right)=\sup_{x\in\cup_{i\in I}A_i}\mu(x)<\frac{1}{2},$$

which implies

$$\mathrm{Cr}\left(\bigcup_{i\in I}A_i\right)\wedge\frac{1}{2}=\mathrm{Cr}\left(\bigcup_{i\in I}A_i\right)=\sup_{i\in I}\mathrm{Cr}(A_i).$$

Therefore, the set function Cr is a credibility measure.

We now prove the uniqueness. If there is another credibility measure Cr$'$ on the ample space $(\Xi^n,\mathcal{P}(\Xi^n))$ such that for any $x=(x_1,\ldots,x_n)\in\Xi^n$, one has $\mathrm{Cr}'(\{x\})=\mu(x)$.

Then, for any $A\in\mathcal{P}(\Xi^n)$, one has

$$\mathrm{Cr}'(A)=\begin{cases}\sup_{x\in A}\mu(x), & \text{if }\ \sup_{x\in A}\mu(x)<1/2\\ 1-\sup_{x\in A^c}\mu(x), & \text{if }\ \sup_{x\in A}\mu(x)=1/2,\end{cases}$$

which implies that Cr$'$ and Cr are identical.

We next prove the validness of theorem in the case of $h>1/2$.

In this case, there is a unique point t with $\mu(t)=h$, and $\sup_{x\in\Xi^n\setminus\{t\}}\mu(x)=1-h$. For any $A\in\mathcal{P}(\Xi^n)$, we define a set function Cr as follows

$$\mathrm{Cr}(A)=\begin{cases}1-\sup_{x\in A^c}\mu(x), & \text{if } t\in A\\ \sup_{x\in A}\mu(x), & \text{if } t\notin A\end{cases}$$

It is easy to check that $\mathrm{Cr}(\{x\})=\mu(x)$ for any $x=(x_1,\ldots,x_n)\in\Xi^n$. In addition,

$$\mathrm{Cr}(\Xi^n)=1-\sup_{x\in\emptyset}\mu(x)=1.$$

Let $A, B \in \mathcal{P}(\Xi^n)$ and $A \subseteq B$. If $t \in A$, then $t \in B$. Thus,

$$\mathrm{Cr}(A) = 1 - \sup_{x \in A^c} \mu(x) \leq 1 - \sup_{x \in B^c} \mu(x) = \mathrm{Cr}(B).$$

If $t \notin A$ but $t \in B$, then

$$\mathrm{Cr}(A) = \sup_{x \in A} \mu(x) \leq 1 - h \leq 1 - \sup_{x \in B^c} \mu(x) = \mathrm{Cr}(B).$$

If $t \notin B$, then $t \notin A$. Thus, one has

$$\mathrm{Cr}(A) = \sup_{x \in A} \mu(x) \leq \sup_{x \in B} \mu(x) = \mathrm{Cr}(B).$$

We now prove the self-duality of Cr. Let $A \in \mathcal{P}(\Xi^n)$. If $t \in A$, then $t \notin A^c$. Thus,

$$\mathrm{Cr}(A) = 1 - \sup_{x \in A^c} \mu(x) = 1 - \mathrm{Cr}(A^c).$$

If $t \notin A$, then $t \in A^c$. Thus,

$$\mathrm{Cr}(A^c) = 1 - \sup_{x \in A} \mu(x) = 1 - \mathrm{Cr}(A).$$

Furthermore, for any family $\{A_i, i \in I\} \subseteq \mathcal{P}(\Xi^n)$ with $\mathrm{Cr}(A_i) \leq 1/2$. Since $\mathrm{Cr}(A_i) \leq 1/2$, we have $t \notin A_i$ for any $i \in I$, which implies that $t \notin \cup_{i \in I} A_i$. As a consequence, one has

$$\sup_{i \in I} \mathrm{Cr}(A_i) = \sup_{i \in I} \sup_{x \in A_i} \mu(x) = \sup_{x \in \cup_{i \in I} A_i} \mu(x).$$

Noting that

$$\mathrm{Cr}\left(\bigcup_{i \in I} A_i\right) \leq 1 - h < \frac{1}{2},$$

thus, we have

$$\mathrm{Cr}\left(\bigcup_{i \in I} A_i\right) \wedge \frac{1}{2} = \mathrm{Cr}\left(\bigcup_{i \in I} A_i\right) = \sup_{x \in \cup_{i \in I} A_i} \mu(x),$$

which implies

$$\mathrm{Cr}\left(\bigcup_{i \in I} A_i\right) \wedge \frac{1}{2} = \sup_{i \in I} \mathrm{Cr}(A_i).$$

Therefore, the set function Cr is a credibility measure.

We next prove the uniqueness. If there is another credibility measure Cr' on the ample space $(\Xi^n, \mathcal{P}(\Xi^n))$ such that for any $x = (x_1, \ldots, x_n) \in \Xi^n$, one has $\mathrm{Cr}'(\{x\}) = \mu(x)$.

Then, for any $A \in \mathcal{P}(\Xi^n)$, one has

$$\mathrm{Cr}'(A) = \begin{cases} 1 - \sup_{x \in A^c} \mu(x), & \text{if } t \in A \\ \sup_{x \in A} \mu(x), & \text{if } t \notin A, \end{cases}$$

which implies that Cr' and Cr are identical. □

4 Conclusions

In the theory of credibility measure, this paper studied the relationship between credibility measures and credibility distribution functions on Euclidian spaces. The major results of the paper are as follows.

Firstly, given a credibility measure on the ample field of an n-ary Euclidian space, we defined the value of credibility distribution function at a point as the credibility measure of the atom containing the point, and discussed the properties of the induced credibility distribution function in Theorem 1.

Secondly, Theorem 2 provided us an interesting result that a credibility measure can be uniquely determined provided a credibility distribution function on an n-ary Euclidian space is given in advance.

From the obtained theoretical results, we concluded that credibility distribution functions can be used to describe fuzzy parameters in various decision-making problems under fuzzy uncertainty.

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