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Why It is Important to Precisiate Goals

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Abstract

After Zadeh and Bellman explained how to optimize a function under fuzzy constraints, there have been many successful applications of this optimization. However, in many practical situations, it turns out to be more efficient to precisiate the objective function before performing optimization. In this paper, we provide a possible explanation for this empirical fact.

Keywords: optimization under uncertainty, fuzzy uncertainty, interval uncertainty, precisiation

1 Formulation of the Problem

Objectives are usually imprecise (fuzzy). In many real-life situations, our objectives are imprecise (fuzzy). A company may want to have a good growth rate, an excellent level of customer satisfaction, etc. A university program may seek excellent quality of graduates, steady growth of the program, more good students and faculty in the program, etc. Many such statements use imprecise (fuzzy) words like "good", "excellent", etc.

At first glance, it makes sense to work with imprecise objectives. Situations when experts use imprecise (fuzzy) words from natural language are ubiquitous in many application areas. To handle such situations, Zadeh came up with fuzzy set and fuzzy logic techniques [10]. These techniques have been successfully applied to many application areas, in particular, to many practical situations in which both the system itself and the corresponding objective function are fuzzy; see, e.g., [1, 2, 5, 8].

From this viewpoint, to solve a real-life problem with fuzzy objectives, it seems reasonable to use these fuzzy techniques.

Somewhat surprisingly, in many practical situations, it is better to first precisiate goals. While a natural idea is to deal directly with the fuzzy objectives, business experience shows that in many cases, it is beneficial to instead "precisiate" the goals – i.e., make them precise – and then solve the optimization problem with the resulting crisp constraints; see, e.g., [6, 9].

Comment. This empirical fact relates to situations when we *can* precisiate, e.g., in business situations, when we can replace crude expert estimates with more precise computation results.

In some practical situations, however, precisiation is not possible: e.g., when the objective function describes such hard-to-crisply-gauge quantities as customer satisfaction (in particular, driving comfort).

In such situation, it is necessary to perform optimization under a fuzzy objective function.

What we do in this paper. In this paper, we provide a possible explanation for the above empirical fact – namely, for the fact that it is often beneficial to precisiate goals.

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2 Analysis of the Problem and the Main Result

Case of interval uncertainty. We will explain the advantage of precisiation on the example of the simplest possible types of fuzzy objective functions – namely, objective functions with interval uncertainty, when for each x:

- we know the interval $[\underline{f}(x), \overline{f}(x)]$ of possible values of the objective function $f(x)$, but
- we have no information about which values from this interval are more probable and which are less probable; see, e.g., [3, 7].

Each interval $[\underline{f}(x), \overline{f}(x)]$ can be represented in the form $[\widetilde{f}(x) - \varepsilon, \widetilde{f}(x) + \varepsilon]$, where $\widetilde{f}(x) \stackrel{\text{def}}{=} (\underline{f}(x) + \overline{f}(x))/2$ is the midpoint of this interval and $\varepsilon \stackrel{\text{def}}{=} (\overline{f}(x) - \underline{f}(x))/2$ is the interval's half-width.

In these terms, knowing the interval $[\widetilde{f}(x) - \varepsilon, \widetilde{f}(x) + \varepsilon]$ means that for every alternative x, the value $f(x)$ of the actual (unknown) objective function belongs to this interval, i.e., equivalently, this value is ε-close to the given approximate value $\widetilde{f}(x)$: $|\widetilde{f}(x) - f(x)| \le \varepsilon$.

A general case of fuzzy uncertainty can be reduced to the interval case. Interval uncertainty is the simplest possible – but, on the other hand, it is general enough. Indeed, every fuzzy number $\mathbf{x}$ can be described as a nested family of intervals – its α-cuts $\mathbf{x}(\alpha) \stackrel{\text{def}}{=} \{x : \mu(x) \ge \alpha\}$. Under reasonable assumptions, processing fuzzy data, i.e., computing the value $\mathbf{y} = f(\mathbf{x}_1, \cdots, \mathbf{x}_n)$, is equivalent to computing, for each α, the corresponding α-cut $\mathbf{y}(\alpha)$ as the range of the given function $f(x_1, \cdots, x_n)$ over the corresponding α-cuts:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \cdots, \mathbf{x}_n(\alpha)) = \{f(x_1, \cdots, x_n) : x_1 \in \mathbf{x}_1(\alpha), \ldots, x_1 \in \mathbf{x}_1(\alpha)\};$$

see, e.g., [5, 8].

Empirical fact: case of interval uncertainty. We consider the situation in which, instead of the actual objective function $f(x)$, we only know an approximate objective function $\widetilde{f}(x)$ for which $|\widetilde{f}(x) - f(x)| \le \varepsilon$ for all x.

We have two options:

- the first option is to select an alternative x based on the approximate objective function $\widetilde{f}(x)$;
- the second option is to elicit the actual objective function $f(x)$, and then select the alternative x which is the best according to the actual objective function.

Of course, if the second option is possible, then its return should be somewhat better – but eliciting the actual objective function often requires a lot of efforts. So, a natural question is whether this time-consuming precisiation is worth the effort.

Intuitively, we expect that since we know the objective function $f(x)$ with accuracy ε, the result of using the first optimization option is ε-close to the actual maximum. So, if we have a reasonably accurate approximation $\widetilde{f}(x)$, with a small ε, the first option should lead to a very good solution. In other words, it seems that precisiation should not be worth the effort.

In practice, however, the result of applying the second option are much better than expected, the advantage is much higher than the intuitive estimate ε. To understand why this happens, let us analyze the corresponding optimization problem.

Optimization: reminder. Ideally, we want to find a value x at which the actual objective function $f(x)$ attains its largest possible value, i.e., for which $f(x) \ge f(y)$ for all $y \ne x$.

If there are several alternatives x which have the same value of the objective function $f(x)$, we would like to obtain all such alternatives – so that we would be able to select, among them, the one that maximizes some other desired characteristic. For example, if we are selecting trajectories of a manned spaceflight to Mars that provide the maximum possible safety, and there are several such trajectories, then we use this non-uniqueness to select a trajectory which requires the smallest possible amount of fuel, or, alternatively, that minimizes the flight time.

From this viewpoint, we would like not to miss all alternatives x for which $f(x) \ge f(y)$ for all y.

How do we describe all desired optimal alternatives based on the approximate objective function. In real life, we do not know the exact values of the objective function, we only know the approximate objective function $\widetilde{f}(x)$ which is known to be ε-close to $f(x)$. How can we make sure that we do not miss any f-optimal alternatives if all we know is this approximate objective function?

The answer to this question is given by the following proposition.

Proposition 1. *For each function $\widetilde{f}(x)$ and for each alternative x_0, the following two conditions are equivalent to each other:*

- *the alternative x_0 is optimal relative to some function $f(x)$ which is ε-close to $\widetilde{f}(x)$;*
- *for every $y \neq x$, we have $\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon$.*

Comment. For reader's convenience, all the proofs are placed in the special Proofs section.

Based on the Proposition, the only way not to miss any f-optimal alternative is to consider all possible alternatives for which $\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon$ for all $y \neq x_0$.

How good are the selections based on the approximate objective function? As we have argued in the previous text, if all we know is an ε-approximate objective function, then we should select an alternative x_0 for which

$$\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon \tag{1}$$

for all $y \neq x_0$.

How good is this selection? How close is the value $f(x_0)$ corresponding to the actual (initially unknown) objective function $f(x)$ the actual maximum

$$M_f \stackrel{\text{def}}{=} \max_x f(x). \tag{2}$$

The answer to this question is given by the following result.

Proposition 2.

- *Let $f(x)$ be a function, let $\widetilde{f}(x)$ be ε-close to $f(x)$, and let x_0 be an alternative for which $\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon$ for all $y \neq x_0$. Then, $f(x_0) \geq M_f - 4\varepsilon$.*
- *Let $\widetilde{f}(x)$ be a function, and let x_0 be an alternative for which:*
  - *$\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon$ for all $y \neq x_0$, and*
  - *$\widetilde{f}(x_0) = \widetilde{f}(y_0) - 2\varepsilon$ for some $y_0 \neq x_0$.*

  Then there exists a function $f(x)$ which is ε-close to $\widetilde{f}(x)$ and for which $f(x_0) = M_f - 4\varepsilon$.

This result explains the empirical fact. This result shows that if we start with an ε-accurate objective function $\widetilde{f}(x)$, we may end up with an alternative which is 4ε-smaller than the desired maximum.

This possible four-times amplification of uncertainty explains why it is often beneficial to precisiate the objective function: for example, even if we have a reasonably accurate description of the objective function, with the accuracy of 20%, the resulting solution may be 80% different from the optimal one – i.e., really bad.

3 What If We Consider Average-Case Accuracy Instead of the Worst-Case One? What If We Consider Naive Approach Instead of a Guaranteed One?

What if we consider average-case accuracy: formulation of the problem. The quality of a selection x_0 can be gauged by the difference $d \stackrel{\text{def}}{=} M_f - f(x_0)$ between the absolute optimum and what we achieve by selecting the alternative x_0.

This difference is always non-negative. In the previous section, we showed that this difference cannot exceed 4ε, and that in the worst-case situation, it can be equal to 4ε.

It is also possible to have $d = 0$: e.g., when $\widetilde{f}(x)$ is actually equal to the actual objective function, and we happened to select an f-optimal alternative as x_0. This is the best-case situation.

In real life, we rarely encounter the best-case and the worst-case situations, so what is the average value of the difference d?

Estimating the average-case accuracy. All we know about the actual difference d is that it is located somewhere on the interval $[0, 4\varepsilon]$. We have no information about the probability of different possible values within this interval. In such a situation, when we have several probability distributions consistent with the available information, it is reasonable to select the most uncertain one, i.e., the one for which the entropy $S \stackrel{\text{def}}{=} -\int \rho(x) \cdot \ln(\rho(x))\,dx$ attains the largest possible value; see, e.g., [4].

It is known that among all possible probability distributions on an interval, the uniform distribution has the largest entropy. For this distribution, the mean value is the midpoint of the interval. For our interval $[0, 4\varepsilon]$, the midpoint is 2ε. Thus, we arrive at the following conclusion.

Average-case accuracy: result. The average value of the difference $d \stackrel{\text{def}}{=} M_f - f(x_0)$ between:

- the absolute optimum M_f and
- the value $f(x_0)$ that we achieve by selecting the alternative x_0

is equal to 2ε.

Discussion. So, even if we consider average-case inaccuracy, the inaccuracy caused by the use of imprecise objective function doubles. This doubling may not be as bad as multiplying by a factor of four, but it can still lead from a reasonable accuracy of the objective function to an unreasonable inaccuracy of the resulting decision. For example, for 20% accuracy in the objective function, we get a 40% accuracy in the resulting alternative – i.e., instead of the optimal value, we get, on average, only 60% of this optimal value: more than 50% less than desired.

Thus, even when we consider average difference, there is still a motivation to preciciate the objective function before performing optimization.

What is we use a naive approach. Instead of trying not to miss all f-optimal alternatives, we can instead naively find an alternative which is optimal relative to the approximate objective function $\widetilde{f}(x)$, i.e., an alternative x_0 for which $\widetilde{f}(x_0) \geq \widetilde{f}(y)$ for all $y \neq x_0$. What is the quality of such a naive selection?

The answer to this equation is provided by the following result.

Proposition 3.

- *If the functions $f(x)$ and $\widetilde{f}(x)$ are ε-close, and x_0 is a $\widetilde{f}$-optimal alternative, then $f(x_0) \geq M_f - 2\varepsilon$.*
- *For every non-constant continuous function $\widetilde{f}(x)$ on a connected compact set, there exists an ε-close function $f(x)$ for which for all $\widetilde{f}$-optimal alternatives x_0, we have $f(x_0) = M_f - 2\varepsilon$.*

Discussion. In other words, if we use the naive approach, then:

- on the one hand, we may miss all actual optima, but
- on the other hand, the worst-case difference $M_f - f(x_0)$ decreases to a half of what it was before: e.g., to only 2ε.

Similarly to what we mentioned above, this is still too high.

If, similar to the above, we use the maximum entropy approach to estimate the average difference, we conclude that for the naive approach, the average difference is equal to ε. This is tolerance – and in line with the original intuitive expectation – but this is only on average. In half of the cases, we get the difference larger than ε – and this is not so good. In other words, the need for precisiation remains even if we consider average case of the naive approach.

4 Proofs

Proof of Proposition 1.

1°. Let us first prove that if the alternative x_0 is f-optimal relative to some function $f(x)$, then $\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon$ for all $y \neq x_0$.

Indeed, f-optimality means that

$$f(x_0) \geq f(y) \tag{3}$$

for all $y \neq x_0$. Now, from the fact that the function $f(x)$ is ε-close to $\widetilde{f}(x)$, we can conclude that

$$\widetilde{f}(x_0) \geq f(x_0) - \varepsilon, \tag{4}$$

which, together with (3), implies that

$$\widetilde{f}(x_0) \geq f(y) - \varepsilon \tag{5}$$

for all $y \neq x_0$. Similarly, from the fact that the function $f(x)$ is ε-close to $\widetilde{f}(x)$, we conclude that

$$f(y) \geq \widetilde{f}(y) - \varepsilon. \tag{6}$$

With (5), this implies that

$$\widetilde{f}(x_0) \geq f(y) - \varepsilon \geq (\widetilde{f}(y) - \varepsilon) - \varepsilon = \widetilde{f}(y) - 2\varepsilon. \tag{7}$$

This is exactly what we want to prove.

2°. Vice versa, let us assume that for some x_0, we have

$$\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon \tag{8}$$

for all $y \neq x_0$. Let us then construct a function $f(x)$ which is ε-close to $\widetilde{f}(x)$ and for which x_0 is f-optimal.

This function $f(x)$ can be constructed as follows:

- for $x = x_0$, we take

$$f(x_0) \stackrel{\text{def}}{=} \widetilde{f}(x_0) + \varepsilon; \tag{9}$$

- for all other alternatives $y \neq x_0$, we take

$$f(y) \stackrel{\text{def}}{=} \widetilde{f}(y) - \varepsilon. \tag{10}$$

From this construction, it is clear that the newly constructed function $f(x)$ is ε-close to the original function $\widetilde{f}(x)$. Let us prove that the given alternative x_0 is indeed f-optimal, i.e., that

$$f(x_0) \geq f(y) \tag{11}$$

for all $y \neq x_0$.

Indeed, from (8) and (9), we conclude that

$$f(x_0) = \widetilde{f}(x_0) + \varepsilon \geq (\widetilde{y} - 2\varepsilon) + \varepsilon = \widetilde{y} - \varepsilon, \tag{12}$$

i.e., that

$$f(x_0) \geq \widetilde{f}(y) - \varepsilon \tag{13}$$

for all $y \neq x_0$. But for these y, we have $f(y) = \widetilde{f}(y) - \varepsilon$, so (13) becomes the desired inequality (11).

The proposition is proven.

Proof of Proposition 2.

1°. Let us first prove that if the function $\widetilde{f}(x)$ is ε-close to the function $f(x)$, and

$$\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon \tag{14}$$

for all $y \neq x_0$, then

$$f(x_0) \geq M_f - 4\varepsilon. \tag{15}$$

Indeed, from the fact that the functions $\widetilde{f}(x)$ and $f(x)$ are ε-close, we can conclude that

$$f(x_0) \geq \widetilde{f}(x_0) - \varepsilon, \tag{16}$$

and thus, taking (14) into account, that

$$f(x_0) \geq \widetilde{f}(x_0) - \varepsilon \geq (\widetilde{f}(y) - 2\varepsilon) - \varepsilon = \widetilde{f}(y) - 3\varepsilon, \tag{17}$$

i.e.,

$$f(x_0) \geq \widetilde{f}(y) - 3\varepsilon \tag{18}$$

for all $y \neq x_0$.

Similarly, from the ε-closeness of $f(x)$ and $\widetilde{f}(x)$, we conclude that

$$\widetilde{f}(y) \geq f(y) - \varepsilon \tag{19}$$

for all $y \neq x_0$. In view of (18), this implies that

$$f(x_0) \geq \widetilde{f}(y) - 3\varepsilon \geq (f(y) - \varepsilon) - 3\varepsilon = f(y) - 4\varepsilon, \tag{20}$$

i.e.,

$$f(x_0) \geq f(y) - 4\varepsilon \tag{21}$$

for all $y \neq x_0$. This same inequality is clearly true for $y = x_0$ as well. Thus, (21) holds for all y, and we can therefore conclude that $f(x_0)$ is larger than or equal to the largest of the right-hand sides of the inequality (21), i.e., that

$$f(x_0) \geq M_f - 4\varepsilon. \tag{22}$$

This is exactly what we wanted to prove.

2°. Let us now assume that for some function $\widetilde{f}(x)$ and for some alternative x_0, we have

$$\widetilde{f}(x_0) \geq \widetilde{f}(y) - 2\varepsilon \tag{23}$$

for all $y \neq x_0$ and

$$\widetilde{f}(x_0) = \widetilde{f}(y_0) - 2\varepsilon \tag{24}$$

for some $y_0 \neq x_0$. Let us prove that there exists a function $f(x)$ which is ε-close to $\widetilde{f}(x)$ and for which

$$f(x_0) = M_f - 4\varepsilon. \tag{25}$$

Indeed, let us define the function $f(x)$ as follows:

- for $x = x_0$, we take
$$f(x_0) \stackrel{\text{def}}{=} \widetilde{f}(x_0) - \varepsilon; \tag{26}$$
- for all other alternatives $y \neq x_0$, in particular, for $y = y_0$, we take
$$f(y) \stackrel{\text{def}}{=} \widetilde{f}(y) + \varepsilon. \tag{27}$$

It is easy to see that thus constructed function $f(x)$ is ε-close to the given function $\widetilde{f}(x)$.

We will prove the desired equality (25) by proving two separate inequalities:

$$f(x_0) \geq M_f - 4\varepsilon \tag{28}$$

and

$$f(x_0) \leq M_f - 4\varepsilon. \tag{29}$$

2.1°. Let us first prove the inequality (28). By combining the formulas (23) and (26), we conclude that

$$f(x_0) = \widetilde{f}(x_0) - \varepsilon \geq (\widetilde{f}(y) - 2\varepsilon) - \varepsilon = \widetilde{f}(y) - 3\varepsilon, \quad (30)$$

i.e., that

$$f(x_0) \geq \widetilde{f}(y) - 3\varepsilon \quad (31)$$

for all $y \neq x_0$.

From the formula (27), we now conclude that $\widetilde{f}(y) = f(y) - \varepsilon$, and thus, (31) turns into

$$f(x_0) \geq \widetilde{f}(y) - 3\varepsilon = (f(y) - \varepsilon) - 3\varepsilon = f(y) - 4\varepsilon, \quad (32)$$

i.e., that

$$f(x_0) \geq f(y) - 4\varepsilon \quad (33)$$

for all $y \neq x_0$. The inequality (33) is clearly also true for $y = x_0$, so we conclude that $f(x_0) \geq M_f - 4\varepsilon$. This is the desired inequality (28).

2.2°. Let us now prove the inequality (29). By combining the formulas (24) and (26), we conclude that

$$f(x_0) = \widetilde{f}(x_0) - \varepsilon = (\widetilde{f}(y_0) - 2\varepsilon) - \varepsilon = \widetilde{f}(y_0) - 3\varepsilon, \quad (34)$$

i.e., that

$$f(x_0) = \widetilde{f}(y_0) - 3\varepsilon. \quad (35)$$

From the formula (27) for $y = y_0$, we conclude that $\widetilde{f}(y_0) = f(y_0) - \varepsilon$, and thus, (35) turns into

$$f(x_0) = \widetilde{f}(y_0) - 3\varepsilon = (f(y_0) - \varepsilon) - 3\varepsilon = f(y_0) - 4\varepsilon. \quad (36)$$

Here,

$$f(y_0) \geq \max_x f(x) = M_f, \quad (37)$$

thus, (36) implies that $f(x_0) \geq M_f - 4\varepsilon$. This is the desired inequality (29).

2.3°. The inequalities (28) and (29) imply the desired equality!(25).

The proposition is proven.

Proof of Proposition 3.

1°. Let us first prove that if the functions $f(x)$ and $\widetilde{f}(x)$ are ε-close, and the alternative x_0 is $\widetilde{f}$-optimal, then

$$f(x_0) \geq M_f - 2\varepsilon. \quad (38)$$

Indeed, $\widetilde{f}$-optimality of x_0 means that

$$\widetilde{f}(x_0) \geq \widetilde{f}(y) \quad (39)$$

for all y. From the fact that the functions $f(x)$ and $\widetilde{f}(x)$ are ε-close, we conclude that

$$f(x_0) \geq \widetilde{f}(x_0) - \varepsilon. \quad (40)$$

Combining this inequality with (39), we conclude that

$$f(x_0) \geq f(\widetilde{y}) - \varepsilon \quad (41)$$

for all y.

Similarly, from the fact that the functions $f(x)$ and $\widetilde{f}(x)$ are ε-close, we can conclude that

$$\widetilde{f}(y) \geq f(y) - \varepsilon. \quad (42)$$

Combining this inequality with (41), we conclude that

$$f(x_0) \geq f(\widetilde{y}) - \varepsilon \geq (f(y) - \varepsilon) - \varepsilon = f(y) - 2\varepsilon, \tag{43}$$

i.e., that

$$f(x_0) \geq f(y) - 2\varepsilon \tag{44}$$

for all y. Since this inequality holds for all y, we conclude that $f(x_0) \geq M_f - 2\varepsilon$. This is the desired formula (38).

2°. Let us now assume that $\widetilde{f}(x)$ is a non-constant function continuous on a connected compact set. Let us construct an ε-close function $f(x)$ for which, for all $\widetilde{f}$-optimal values x_0, we have

$$f(x_0) = M_f - 2\varepsilon. \tag{45}$$

This function will be constructed as follows:

- for each $\widetilde{f}$-optimal alternative x_0, we take
$$f(x_0) = \widetilde{f}(x_0) - \varepsilon; \tag{46}$$
- for all other alternatives y, we take
$$f(y) = \widetilde{f}(y) + \varepsilon. \tag{47}$$

Since the function $\widetilde{f}(x)$ is continuous on a compact set, there exists at least one point x_M on which its maximum is attained, i.e., for which $\widetilde{f}(x_M) = M_{\widetilde{f}} \stackrel{\text{def}}{=} \max\limits_x \widetilde{f}(x)$.

Since the function $\widetilde{f}(x)$ is not constant, there exists values which are smaller than its maximum $M_{\widetilde{f}}$. Since the function $\widetilde{f}(x)$ is continuous, and its domain is connected, for every $\delta > 0$, there exists an alternative x_δ for which

$$M_{\widetilde{f}} - \delta < \widetilde{f}(x_\delta) < M_{\widetilde{f}}. \tag{48}$$

Since $\widetilde{f}(x_\delta) \neq M_{\widetilde{f}}$, by our construction of the function $f(x)$ (formula (47)), we get

$$f(x_\delta) = \widetilde{f}(x_\delta) + \varepsilon. \tag{49}$$

By combining (48) and (49), we conclude that

$$f(x_\delta) \geq M_{\widetilde{f}} - \delta + \varepsilon. \tag{50}$$

Thus, for the maximum $M_f \geq f(x_\delta)$ of the new function $f(x)$, we get

$$M_f \geq M_{\widetilde{f}} - \delta + \varepsilon. \tag{51}$$

This inequality holds for all $\delta > 0$. By tending to the limit $\delta \to 0$, we conclude that

$$M_f \geq M_{\widetilde{f}} + \varepsilon, \tag{52}$$

thus

$$M_{\widetilde{f}} \leq M_f - \varepsilon. \tag{53}$$

On the other hand, for the $\widetilde{f}$-optimal alternative x_M, i.e., the alternative for which $\widetilde{f}(x_M) = M_{\widetilde{f}}$, our construction (46) implies that

$$f(x_M) = \widetilde{f}(x_M) - \varepsilon = M_{\widetilde{f}} - \varepsilon. \tag{54}$$

With (53), this implies that

$$f(x_M) \leq (M_f - \varepsilon) - \varepsilon = M_f - 2\varepsilon. \tag{55}$$

We already know, from Part 1 of this proof, that we have

$$f(x_M) \geq M_f - 2\varepsilon. \tag{56}$$

The two inequalities (55) and (56) imply that $f(x_M) = M_f - 2\varepsilon$, which is exactly the desired equality (45).

The proposition is proven.

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References

[1] Bellman, R.E., and L.A. Zadeh, Decision making in a fuzzy environment, *Management Science*, vol.17, no.4, pp.B141–B164, 1970.

[2] Bouchon-Meunier, B., Kreinovich, V., Lokshin, A., and H.T. Nguyen, On the formulation of optimization under elastic constraints (with control in mind), *Fuzzy Sets and Systems*, vol.81, no.1, pp.5–29, 1996.

[3] Jaulin, L., Kieffer, M., Didrit, O., and E. Walter, *Applied Interval Analysis*, Springer Verlag, London, 2001.

[4] Jaynes, E.T., and G.L. Bretthorst, *Probability Theory: The Logic of Science*, Cambridge University Press, Cambridge, UK, 2003.

[5] Klir, G., and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[6] Marr, B., *Key Performance Indicators: The 75+ Measures Every Manager Needs to Know*, Financial Times/Prentice Hall, 2012.

[7] Moore, R.E., Kearfott, R.B., and M.J. Cloud, *Introduction to Interval Analysis*, SIAM Press, Philadelphia, Pennsylviania, 2009.

[8] Nguyen, H.T., and E.A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.

[9] Parmenter, D., *Key Performance Indicators*, John Wiley & Sons, 2007.

[10] Zadeh, L.A., Fuzzy sets, *Information and Control*, vol.8, pp.338–353, 1965.