Journal of Uncertain Systems
Vol.9, No.3, pp.230-240, 2015
Online at: www.jus.org.uk

Modeling Credibilistic Data Envelopment Analysis under Fuzzy Input and Output Data

Xue-Qin Feng[1], Ming-Qiang Meng[2], Yan-Kui Liu[3,*]
[1] *College of Economics, Hebei University, Baoding 071002, Hebei, China*
[2] *Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College Shijiazhuang 050003, China*
[3] *College of Mathematics & Information Science, Hebei University, Baoding 07100201, Hebei, China*

Received 8 December 2013; Revised 16 Jun 2015

Abstract

Data envelopment analysis (DEA) is an approach to measuring the relative efficiencies of a set of decision making units (DMUs) that use multiple inputs to produce multiple outputs. In this paper, we study the input-oriented and the output-oriented fuzzy DEA models, in which the input data and output data are characterized by fuzzy variables with known possibility distributions. The definitions of expectation efficiency of the evaluated DMUs are introduced. Although in some special cases the credibility constraints can be turned into their crisp equivalents, we cannot do so for the expectation objective function. As a result, for general fuzzy inputs and outputs, the fuzzy DEA models cannot be turned into their deterministic mathematical programming problems. In addition, since the objective functions of the input-oriented and the output-oriented DEA models are the expected ratio of weighted fuzzy outputs to weighted fuzzy inputs, and the expected ratio of weighted fuzzy inputs to weighted fuzzy outputs, respectively, they have no convexity property in general case. Therefore, conventional numerical optimization methods cannot be used to solve the fuzzy DEA models with general fuzzy inputs and outputs. To avoid this difficulty, we suggest an approximation approach (AA) to evaluating expectation objective and credibility constraint functions, and the reasonableness of AA is demonstrated by discussing the related convergence properties. After that, we design a hybrid heuristic algorithm for solving our fuzzy DEA models. The hybrid algorithm integrates AA, neural network (NN), and the simulated annealing (SA) algorithm. Finally, one numerical example is provided to illustrate the effectiveness of the approximation-based SA algorithm and the relative efficiency is analyzed via numerical experiments.

Keywords: fuzzy DEA model, expectation efficiency, approximation approach, neural network, simulated annealing

1 Introduction

DEA, initially proposed by Charnes et al. [3], has been widely used to evaluate the relative efficiency of a set of homogeneous DMUs with multiple inputs and multiple outputs. In the original DEA models, the input and output data are assumed to be measured exactly. One of the advantages of the DEA model is that it does not require either a priori weights or the explicit specification of functional relations between the multiple inputs and outputs. On the basis of CCR model [3], various extensions of the CCR model have been proposed, including BCC model [1], FDH (free disposal hull) model [17], SBM (slacks-based measure of efficiency) model [22], RAM model [4], and other well-known DEA models [2].

However, there is a weakness in deterministic DEA models, they do not allow imprecise data in inputs and outputs, i.e., the data of inputs and outputs of the evaluated DMUs are assumed to be measured with crisp positive values on a ratio scale. However, as some researchers pointed out, this assumption may not always be true in real-world problems. As a result, DEA efficiency evaluation may be sensitive to such imprecise data. In order to incorporate uncertain input and output data to DEA analysis, many researchers applied their

*Corresponding author.
Email: lyk@hbu.edu.cn (Y.-K. Liu).

minds to deal with the imprecise data of DEA in fuzzy environment. For example, Sengupta [21] explored the use of fuzzy set-theoretic measures in the context of DEA; Maeda and Tanaka [5] formulated a DEA model with an interval efficiency which consists of efficiencies obtained from the optimistic and pessimistic viewpoints; Hatami-Marbini and Tavana [7] proposed an alternative fuzzy outranking method by extending the Electre I method to take into account the uncertain, imprecise and linguistic assessments provided by a group of decision makers; Qin et al. [18, 19] studied fuzzy DEA models via parametric programming method, where the parameters characterize the degree of uncertainty of the fuzzy coefficients, and Meng [15] presented a new satisficing DEA model with credibility criterion. The interested reader may refer to [6], in which the authors provided a taxonomy and review of the fuzzy DEA methods.

Different from the work mentioned above, the main purpose of this paper is to construct the input-oriented and the output-oriented credibililistic DEA models by adopting the expectation criterion in the objective function. The theoretical foundation of our credibilistic DEA models is credibility measure theory [10, 11, 13]. It is well-known that probability measure is additive and the mathematical expectation of random variable is linear. However, credibility measure is nonadditive and the expected value operator of fuzzy variable is nonlinear in general. Therefore, the solution methods developed for stochastic DEA cannot be used for our credibilistic DEA models. In this paper, the objective functions in our input-oriented and output-oriented credibilistic DEA models are the expected ratio of weighted fuzzy outputs to weighted fuzzy inputs, and the expected ratio of weighted fuzzy inputs to weighted fuzzy outputs, respectively, which are different from those discussed in the literature, and the proposed credibilistic DEA models are in general cannot be turned into deterministic programming problems. Therefore, conventional numerical optimization methods cannot be used to solve our credibilistic DEA models. To avoid this difficulty, we adopt AA to evaluate expectation objective and credibility constraint functions, and discuss the convergence of the evaluation method. After that, we design a hybrid SA algorithm, which integrates AA, NN and the SA algorithm, to solve the developed credibilistic DEA models.

The rest of this paper is organized as follows. In Section 2, we firstly present the new input-oriented and the output-oriented credibilistic DEA models; then introduce the definition of expectation efficiency of the evaluated DMUs. The equivalent problems of the proposed credibilistic DEA models are discussed in Section 3. In Section 4, we design a hybrid heuristic algorithm, which integrates AA, NN and the SA algorithm, for general credibilistic DEA models. In Section 5, we provide one numerical example to illustrate the effectiveness of the approximation-based SA algorithm and analyze the DMUs' relative efficiency via numerical experiments. Section 6 draws our conclusions.

2 Formulation of Credibilistic DEA Models

In this section, we will develop a credibility approach to input-oriented and the output-oriented fuzzy DEA models, in which the input and output data are characterized by fuzzy variables with known possibility distributions. The aim of the proposed credibilistic DEA model is to optimize the expected value of objective function subject to credibility constraints with prescribed credibility levels.

Firstly, on the basis of fuzzy expected value operator and credibility measure, we can formulate the following input-oriented credibilistic DEA model,

$$
\begin{array}{ll}
\max\limits_{u,v} & V^I_{EDEA} = E\left[v^T\tilde{y}_0/u^T\tilde{x}_0\right] \\
\text{subject to} & \mathrm{Cr}\{u^T\tilde{x}_i - v^T\tilde{y}_i \geq 0\} \geq \alpha_i, i = 1,2,\ldots,n \\
& u \geq 0, u \neq 0 \\
& v \geq 0, v \neq 0,
\end{array} \tag{1}
$$

where

$\tilde{x}_0$: the fuzzy inputs column vector consumed by DMU_0;

$\tilde{x}_i$: the fuzzy inputs column vector consumed by DMU_i;

$\tilde{y}_0$: the fuzzy outputs column vector produced by DMU_0;

$\tilde{y}_i$: the fuzzy outputs column vector produced by DMU_i, and

α_i: the predetermined credibility level corresponding to the ith constraint for $i = 1, 2, \ldots, n$.

In model (1), we wish to find the decision (u, v) with maximum expected return value $E\left[v^T\tilde{y}_0/u^T\tilde{x}_0\right]$, while the fuzzy event $u^T\tilde{x}_i - v^T\tilde{y}_i \geq 0$ $(i = 1, 2, \ldots, n)$ is satisfied with at least credibility α_i $(i = 1, 2, \ldots, n)$. What's more, model (1) is an input-oriented DEA one to evaluate the DMU_0 and we need to solve n times with $(\tilde{x}_0, \tilde{y}_0) = (\tilde{x}_i, \tilde{y}_i)$ $(i = 1, 2, \ldots, n)$ for the evaluated n DMUs.

Secondly, we can employ fuzzy expected operator and credibility measure to build the output-oriented credibilistic DEA model as follows,

$$\begin{array}{ll} \min\limits_{u,v} & V_{EDEA}^{O} = E\left[u^T\tilde{x}_0/v^T\tilde{y}_0\right] \\ \text{subject to} & \mathrm{Cr}\{u^T\tilde{x}_i - v^T\tilde{y}_i \geq 0\} \geq \alpha_i, i = 1, 2, \ldots, n \\ & u \geq 0, u \neq 0 \\ & v \geq 0, v \neq 0. \end{array} \tag{2}$$

In model (2), α_i is a predetermined credibility level corresponding to the ith constraint for $i = 1, 2, \ldots, n$. We may interpret the output-oriented fuzzy DEA model (2) in a similar way as model (1).

As we know, we use the optimal value to determine the efficiency of DMU_0 in the deterministic environment. Before we define the efficiency in fuzzy environment, we first recall the concept of efficiency in the deterministic environment. For the input-oriented and the output-oriented CCR models, if the optimal value equals to unity and there exists at least one optimal solution whose each component is larger than 0, then DMU_0 is CCR-efficient; if the optimal value equals to unity and at least one element of the optimal solution is zero, then DMU_0 is CCR-weak-efficient; Otherwise, it is said to be CCR-inefficient. Correspondingly, we explain the efficiency of DMU_0 by the expected value in the input-oriented and the output-oriented credibilistic DEA models at the specified credibility levels.

Firstly, for input-oriented credibilistic DEA model (1), we formally define its efficiency as follows.

Definition 2.1. For input-oriented credibilistic DEA model (1), DMU_0 is expectation efficient if its optimal value $V_{EDEA}^{I*} \geq 1$ and there exists at least one optimal solution (u^*, v^*) with $u^* > 0, v^* > 0$.

DMU_0 is expectation weak efficient if its optimal value $V_{EDEA}^{I*} \geq 1$ and at least one element of the optimal solution (u^*, v^*) is zero. Otherwise, DMU_0 is said to be expectation inefficient.

Secondly, for output-oriented credibilistic DEA model (2), we have the following formal definition about expectation efficiency.

Definition 2.2. For output-oriented credibilistic DEA model (2), DMU_0 is expectation-efficient if its optimal value $V_{EDEA}^{O*} \leq 1$ and there exists at least one optimal solution (u^*, v^*) with $u^* > 0, v^* > 0$.

DMU_0 is expectation-weak-efficient if its optimal value $V_{EDEA}^{O*} \leq 1$ and at least one element of the optimal solution (u^*, v^*) is zero. Otherwise, DMU_0 is said to be expectation-inefficient.

Remark: Definitions 2.1 and 2.2 are the generalizations of the deterministic CCR efficiency.

Using the concepts of expectation efficiency of the evaluated DMU_0, in the following section, we will discuss the equivalent problems of models (1) and (2).

3 The Equivalent Credibilistic DEA Models

When the number of fuzzy inputs and fuzzy outputs increase, we can see that models (1) and (2) may include a large number of fuzzy variables. Therefore, to find out the optimal solutions to models (1) and (2), we are required to give efficient methods of computing the expectation objective and credibility constraint functions. For general fuzzy inputs and outputs, we cannot achieve these two goals simultaneously. In the following, we consider a special case, in which only the credibility constraints can be turned into their crisp equivalents.

Let $\xi = (a_1, a_2, a_3, a_4)$ and $\eta = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy variables. If ξ and η are mutually independent [14], then one has

$$\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4),$$

and

$$\beta\xi + \gamma\eta = (\beta a_1 + \gamma b_1, \beta a_2 + \gamma b_2, \beta a_3 + \gamma b_3, \beta a_4 + \gamma b_4)$$

for any $\beta, \gamma \geq 0$. As a consequence, we have the following property of credibility constraints.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent trapezoidal fuzzy variables with $\xi_i = (r_i^1, r_i^2, r_i^3, r_i^4)$, and $k_i > 0, i = 1, 2, \ldots, n$. According to the definition of credibility measure, it is easy to check the credibility has the following properties related to trapezoidal fuzzy variables.

(i) When $\alpha < 0.5$, $\mathrm{Cr}\{\sum_{i=1}^n k_i\xi_i \geq 0\} \geq \alpha$ if and only if $(1-2\alpha)\sum_{i=1}^n k_i r_i^4 + 2\alpha \sum_{i=1}^n k_i r_i^3 \geq 0$;

(ii) When $\alpha \geq 0.5$, $\mathrm{Cr}\{\sum_{i=1}^n k_i\xi_i \geq 0\} \geq \alpha$ if and only if $(2\alpha-1)\sum_{i=1}^n k_i r_i^1 + 2(1-\alpha) \sum_{i=1}^n k_i r_i^2 \geq 0$.

For the sake of convenience, we denote $x_i^{r_i} = (x_{1i}^{r_i}, x_{2i}^{r_i}, \cdots, x_{mi}^{r_i})$, $y_i^{r_i} = (y_{1i}^{r_i}, y_{2i}^{r_i}, \cdots, y_{ri}^{r_i})$. When $\alpha_i \geq 0.5$ $(i = 1, \ldots, n)$, according to the properties of credibility, the credibilistic DEA models (1) and (2) become the following equivalent models (3) and (4), respectively.

$$\begin{aligned} \max_{u,v} \quad & V_{EDEA}^I = E\left[v^T\tilde{y}_0/u^T\tilde{x}_0\right] \\ \text{subject to} \quad & (2\alpha_i - 1)(u^T x_i^{r_1} - v^T y_i^{r_4}) + 2(1-\alpha_i)(u^T x_i^{r_2} - v^T y_i^{r_3}) \geq 0, i \leq n \\ & u \geq 0, u \neq 0 \\ & v \geq 0, v \neq 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \min_{u,v} \quad & V_{EDEA}^O = E\left[u^T\tilde{x}_0/v^T\tilde{y}_0\right] \\ \text{subject to} \quad & (2\alpha_i - 1)(u^T x_i^{r_1} - v^T y_i^{r_4}) + 2(1-\alpha_i)(u^T x_i^{r_2} - v^T y_i^{r_3}) \geq 0, i \leq n \\ & u \geq 0, u \neq 0 \\ & v \geq 0, v \neq 0, \end{aligned} \tag{4}$$

where α_i is a predetermined credibility level corresponding to the ith constraint for $i = 1, 2, \ldots, n$.

So far, for the case of trapezoidal inputs and outputs, we have discussed the method of turning the credibility constraints into their equivalent deterministic forms which facilitate us to check the credibility constraints in this special case. However, we cannot do so in general case. Consequently, to design solution methods for general models (1) and (2), we have to discuss methods of computing expectation objective functions in these two models, which is usually difficult compared with credibility constraint functions, even in the case when fuzzy inputs and outputs are trapezoidal fuzzy variables. The issues will be discussed in the next section.

4 Solution Methods

4.1 Computing Expectation Objective and Credibility Functions

In order to solve credibilistic DEA models (1) and (2), for any given weight vector (u, v), we are required to compute the expectation objective functions

$$(u,v) \to E\left[u^T\tilde{x}_0/v^T\tilde{y}_0\right], (u,v) \to E\left[v^T\tilde{y}_0/u^T\tilde{x}_0\right],$$

and the credibility function of fuzzy event $u^T\tilde{x}_0 - v^T\tilde{y}_0 \geq 0$,

$$(u,v) \to \mathrm{Cr}\left\{u^T\tilde{x}_0 - v^T\tilde{y}_0 \geq 0\right\}.$$

In the following, we discus the general case that fuzzy input data $\tilde{x}_0$ and fuzzy output data $\tilde{y}_0 \geq 0$ are continuous fuzzy variables with general possibility distributions.

In the input-oriented and the output-oriented credibilistic DEA models (1) and (2), when the inputs and outputs are general continuous fuzzy variables for each DMU, we cannot get the analytical expressions of expectation objective and credibility constraint functions. In this case, the exact values of expectation objective and credibility constraint functions are usually difficult to compute. To overcome this difficulty, we adopt AA proposed in [12] to evaluate the expectation objective and credibility constraint functions.

Suppose that $(\tilde{x}_0^T, \tilde{y}_0^T) = (\tilde{x}_{1,0}, \tilde{x}_{2,0}, \cdots, \tilde{x}_{m,0}, \tilde{y}_{m+1,0}, \tilde{y}_{m+2,0}, \cdots, \tilde{y}_{m+r,0})$ is a continuous fuzzy vector with an infinite support $\Xi = \Pi_{j=1}^{m+r}[a_j, b_j]$, where $[a_j, b_j]$ is the support of $\tilde{x}_{j,0}$ $(1 \le j \le m)$ or $\tilde{y}_{j,0}$ $(m+1 \le j \le m+r)$. In this case, we adopt AA to approximate the possibility distribution of $(\tilde{x}_0^T, \tilde{y}_0^T)$ by a sequence of possibility distributions of discrete fuzzy vectors $(\tilde{x}_t^T, \tilde{y}_t^T)$. The discretization method is described as follows.

For each integer t, we define the discrete fuzzy vector

$$(\tilde{x}_t^T, \tilde{y}_t^T) = (\tilde{x}_{t,1}, \tilde{x}_{t,2}, \cdots, \tilde{x}_{t,m}, \tilde{y}_{t,m+1}, \tilde{y}_{t,m+2}, \cdots, \tilde{y}_{t,m+r})$$

in the following way. For each $j \in \{1, 2, \ldots, m\}$, define fuzzy variable $\tilde{x}_{t,j} = g_{t,j}(\tilde{x}_{j,0})$ and for each $j \in \{m+1, m+2, \ldots, m+r\}$, define fuzzy variable $\tilde{y}_{t,j} = g_{t,j}(\tilde{y}_{j,0})$ as

$$g_{t,j}(v_j) = \sup\left\{\frac{k_j}{t} \mid k_j \in Z, \text{ s.t. } \frac{k_j}{t} \le v_j\right\}$$

for $v_j \in [a_j, b_j]$, where Z is the set of integers.

Note that fuzzy variables $\tilde{x}_{j,0}$ and $\tilde{y}_{j,0}$ take their values in $[a_j, b_j]$, while $\tilde{x}_{t,j}$ and $\tilde{y}_{t,j}$ take values k_j/t for $k_j = [ta_j], [ta_j]+1, \ldots, [tb_j]$, where $[r]$ is the maximal integer such that $[r] \le r$. In addition, for each k_j, as $\tilde{x}_{j,0}$ and $\tilde{y}_{j,0}$ take their values in $[k_j/t, (k_j+1)/t)$, fuzzy variables $\tilde{x}_{t,j}$ and $\tilde{y}_{t,j}$ take the value k_j/t only. Hence, the possibility distributions of fuzzy variables $\tilde{x}_{t,j}$ and $\tilde{y}_{t,j}$ are determined by

$$v_{t,j}\left(\frac{k_j}{t}\right) = \mathrm{Pos}\left\{\gamma \mid \frac{k_j}{t} \le \tilde{x}_{j,0}(\gamma) < \frac{k_j+1}{t}\right\}, \ j = 1, 2, \ldots, m,$$

and

$$v_{t,j}\left(\frac{k_j}{t}\right) = \mathrm{Pos}\left\{\gamma \mid \frac{k_j}{t} \le \tilde{y}_{j,0}(\gamma) < \frac{k_j+1}{t}\right\}, \ j = m+1, m+2, \ldots, m+r$$

for $k_j = [ta_j], [ta_j]+1, \ldots, [tb_j]$. Consequently, for each $\gamma \in \Gamma$, we have

$$\tilde{x}_{j,0}(\gamma) - \frac{1}{t} < \tilde{x}_{t,j}(\gamma) \le \tilde{x}_{j,0}(\gamma), \ j = 1, 2, \ldots, m,$$

and

$$\tilde{y}_{j,0}(\gamma) - \frac{1}{t} < \tilde{y}_{t,j}(\gamma) \le \tilde{y}_{j,0}(\gamma), \ j = m+1, m+2, \ldots, m+r,$$

which can be equivalently rewritten as

$$|\tilde{x}_{t,j}(\gamma) - \tilde{x}_{j,0}(\gamma)| < \frac{1}{t}, \ \ |\tilde{y}_{t,j}(\gamma) - \tilde{y}_{j,0}(\gamma)| < \frac{1}{t}.$$

In addition, $(\tilde{x}_t^T, \tilde{y}_t^T)$ and $(\tilde{x}_0^T, \tilde{y}_0^T)$ are $(m+r)$-ary fuzzy vectors, $\tilde{x}_{t,j}$ and $\tilde{x}_{j,0}$ are their jth components for $j = 1, 2, \ldots, m$, and $\tilde{y}_{t,j}$ and $\tilde{y}_{j,0}$ are their jth components for $j = m+1, m+2, \ldots, m+r$, respectively. Hence, we have

$$\begin{aligned}&\|(\tilde{x}_t^T, \tilde{y}_t^T)(\gamma) - (\tilde{x}_0^T, \tilde{y}_0^T)(\gamma)\| \\ &= \sqrt{\sum_{j=1}^{m}(\tilde{x}_{t,j}(\gamma) - \tilde{x}_{j,0}(\gamma))^2 + \sum_{j=m+1}^{m+r}(\tilde{y}_{t,j}(\gamma) - \tilde{y}_{j,0}(\gamma))^2} \le \frac{\sqrt{m+r}}{t}, \gamma \in \Gamma,\end{aligned}$$

which implies that the sequence $\{(\tilde{x}_t^T, \tilde{y}_t^T)\}$ of discrete fuzzy vectors converges uniformly to the continuous fuzzy vector $(\tilde{x}_0^T, \tilde{y}_0^T)$.

The process to evaluate the expectation objective function by AA is summarized as follows.

Algorithm 4.1. Evaluating expectation objective function by AA

Step 1. Generate K points $((\hat{x}_t^k)^T, (\hat{y}_t^k)^T) = (\hat{x}_{t,1}^k, \cdots, \hat{x}_{t,m}^k, \hat{y}_{t,m+1}^k, \cdots, \hat{y}_{t,m+r}^k)$ uniformly from the support of $(\tilde{x}_0^T, \tilde{y}_0^T)$ for $k = 1, 2, \ldots, K$.

Step 2. Compute $v_k = v_{t,1}(\hat{x}_{t,1}^k) \wedge \cdots \wedge v_{t,m+r}(\hat{y}_{t,m+r}^k)$ and $f_k = f(\hat{x}_t^k, \hat{y}_t^k)$ for $k = 1, 2, \ldots, K$, where $f(\tilde{x}_0, \tilde{y}_0) = v^T\tilde{y}_0/u^T\tilde{x}_0$ for model (1) or $f(\tilde{x}_0, \tilde{y}_0) = u^T\tilde{x}_0/v^T\tilde{y}_0$ for model (2).

Step 3. Rearrange the subscript k of v_k and f_k such that $f_1 \leq f_2 \leq \cdots \leq f_K$.

Step 4. Calculate the weights w_k, where

$$w_k = \frac{1}{2}(\max_{i=1}^{k} v_i - \max_{i=0}^{k-1} v_i) + \frac{1}{2}(\max_{i=k}^{K} v_i - \max_{i=k+1}^{K+1} v_i)$$

with $v_0 = v_{K+1} = 0$.

Step 5. Return the expected value by $E = \sum_{k=1}^{K} w_k f_k$.

The convergence of the evaluation method is guaranteed by the following convergent result. As a consequence, the value of expectation objective $E\left[v^T\tilde{y}_0/u^T\tilde{x}_0\right]$ or $E\left[u^T\tilde{x}_0/v^T\tilde{y}_0\right]$ can be estimated by the approximating expectation objective value $E\left[v^T\tilde{y}_t/u^T\tilde{x}_t\right]$ or $E\left[u^T\tilde{x}_t/v^T\tilde{y}_t\right]$ provided t is sufficiently large.

Theorem 4.1. *Consider credibilistic DEA models (1) and (2). If the fuzzy inputs-outputs $(\tilde{x}_0^T, \tilde{y}_0^T)$ is an $(m+r)$-ary continuous and bounded fuzzy vector, and the sequence $(\tilde{x}_t^T, \tilde{y}_t^T)$ of fuzzy vectors is the discretization of the fuzzy vector $(\tilde{x}_0^T, \tilde{y}_0^T)$, then we have*

$$\lim_{t\to\infty} E[v^T\tilde{y}_t/u^T\tilde{x}_t] = E[v^T\tilde{y}_0/u^T\tilde{x}_0], \text{ and } \lim_{t\to\infty} E[u^T\tilde{x}_t/v^T\tilde{y}_t] = E[u^T\tilde{x}_0/v^T\tilde{y}_0].$$

Proof. Since the fuzzy inputs-outputs $(\tilde{x}_0^T, \tilde{y}_0^T)$ is an $(m+r)$-ary continuous and bounded positive fuzzy vector, and for any feasible solution (u, v), the functions $v^T\tilde{y}_0/u^T\tilde{x}_0$ or $u^T\tilde{x}_0/v^T\tilde{y}_0$ are continuous on $\Re^{m+r}$. By the suppositions of this theorem, the conditions of [12, Theorem 3] are satisfied. As a consequence, the assertions in this theorem hold. □

In the input-oriented and the output-oriented credibilistic DEA models (1) and (2), when the inputs and outputs are general continuous fuzzy variables for each DMU, we cannot turn the credibility constraint functions into their crisp deterministic forms. In this case, we can also evaluate the credibility constraint functions by AA. The convergence of the approach is included in the following result.

Theorem 4.2. *Consider credibilistic DEA models (1) and (2). If the fuzzy inputs-outputs $(\tilde{x}_0^T, \tilde{y}_0^T)$ is an $(m+r)$-ary continuous and bounded fuzzy vector, and the sequence $(\tilde{x}_t^T, \tilde{y}_t^T)$ of fuzzy vectors is the discretization of the fuzzy vector $(\tilde{x}_0^T, \tilde{y}_0^T)$, then we have*

$$\lim_{t\to\infty} \mathrm{Cr}\left\{u^T\tilde{x}_t - v^T\tilde{y}_t \geq 0\right\} = \mathrm{Cr}\left\{u^T\tilde{x}_0 - v^T\tilde{y}_0 \geq 0\right\},$$

provided that credibility function $\mathrm{Cr}\{u^T\tilde{x}_0 - v^T\tilde{y}_0 \geq r\}$ *is continuous at* $r = 0$.

Proof. By the construction of $(\tilde{x}_t^T, \tilde{y}_t^T)$, we know that the sequence $\{(\tilde{x}_t^T, \tilde{y}_t^T)\}$ of discrete fuzzy vectors converges uniformly to the continuous fuzzy vector $(\tilde{x}_0^T, \tilde{y}_0^T)$. For any given inputs and outputs weight vectors u and v, since $u^T\tilde{x}_t - v^T\tilde{y}_t$ is a linear combination of discrete fuzzy variables

$$(\tilde{x}_{t,1}, \tilde{x}_{t,2}, \cdots, \tilde{x}_{t,m}, \tilde{y}_{t,m+1}, \tilde{y}_{t,m+2}, \cdots, \tilde{y}_{t,m+r}),$$

the sequence $\{u^T\tilde{x}_t - v^T\tilde{y}_t\}$ of discrete fuzzy variables converges uniformly to the continuous fuzzy variable $u^T\tilde{x}_0 - v^T\tilde{y}_0$. It is known that in credibility theory uniform convergence implies convergence in distribution for sequence of fuzzy variables, we deduce that $\{u^T\tilde{x}_t - v^T\tilde{y}_t\}$ converges in distribution to $u^T\tilde{x}_0 - v^T\tilde{y}_0$. Therefore, by [12, Theorem 1], the assertion of the theorem is valid provided that credibility function $\mathrm{Cr}\{u^T\tilde{x}_0 - v^T\tilde{y}_0 \geq r\}$ is continuous at $r = 0$. □

4.2 Approximation-based SA Algorithm

For simplicity, we take $E(u, v, \tilde{x}_0, \tilde{y}_0)$ as the objective functions for credibilistic DEA models (1) and (2) in the rest of this paper. We now introduce the SA algorithm to solve models (1) and (2). The name of the SA algorithm derived from an analogy between the simulation of the annealing process of solids originally proposed by Metropolis et al. [16] and the strategy of solving combinatorial optimization problems firstly provided by Kirkpatrick et al. [9]. In the annealing process, a solid is heated to a high temperature and cooled slowly to low it to crystallize. As the heating process allows the atoms to move randomly, if the cooling process is done too slowly, it gives the atoms enough time to redistribute themselves in a regular crystalline lattice structure as they lose mobility until a minimum energy state is reached, which is also named stability. In the analogy, the state of solid corresponding to the feasible solution of the combinatorial optimization problems, the energy at every state corresponding to the improvement of the objective function and the minimum energy state will be the optimal solution of the combinatorial optimization problems.

The SA algorithm, a very excellent global optimization algorithm, is a stochastic approach to direct the search. In the SA algorithm, an optimization problem starts with a high temperature. After generating an initial solution, it attempts to move from the current solution to one of its neighborhood solutions, which are used to change the values of decision variables, and made according to a user-defined annealing schedule. Then the changes in the objective function values (ΔE) are computed. If the new solution results in better objective value, it is accepted. However, if the new solution yields worse objective value, it can still be accepted according to the probability function, $P(\Delta E) = \min\{1, \exp(-\Delta E/t_k)\}$, where t_k is the current temperature. This check is performed by firstly generating a random real number from the interval (0, 1). If the value is less than or equal to the probability value, the new solution is accepted; otherwise, it is rejected. By accepting worse solutions, the SA algorithm can avoid being trapped in local optimal solution. the SA algorithm repeats the above process $MarkovLength$ times at each temperature to reach the thermal equilibrium, where $MarkovLength$ is a control parameter, which was usually called the Markov chain length. The parameter t_k is gradually decreased by a cooling function as the SA algorithm proceeds until the stopping condition is satisfied. However, three parameters must be specified in implementing the SA algorithm: an initial temperature; a temperature decreasing rule, and the number of iterations to be performed at each temperature stage, i.e., the Markov Chain length.

The approximation-based SA algorithm for solving models (1) and (2) is summarized as:

Algorithm 4.2. Approximation-based SA algorithm

Step 1. Set the initialized temperature $t_0 := t_{max}$ and generate initialized solution (u_0, v_0), then evaluate its corresponding objective function value $E(u_0, v_0, \tilde{x}_0, \tilde{y}_0)$ by AA.

Step 2. Randomly sample a point (u, v) from the given feasible region, then evaluate by AA the expectation objective function $E(u, v, \tilde{x}_0, \tilde{y}_0)$ and $\Delta E = E(u_0, v_0, \tilde{x}_0, \tilde{y}_0) - E(u, v, \tilde{x}_0, \tilde{y}_0)$ in model (1) or $\Delta E = E(u, v, \tilde{x}_0, \tilde{y}_0) - E(u_0, v_0, \tilde{x}_0, \tilde{y}_0)$ in model (2).

Step 3. Check whether $\min\{1, \exp(-\Delta E/t_k)\} \geq$ random(0,1). If yes, then $(u_0, v_0) := (u, v)$; otherwise remains the current state (u_0, v_0).

Step 4. At the same temperature, check whether the inner-circulation (I-C) satisfies the stopping criterion. If yes, go to the next step; otherwise go back to Step 2.

Step 5. Decrease the temperature slowly by $t_{k+1} := \lambda t_k$, where $\lambda \in (0, 1)$.

Step 6. Check whether the outer-circulation (O-C) satisfies the stopping criterion. If yes, then output the optimal solution and optimal value, stop; otherwise, go back to Step 2.

For designing the SA algorithm, some principle factors should be considered.

Initial solution: An initial solution is a starting solution or point that will be used in the search process and considered as a random solution. In this paper, we use (u_0, v_0) to denote it, which is the best one of the one hundred random points produced in the feasible field.

Initial temperature: An initial temperature $t_0 = t_{max}$ is used to control the series of atoms' moves in the search process. In general, the initial temperature should be high enough to allow all candidate solutions to be accepted.

Temperature decreasing rule: The temperature decreasing rule in this paper is decided by the parameter λ, which is also used to control the series of atoms' moves in the search process. The parameter λ is the rate at which the temperature is decreased. In this paper, we set the parameter $\lambda = 0.99$, and the temperature is decreased through multiplication by a fixed factor λ while starting from the initial temperature t_0.

Neighboring solutions: Neighboring solutions are the set of feasible solutions that can be generated from the current solution. Each feasible solution can be directly reached from current solution by a move and resulted neighboring solution. Here, we adopt the following method to generate the neighboring solutions.

$$Neighboring\ solutions \sim \mathcal{N}(current\ solution, 0.5)$$

where $\mathcal{N}(\cdot,\cdot)$ means normal distribution.

Stopping criteria: The SA algorithm has inside and outside circulations, where the inside circulation controls the achievement to equilibrium state in the current temperature and the outside circulation controls the temperature decreasing rate, hence there are two stopping criteria:

$C1$) In the outer-circulation, the number of temperature transitions is used as a stopping rule. Furthermore, when the parameter $t_k < \varepsilon$ in the SA algorithm, the algorithm can be terminated, where ε is a very small positive constant (e.g., $\varepsilon = 10^{-8}$ in this paper) or calculated by other parameters.

$C2$) In the inner-circulation, the control parameter $MarkovLength$, usually called the Markov chain length, controls the completion of the inner loop. In addition, the users can determine this period.

4.3 Approximation-based Hybrid SA Algorithm

Till now, we have discussed the methods of evaluating expectation objective and credibility constraint functions included in models (1) and (2). However, the approximation-based SA algorithm is a time-consuming process provided the number of realizations of fuzzy inputs and outputs. To speed up the solution process, we desire to replace the expectation objective and credibility constraint functions by an NN since a trained NN has the ability to approximate integrable functions [20]. We employ the fast BP algorithm [8] to train a feed-forward NN to approximate the expectation objective or credibility constraint functions. Usually, an NN with two hidden layers is better in generation than the NN with one hidden layer. But in most applications, an NN with one hidden layer is enough to be a universal approximator for any integrable functions. Thus, in this paper, we only consider the NN with the input layer, one hidden layer and the output layer connected in a feed-forward way. For this purpose, we design an approximation-based hybrid SA algorithm, which integrates AA, NN and the SA algorithm, to solve general fuzzy expected value DEA models (1) and (2). In this hybrid algorithm, the AA is used to evaluate the values of expectation objective and credibility constraint functions; NN is employed to approximate the expectation objective and credibility constraint functions, and the SA algorithm is used to find the optimal solutions to the DEA models. More precisely, we first employ AA to generate a set of input-output data for expectation objective and credibility constraint functions. Then, using the generated input-output data, we train an NN by BP algorithm to approximate the expectation objective and credibility constraint functions. We repeat the BP algorithm until the error for all vectors in the training set is reduced to an acceptable value. After that, we use new data, which are not learned by the NN, to test the trained NN. We don't stop the training process until we satisfy with the test results. After the NN is well-trained, it is embedded into a SA algorithm. In the process of the SA algorithm, the output values of the NN represent the approximate values of the expectation objective and credibility constraint functions. Therefore, during solution process, it is not necessary to adopt AA to evaluate expectation objective and credibility constraint functions so that much time can be saved. The process of the approximation-based hybrid SA algorithm for solving the proposed fuzzy expected value DEA model is summarized as:

Algorithm 4.3. Approximation-based hybrid SA algorithm

Step 1. Generate a set of input-output data for expectation objective function

$$U : (u, v) \rightarrow \quad E(u, v, \tilde{x}_0, \tilde{y}_0)$$

by AA.

Step 2. Train an NN by BP algorithm to approximate the expectation objective function $E(u, v, \tilde{x}_0, \tilde{y}_0)$ according to the generated training input-output data.

Step 3. Initialize the temperature and generate initialized state, and compute its corresponding expectation objective function value by the trained NN.

Step 4. Check whether the outer-circulation satisfies the stopping criterion. If yes, then output the optimal solution and optimal value; otherwise, go to the next step.

Step 5. Generate a new state by the state-produced function, and calculate its corresponding expectation objection function value by the trained NN.

Step 6. Check whether the new state is accepted by the acceptance function. If yes, update the current state; otherwise, remain the current state and go to the next step.

Step 7. Check whether the inner-circulation satisfies the stopping criterion. If yes, decrease the temperature slowly and then go back to Step 4; otherwise, go back to Step 5.

5 Computational Results and Efficiency Analysis

In this section, one numerical example is provided to illustrate the feasibility and effectiveness of the approximation-based hybrid SA algorithm. The numerical experiments are performed on a personal computer, and the parameters involved in the SA algorithm are set as follows: The initial temperature $t_{\max} = 10$; The parameter λ in temperature decreasing rule is 0.99; The parameter ε in the outer-circulation is 10^{-8}, and the parameter $MarkovLength$ in the inner-circulation is 3000. The required fuzzy input data and fuzzy output data for our numerical experiments are collected in Table 1, which are characterized by triangular fuzzy variables. In addition, the credibility levels for all DMUs in models (1) and (2) are assumed to be equal, i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$.

Table 1: DMUs with two fuzzy inputs and two fuzzy outputs

DMU_i	Input 1	Input 2	Output 1	Output 2
i=1	(3.0, 3.5, 4.0)	(1.9, 2.1, 2.3)	(2.6, 2.8, 3.0)	(3.8, 4.1, 4.4)
i=2	(2.7, 2.7, 2.7)	(1.4, 1.5, 1.6)	(2.5, 2.5, 2.5)	(3.5, 3.7, 3.9)
i=3	(4.1, 4.5, 4.9)	(2.0, 2.4, 2.8)	(2.9, 3.4, 3.9)	(4.5, 5.3, 6.1)
i=4	(3.2, 3.9, 4.6)	(2.0, 2.1, 2.2)	(2.8, 3.2, 3.6)	(5.5, 5.7, 5.9)
i=5	(5.4, 6.0, 6.6)	(3.4, 3.9, 4.4)	(4.2, 4.9, 5.6)	(6.8, 7.7, 8.6)

Firstly, when the fuzzy input and output data are mutually independent triangular fuzzy variables, the credibility constraints can be converted to their crisp equivalents by the methods discussed in subsection 3. However, we cannot do so for the expectation objective functions.

Then, we adopt AA to evaluate the values of the expectation objective functions (10000 discretization points in the numerical experiments), and repeat this process to produce a set of 2000 input-output data for the input-oriented expectation objective function

$$U_1 : (u, v) \to \quad E(u, v, \tilde{x}_0, \tilde{y}_0) = E\left[v^T \tilde{y}_0 / u^T \tilde{x}_0\right],$$

and the output-oriented expectation objective function

$$U_2 : (u, v) \to \quad E(u, v, \tilde{x}_0, \tilde{y}_0) = E\left[u^T \tilde{x}_0 / v^T \tilde{y}_0\right],$$

respectively.

After that, according to the sets of input-output data, we train two feedforward NNs to approximate the expectation objective functions U_1 and U_2, respectively. In the numerical experiments, the trained NNs have the same structures: four input neurons representing the values of decision variables u_1, u_2, v_1 and v_2; five hidden neurons, and one output neuron representing the output value of expectation objective function.

Finally, the trained NNs are embedded respectively into the SA algorithm to produce the AA and NN-based SA algorithm to search for the optimal solutions to the input-oriented and the output-oriented expected value DEA models, respectively.

The results of evaluating the DMUs with credibility level $\alpha = 0.95$ in the input-oriented DEA model (1) are reported in Table 2, which can be interpreted in the following way. DMU_1, DMU_3 and DMU_5 are expectation-inefficient with expected values 0.873778, 0.935441 and 0.949802, respectively. Therefore, if those DMUs want to improve their relative efficiency values to change their inferior positions in competition, they must decrease their input data because model (1) is an input-oriented fuzzy expectation DEA model. On the other hand, the other two DMUs are expectation-efficient. From Table 2, we can also see that DMU_2 is the most efficient, follows with DMU_4 and DMU_5. Hence, the efficient values give more information about the DMUs and can make the decision maker have a reasonable choice to increase its efficiency in competition.

Table 2: Results of evaluation with $\alpha = 0.95$ in model (1)

DMUs	Optimal solution(u,v)	Optimal value	Evaluating analysis
DMU_1	(0.0043, 0.0253, 0.0921, 0.0049)	0.873778	Inefficiency
DMU_2	(0.0107, 0.0279, 0.0165, 0.0053)	1.033811	Efficiency
DMU_3	(0.0018, 0.0385, 0.0204, 0.0005)	0.935441	Inefficiency
DMU_4	(0.0059, 0.0132, 0.0067, 0.0035)	1.005339	Efficiency
DMU_5	(0.0014, 0.0114, 0.0000, 0.0037)	0.949802	Inefficiency

Table 3 collects the results of evaluating the DMUs with credibility level $\alpha = 0.95$ in the output-oriented DEA model (2). From Table 3, we can see that DMU_1, DMU_3 and DMU_5 are expectation-inefficient with expected values 1.129857, 1.148728 and 1.102718, respectively. The relative expectation-inefficient DMUs have some disadvantages in the output data which should be changed in order to vary their weakness positions in competition because of the output-oriented credibilistic DEA model (2). On the other hand, DMU_2 and DMU_4 are expectation-efficient. From Table 3, we can also see that DMU_2 is the most efficient, follows with DMU_4 and DMU_5.

Table 3: Results of evaluation with $\alpha = 0.95$ in model (2)

DMUs	Optimal solution(u,v)	Optimal value	Evaluating analysis
DMU_1	(0.0025, 0.0214, 0.0054, 0.0048)	1.129857	Inefficiency
DMU_2	(0.0036, 0.0172, 0.0103, 0.0016)	0.974755	Efficiency
DMU_3	(0.0058, 0.0147, 0.0119, 0.0006)	1.148728	Inefficiency
DMU_4	(0.0032, 0.0451, 0.0046, 0.0142)	0.998820	Efficiency
DMU_5	(0.0120, 0.0062, 0.0136, 0.0005)	1.102718	Inefficiency

According to the computational results shown in Tables 2 and 3, we know that the identical DMU has the same relative efficiency which are evaluated by the two types of proposed fuzzy expected value DEA models with the same credibility levels. For the identical expectation-inefficient DMU in the two types of fuzzy DEA models, however, if it wants to improve its inferior position in competition, it should take different measures according to the model in which it is evaluated. Hence, the decision makers play a very important role in the evaluating relative efficiency using the credibilistic DEA models. They can reasonably choose the fuzzy DEA model in the light of their pursing orientation to deal with the relative efficiency evaluation of the DMUs.

6 Conclusions

On the basis of credibility measure theory, this paper studied fuzzy DEA from a new viewpoint. The major new results of this paper included the following several aspects.

Firstly, we developed the input-oriented and the output-oriented credibilistic DEA models with fuzzy inputs and fuzzy outputs to estimate the relative efficiency of the evaluated DMUs, in which we employed the expected value operator to construct the objective functions and credibility measure to construct the constraint functions. The definitions of expectation efficiency of the evaluated DMUs to the proposed expected value DEA models were introduced.

Secondly, for general credibilistic DEA models, we adopted AA to evaluate the expectation objective and credibility constraint functions. The reasonableness of the method are ensured by two related convergent results (Theorems 4.1 and 4.2).

Finally, an approximation-based hybrid algorithm was designed for solving our credibilistic EDA models. The hybrid algorithm is the integration of AA, NN, and the SA algorithm. One numerical example was provided to illustrate the feasibility and effectiveness of the approximation-based hybrid SA algorithm, and the DMUs' relative efficiency was analyzed via numerical experiments.

Acknowledgments

The authors wish to thank the anonymous referees for their valuable suggestions that have been incorporated into this paper. This work was supported by the National Natural Science Foundation of China No.61374184.

References

[1] Banker, R.D., Charnes, A., and W.W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science*, vol.30, pp.1078–1092, 1984.

[2] Charnes, A., Cooper, W.W., Lewin, A.Y., and L.M. Seiford, *Data Envelopment Analysis: Theory, Methodology, and Application*, Kluwer Academic Publishers, London, 1994

[3] Charnes, A., Cooper, W.W., and E. Rhodes, Measuring the efficiency of decision making units, *European Journal of Operational Research*, vol.2, pp.429–444, 1978.

[4] Cooper, W.W., Park, K.S., and J.T. Pastor, RAM: a range adjusted measure of inefficiency for use with additive models, and relations to other models and measures in DEA, *Journal of Productivity Analysis*, vol.11, pp.5–24, 1999.

[5] Entani, T., Maeda, Y., and H. Tanaka, Dual models of interval DEA and its extension to interval data, *European Journal of Operational Research*, vol.136, pp.32–45, 2002.

[6] Hatami-Marbini, A., Emrouznejad, A., and M. Tavana, A taxonomy and review of the fuzzy data envelopment analysis literature: two decades in the making, *European Journal of Operational Research*, vol.214, pp.457–472, 2011.

[7] Hatami-Marbini, A., and M. Tavana, An extension of the Electre I method for group decision-making under a fuzzy environment, *Omega*, vol.39, pp.373–386, 2011.

[8] Karayiannis, N.B., and A.N. Venetsanopoulos, Fast learning algorithms for neural networks, *IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing*, vol.39, pp.453–474, 1992.

[9] Kirkpatrick, S., Gelatt, J., and M. Vecchi, Optimization by simulated annealing, *Science*, vol.220, pp.671–680, 1983.

[10] Liu, B., Credibility theory, *Studies in Fuzziness and Soft Computing*, vol.154, pp.81–156, 2007.

[11] Liu, B., and Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, vol.10, pp.445–450, 2002.

[12] Liu, Y., Convergent results about the use of fuzzy simulation in fuzzy optimization problems, *IEEE Transactions on Fuzzy Systems*, vol.14, pp.295–304, 2006.

[13] Liu, Y., Chen, Y., Liu, Y., and R. Qin, *Fuzzy Optimization Methods with Applications*, Science Press, Beijing, 2013.

[14] Liu, Y., and J. Gao, The independence of fuzzy variables with applications to fuzzy random optimization, *International Journal of Uncertainty, Fuzziness & Knowlege-Based Systems*, vol.15, pp.1–20, 2007.

[15] Meng, M., A hybrid particle swarm optimization algorithm for satisficing data envelopment analysis under fuzzy chance constraints, *Expert Systems with Applications*, vol.41, pp.2074–2082, 2014.

[16] Metropolis, P., Rosenbluth, A., and M. Rosenbluth, Equation of state calculations by fast computing machines, *Journal of Chemical Physics*, vol.21, pp.1087–1092, 1953.

[17] Petersen, N.C., Data envelopment analysis on a relaxed set of assumptions, *Management Science*, vol.36, pp.305–313, 1990.

[18] Qin, R., Liu, Y., and Z. Liu, Methods of critical value reduction for type-2 fuzzy variables and their applications, *Journal of Computational and Applied Mathematics*, vol.235, pp.1454–1481, 2011.

[19] Qin, R., Liu, Y., and Z. Liu, Modeling fuzzy data envelopment analysis by parametric programming method, *Expert Systems with Applications*, vol.38, pp.8648–8663, 2011.

[20] Scarselli, F., and A.C. Tsoi, Universal approximation using feedforward neural networks: a survey of some existing methods, and some new results, *Neural Networks*, vol.11, pp.15–37, 1998.

[21] Sengupta, J.K., A fuzzy systems approach in data envelopment analysis, *Computer and Mathematics with Applications*, vol.24, pp.259–266, 1992.

[22] Tone, K., A slack-based measure of efficiency in data envelopment analysis, *European Journal of Operational Research*, vol.130, pp.498–509, 2001.