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Approximate Solution of First Order Nonlinear Fuzzy Initial Value Problem with Two Different Fuzzifications

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Abstract

In this work, the homotopy perturbation method is developed and formulated to find an approximate-analytical solution of fuzzy initial value problems involving a nonlinear first order ordinary differential equation. HPM allows for the solution of FIVPs to be calculated in the form of an infinite series in which the components can be easily computed. An efficient algorithm is proposed of HPM on the basis of the principles and definitions of fuzzy sets theory and the capability of the method is illustrated by solving a nonlinear first order fuzzy Riccati equation with two different fuzzifications. The results indicate that the method is very effective and simple to apply and presented solution in the form of tables and figures.

Keywords: fuzzy numbers, fuzzy differential equations, homotopy perturbation method

1 Introduction

Fuzzy set theory is a powerful tool for the modeling of vagueness, and for processing uncertainty or subjective information on mathematical models. The use of fuzzy sets can be an effective tool for a better understanding of the studied phenomena. Many dynamical real life problems may be formulated as a mathematical model involving as a system of ordinary or partial differential equations. Fuzzy differential equations are a useful tool to model a dynamical system when information about its behavior is inadequate and uncertain. Fuzzy initial value problem (FIVP) involving ordinary differential equations are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including, population models [2, 8, 25, 26], mathematical physics [24], and medicine [1, 7]. An initial value problem involving first order linear fuzzy differential equation can be considered to be the simplest case to test the effectiveness of proposed methods for solving fuzzy differential equations. In many cases, the analytical solution may be difficult to be evaluated and therefore numerical and approximate analytical techniques may be necessary to obtain the solution.

The Homotopy perturbation Method introduced by He [15, 16, 17, 18] has been used by many mathematicians and engineers to solve various functional equations. In this method the solution is considered as the sum of an infinite series which converges rapidly. Using ideas from the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a small parameter. Approximate-analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), and Differential Transform Method (DTM) were used to solve first order fuzzy initial value problems involving ordinary differential equations. In [12], the HPM was used to solve first order linear fuzzy initial value problems. Allahviranloo et al. [4] implemented the DTM on first order linear FIVPs. The ADM [3, 6, 13] was employed to solve first order linear and nonlinear fuzzy initial value problems. Also VIM [7, 13] was used to solve first order linear and nonlinear fuzzy initial value problems. Our aim in this study is to formulate HPM to solve first order fuzzy Riccati equation and analyze the obtained results with two different fuzzifications. The main thrust of this technique is that the solution which is expressed as an infinite series converges fast to exact solutions. To the best of our knowledge, this is the first attempt at solving and analyzing the first order nonlinear FIVP using HPM with two different fuzzifications. The structure of this paper is organized as follows. We will start in section 2 with some preliminary concepts about fuzzy numbers. In section 3, we reviewed the concept of HPM and formulated it to obtain

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a reliable approximate solution to first order FIVPs. In section 4, we employ HPM on test example involving first order fuzzy Riccati equation and finally, in section 5, we give the conclusions of this study.

2 Preliminaries

The *r*-level (or *r*-cut) [9] set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$, i.e.

$$\tilde{A}_r = \{x \in X | \mu_{\tilde{A}} > r, r \in [0,1]\}.$$

Definition 2.1 Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number [10] μ is called a triangular fuzzy number if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$, and its membership function has the following form:

$$\mu(x;\alpha,\beta,\gamma) = \begin{cases} 0, & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta \\ \dfrac{\gamma - x}{\gamma - \beta}, & \text{if } \beta \leq x \leq y \\ 1, & \text{if } x > \gamma. \end{cases}$$

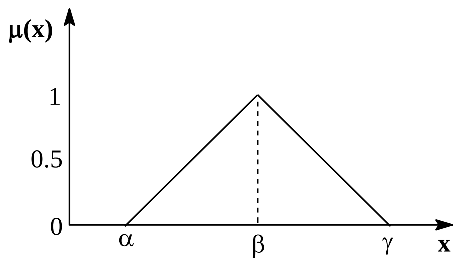

Figure1: Triangular fuzzy number

The r-level sets of triangular fuzzy numbers are

$$[\mu]_r = [\alpha + r\,(\beta - \alpha), \gamma - r\,(\gamma - \beta)], r \in [0, 1].$$

In this paper the class of all fuzzy subsets of $\mathbb{R}$ will be denoted by $\tilde{E}$ and satisfy the following properties [10, 22]:

1. $\mu(t)$ is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$,
2. $\mu(t)$ is convex fuzzy set, i.e.$\mu(\lambda t + (1-\lambda)s) \geq \min\{\mu(t), \mu(s)\}$ $\forall t, s \in \mathbb{R}, \lambda \in [0,1]$,
3. μ upper semi-continuous on $\mathbb{R}$, and $\overline{\{t \in \mathbb{R}: \mu(t) > 0\}}$ is compact.

$\tilde{E}$ is called the space of fuzzy numbers and $\mathbb{R}$ is a proper subset of $\tilde{E}$.

Define the r-level sets $x \in \mathbb{R}, [\mu]_r = \{x \setminus \mu(x) \geq r\}, 0 \leq r \leq 1$ where$[\mu]_0 = \{x \setminus \mu(x) > 0\}$ is compact [30] which is a closed bounded interval and denoted by $[\mu]_r = (\underline{\mu}(t), \overline{\mu}(t))$. In the parametric form, a fuzzy number is represented by an ordered pair of functions $(\underline{\mu}(t), \overline{\mu}(t))$, $r \in [0,1]$ which satisfies [19]:

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over [0,1].
2. $\overline{\mu}(t)$ is a bounded left continuous non-increasing function over [0,1].
3. $\underline{\mu}(t) \leq \overline{\mu}(t), r \in [0,1]$.

A crisp number *r* is simply represented by $\underline{\mu}(r) = \overline{\mu}(r) = r, r \in [0,1]$.

Definition 2.2 ([27]) If $\tilde{E}$ be the set of all fuzzy numbers, we say that $f(t)$ is a fuzzy function if $f: \mathbb{R} \to \tilde{E}$.

Definition 2.3 ([11]) A mapping $f: T \to \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote *r*-level set by

$$[\tilde{f}(t)]_r = \left[\underline{f}(t;r), \overline{f}(t;r)\right], t \in T, r \in [0,1].$$

The *r*-level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their *r*-level sets based on the resolution identity theorem.

Definition 2.5 ([29, 30]) Each function $f: X \to Y$ induces another function $\tilde{f}: F(X) \to F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} Sup_{x \in f^{-1}(y)} U(x), & \text{if } y \in range\ (f) \\ 0, & \text{if } y \notin range\ (f). \end{cases}$$

This is called the Zadeh extension principle.

Definition 2.6 ([19]) Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$such that $\tilde{x} = \tilde{y} + \tilde{z}$, then z is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.7 ([29]) If $\tilde{f}: I \to \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$, we say that $\tilde{f}$ Hukuhara Differentiable at y_0. If there exists an element $[\tilde{f}']_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h\to 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.$$

The fuzzy set $[\tilde{f}'(y_0)]_r$ is called the Hukuhara derivative of $[\tilde{f}']_r$ at y_0.

These limits are taken in the space $(\tilde{E}, D)$ if t_0 or T, then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r-level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r$, $\forall\, r \in [0,1]$. By consideration of definition of the metric D all the r-level set $[\tilde{f}(0)]_r$ are Hukuhara differentiable at y_0, with Hukuhara derivatives $[\tilde{f}'(y_0)]_r$, when $\tilde{f}: I \to \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $[\tilde{f}'(y_0)]_r$ it' lead to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0,1]$ which satisfies the above limits, i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$, then all its r-levels $[\tilde{f}'(t)]_r$ are Hukuhara differentiable at t_0.

Definition 2.8 ([22]) Define the mapping $\tilde{f}': I \to \tilde{E}$ and $y_0 \in \mathrm{I}$, where $I \in [t_0, T]$. We say that $\tilde{f}'$ Hukuhara differentiable $t \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)$, $\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative.

Theorem 2.1 ([27]) Let $\tilde{f}: [t_0 + \alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote by

$$[\tilde{f}'(t)]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right].$$

Then the boundary functions $\underline{f}'(t;r), \overline{f}'(t;r)$ are differentiable $\forall\, r \in [0,1]$ such that

$$[\tilde{f}'(t)]_r = \left[(\underline{f}(t;r))', (\overline{f}(t;r))'\right].$$

Definition2.5 ([19]) The fuzzy integral of fuzzy process, $\tilde{f}(t;r)$, $\int_a^b \tilde{f}(t;r)dt$ for $a, b \in T$and $r \in [0,1]$ is defined by:

$$\int_a^b \tilde{f}(t;r)dt = \left[\int_a^b \underline{f}(t;r)dt, \int_a^b \overline{f}(t;r)dt\right].$$

3 Fuzzification and Defuzzification of HPM

The general structure of HPM for solving crisp ordinary differential equations was described in [15, 16, 17, 18]. To solve the first order FIVP, there is a need to fuzzify and then defuzzify HPM. Consider the following general first order FIVP

$$\tilde{y}'(t) = f(t, \tilde{y}(t)) + \tilde{w}(t), \quad t \in [t_0, T], \tag{1}$$

$$\tilde{y}(t_0) = \tilde{y}_0. \tag{2}$$

According to section 2, $\tilde{y}$ is a fuzzy function of the crisp variable t with f being a fuzzy function of the crisp variable t and the fuzzy variable $\tilde{y}$. Here $\tilde{y}'$ is the fuzzy first order Hukuhara-derivative and $\tilde{y}(t_0)$ is triangular fuzzy number.

Denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$ for $t \in [t_0, T]$ and $r \in [0,1]$. It means that the r-level sets of $\tilde{y}(t)$ can be defined as:

$$[\tilde{y}(t)]_r = \left[\underline{y}(t;r), \overline{y}(t;r)\right],$$

$$\left[\tilde{y}'(t)\right]_r = \left[\underline{y}'(t;r), \overline{y}'(t_0;r)\right],$$

$$[\tilde{y}(t_0)]_r = \left[\underline{y}(t_0;r), \overline{y}(t_0;r)\right],$$

where $w(t)$ is crisp or fuzzy inhomogeneous term such that $[\widetilde{w}(t)]_r = \left[\underline{w}(t;r), \overline{w}(t;r)\right]$.

Since $y'(t) = f(t, y(t)) + w(t)$, then $\left[\tilde{f}(t,\tilde{y})\right]_r = \left[\underline{f}(t,\tilde{y};r), \overline{f}(t,\tilde{y};r)\right]$. Now using Zadeh extension principles we have the followings:

$$\tilde{f}(t,\tilde{y}(t;r)) = \left[\underline{f}(t,\tilde{y}(t;r)), \underline{f}(t,\tilde{y}(t;r))\right] \quad (3)$$

where

$$\underline{f}(t,\tilde{y}(t;r)) = \mathcal{F}\left(t, \underline{y}(t;r), \overline{y}(t;r)\right) = \mathcal{F}(t,\tilde{y}(t;r)),$$

$$\overline{f}(t,\tilde{y}(t;r)) = \mathcal{G}\left(t, \underline{y}(t;r), \overline{y}(t;r)\right) = \mathcal{G}(t,\tilde{y}(t;r)),$$

Then

$$\underline{y}'(t;r) = \mathcal{F}(t,\tilde{y}(t;r)) + \underline{w}(t;r), \quad (4)$$

$$\overline{y}'(t;r) = \mathcal{G}(t,\tilde{y}(t;r)) + \overline{w}(t;r), \quad (5)$$

where the membership function of $\mathcal{F}(t,\tilde{y}(t;r)) + \underline{w}(t;r)$ and $\mathcal{G}(t,\tilde{y}(t;r)) + \overline{w}(t;r)$ can be defined as

$$\mathcal{F}(t,\tilde{y}(t;r)) + \underline{w}(t;r) = min\{\tilde{y}'(t,\tilde{\mu}(r)) : \mu | \mu \in [\tilde{y}(t;r)]_r\},$$

$$\mathcal{G}(t,\tilde{y}(t;r)) + \overline{w}(t;r) = max\{\tilde{y}'(t,\tilde{\mu}(r)) : \mu | \mu \in [\tilde{y}(t;r)]_r\}.$$

From the above analysis, we can rewrite Eqs. (4) and (5) in the following forms

$$\begin{cases} \underline{\mathcal{L}y}(t;r) = \mathcal{F}(t,\tilde{y}(t;r)) + \underline{w}(t;r), \quad t \in [t_0, T] \\ \underline{y}(t_0;r) = \underline{y}_0 \\ r \in [0,1], \end{cases} \quad (6)$$

$$\begin{cases} \overline{\mathcal{L}}\overline{y}(t;r) = \mathcal{G}(t,\tilde{y}(t;r)) + \overline{w}(t;r), \quad t \in [t_0, T] \\ \overline{y}(t_0;r) = \overline{y}_0 \\ r \in [0,1] \end{cases} \quad (7)$$

where $\tilde{\mathcal{L}} = [\underline{\mathcal{L}}, \overline{\mathcal{L}}]$ are the linear operators with $\tilde{\mathcal{L}} = \frac{d}{dt}$ and $\mathcal{F}$, $\mathcal{G}$ are nonlinear operators followed by the inverse operators $\tilde{\mathcal{L}}^{-1} = \int_0^t \widetilde{[.]}_r \, d\tau$ and applying it on Eqs.(6) and (7). According to the HPM, a homotopy form is constructed into Eqs. (6) and (7) which satisfies the following relation

$$\underline{\mathcal{H}}(t,p;r) = \underline{\mathcal{L}}\left[\underline{y}(t;r) - \underline{y}_0(t;r)\right] + p\left[\underline{\mathcal{L}y}_0(t;r) - \mathcal{F}(t,\tilde{y}(t;r)) - \underline{w}(t;r)\right] = 0, \quad (8)$$

$$\overline{\mathcal{H}}(t,p;r) = \overline{\mathcal{L}}\left[\overline{y}(t;r) - \overline{y}_0(t;r)\right] + p\left[\overline{\mathcal{L}}\overline{y}_0(t;r) - \mathcal{G}(t,\tilde{y}(t;r)) - \overline{w}(t;r)\right] = 0 \quad (9)$$

where $p \in [0, 1]$ is an embedding parameter and $\underline{y}_0(t;r)$, and $\overline{y}_0(t;r)$ are initials guesses and can be defined as follows:

$$\begin{cases} \underline{y}_0(t;r) = \underline{y}_0 \\ \overline{y}_0(t;r) = \overline{y}_0. \end{cases} \quad (10)$$

Since $p \in [0,1]$ is an embedding parameter, $\underline{y}_0(t;r), \overline{y}_0(t;r)$ is an initial approximation of Eqs. (4) and (5), which satisfies the initial condition (2). For all $r \in [0,1]$, $t \in [t_0, T]$ and from Eqs. (8) and (9), we have

$$\begin{cases} \underline{\mathcal{H}}(t,0;r) = \underline{\mathcal{L}}\left[\underline{y}(t;r) - \underline{y}_0(t;r)\right] = 0 \\ \underline{\mathcal{H}}(t,1;r) = \left[\underline{\mathcal{L}y}_0(t;r) - \mathcal{F}(t,\tilde{y}(t;r)) - \underline{w}(t;r)\right] = 0, \end{cases} \quad (11)$$

also for the upper bound we have

$$\begin{cases}\overline{\mathcal{H}}(t,0;r) = \bar{\mathcal{L}}\left[\overline{y}(t;r) - \overline{y}_0(t;r)\right] = 0 \\ \overline{\mathcal{H}}(t,1;r) = \left[\bar{\mathcal{L}}\overline{y}_0(t;r) - \mathcal{G}\big(t,\tilde{y}(t;r)\big) - \overline{w}(t;r)\right] = 0,\end{cases} \tag{12}$$

where Eqs. (11) and (12) are called the homotopy deformation [17]. The embedding parameter p is a small parameter used to construct HPM series such that from Eqs. (11) and (12) we can represent each $\tilde{y}(t;r)$ in these equations as follows:

$$\tilde{y}(t;r) = \sum_{k=0}^{\infty} p^k \tilde{y}_k(t;r). \tag{13}$$

Now substituting Eq. (13) into Eqs. (6) and (7) and collecting terms of the same powers of p, we have:

$$\begin{aligned}
&p^0: \left\{\underline{y}_0(t;r) = \underline{y}_0,\right. \\
&p^1: \begin{cases}\underline{\mathcal{L}}\left[\underline{y}_1(t;r) + \underline{y}_0(t;r)\right] - F\big(t,\tilde{y}_0(t;r)\big) - \underline{w}(t;r) = 0 \\ \underline{y}_1(t_0;r) = 0,\end{cases} \\
&p^2: \begin{cases}\underline{\mathcal{L}}\,\underline{y}_2(t;r) - F\big(t,\tilde{y}_1(t;r)\big) = 0 \\ \underline{y}_2(t_0;r) = 0,\end{cases} \\
&\vdots \\
&p^{k+1}: \begin{cases}\underline{\mathcal{L}}\,\underline{y}_{k+1}(t;r) - F\big(t,\tilde{y}_k(t;r)\big) = 0 \\ \underline{y}_k(t_0;r) = 0.\end{cases}
\end{aligned} \tag{14}$$

Similarly for the upper bound of Eq. (1),

$$\begin{aligned}
&p^0: \left\{\overline{y}_0(t;r) = \overline{y}_0,\right. \\
&p^1: \begin{cases}\bar{\mathcal{L}}\left[\overline{y}_1(t;r) + \overline{y}_0(t;r)\right] - \mathcal{G}\big(t,\tilde{y}_0(t;r)\big) - \underline{w}(t;r) = 0 \\ \overline{y}_1(t_0;r) = 0,\end{cases} \\
&p^2: \begin{cases}\bar{\mathcal{L}}\overline{y}_2(t;r) - \mathcal{G}\big(t,\tilde{y}_1(t;r)\big) = 0 \\ \overline{y}_2(t_0;r) = 0,\end{cases} \\
&\vdots \\
&p^{k+1}: \begin{cases}\bar{\mathcal{L}}\overline{y}_{k+1}(t;r) - \mathcal{G}\big(t,\tilde{y}_k(t;r)\big) = 0 \\ \overline{y}_k(t_0;r) = 0.\end{cases}
\end{aligned} \tag{15}$$

From Eqs. (14) and (15) the approximate analytical solution of Eq. (1) is given by setting $p = 1$ as follows:

$$\tilde{y}(t;r) = \varphi_m(t;r) = \sum_{i=0}^{m-1} \tilde{y}_i(t;r). \tag{16}$$

Now the exact solutions of Eq. (1), can be obtained by setting p = 1. Therefore

$$\tilde{Y}(t;r) = \lim_{p\to 1} \tilde{y}(t;r) = \lim_{p\to 1}\left\{\sum_{i=0}^{\infty} p^i \tilde{y}_i(t;r)\right\} = \sum_{i=0}^{\infty} \tilde{y}_i(t;r). \tag{17}$$

4 Results and Discussion

Consider the first order nonlinear crisp Riccati equation [23]. Then from section 2 and [23] the fuzzy version of the crisp Riccati equation can be written as follows

$$\begin{aligned}\tilde{y}'(t) &= [\tilde{y}(t)]^2 + t^2, \quad t \geq 0, \\ \tilde{y}(0) &= [\tilde{0}]\,,\ \forall\, r \in [0,1].\end{aligned} \tag{18}$$

In this problem we have freedom to defuzzify the initial conditions. According to the definitions of fuzzy numbers in section 2, we let $[\tilde{0}]_r$ be triangular fuzzy numbers such that for all $r \in [0,1]$, we have:

First defuzzification

$$[\tilde{0}]_r = [-0.1,0,0.1]_r = [0.1r - 0.1, 0.1 - 0.1r], \tag{19}$$

Second defuzzification

$$[\tilde{0}]_r = [-1,0,1]_r = [r - 1, 1 - r]. \tag{20}$$

Now we define the linear operator of Eq. (18) is $\tilde{\mathcal{L}} = \frac{d}{dt}$ with the inverse operator $\tilde{\mathcal{L}}^{-1} = \int_0^t \widetilde{[.]}_r\, d\tau$, then the initial approximation guesses for the first defuzzification is

$$\tilde{y}_0(t;r) = [0.1(r-1), 0.1(1-r)], \quad (21)$$

and the initial approximation guesses for the second defuzzification is

$$\tilde{y}_0(t;r) = [(r-1), (1-r)]. \quad (22)$$

According to the HPM in section 3, the n components of $\tilde{y}_k(t;r)$ for $k = 1,2,..,n$ and $r \in [0,1]$ can be determined by solving the followings

$$\begin{aligned}
&p^0: \left\{ \underline{y}_0(t;r) = [\underline{0}]_r, \right. \\
&p^1: \begin{cases} \underline{y}_1(t;r) = \underline{\mathcal{L}}^{-1}\left[\underline{y}_0{}^2(t;r) + t^2\right] \\ \underline{y}_1(t_0;r) = 0, \end{cases} \\
&p^2: \begin{cases} \underline{y}_2(t;r) = \underline{\mathcal{L}}^{-1}\left[\underline{y}_1(t;r)\underline{y}_0(t;r)\right] \\ \underline{y}_2(t_0;r) = 0, \end{cases} \\
&\vdots \\
&p^{k+1}: \begin{cases} \underline{y}_{k+1}(t;r) = \underline{\mathcal{L}}^{-1} \sum_{k=0}^{n-1} \underline{y}_k(t;r)\, \underline{y}_{n-1-k}(t;r) \\ \underline{y}_k(t_0;r) = 0. \end{cases}
\end{aligned} \quad (23)$$

Similarly for the upper bound

$$\begin{aligned}
&p^0: \left\{ \overline{y}_0(t;r) = [\overline{0}]_r, \right. \\
&p^1: \begin{cases} \overline{y}_1(t;r) = \bar{\mathcal{L}}^{-1}\left[\overline{y}_0{}^2(t;r) + t^2\right] \\ \overline{y}_1(t_0;r) = 0, \end{cases} \\
&p^2: \begin{cases} \overline{y}_2(t;r) = \bar{\mathcal{L}}^{-1}\left[\overline{y}_0(t;r)\overline{y}_1(t;r)\right] \\ \overline{y}_2(t_0;r) = 0, \end{cases} \\
&\vdots \\
&p^{k+1}: \begin{cases} \overline{y}_{k+1}(t;r) = \bar{\mathcal{L}}^{-1}\left[\sum_{k=0}^{n-1} \overline{y}_k(t;r)\, \overline{y}_{n-1-k}(t;r)\right] \\ \overline{y}_k(t_0;r) = 0. \end{cases}
\end{aligned} \quad (24)$$

From Eqs. (23) and (24), we obtain 5-order HPM series solution in of Eq. (18) as shown in the following table.

Table 1: Approximate solution of Eq. (18) by 5-order of HPM with two different defuzzification at $t = 0.5$ for all $r \in [0,1]$

	First defuzzification		Second defuzzification	
r	$\underline{y}(0.5;r)$	$\overline{y}(0.5;r)$	$\underline{y}(0.5;r)$	$\overline{y}(0.5;r)$
0	−0.05443603300114	0.148169792891714	−0.620208173626142	2.033056335302429
0.2	−0.03593230984128	0.126005660473000	−0.532667579876142	1.384804027266715
0.4	−0.01706855000822	0.104299492727492	−0.423952351205507	0.907926354151635
0.6	0.002165852578048	0.083037245735190	−0.294887487614237	0.547248315957191
0.8	0.021781926717524	0.062205442296095	−0.141897989102332	0.266394912683381
1	0.041791144330206	0.041791144330206	0.041791144330206	0.041791144330206

Also we can show this conclusion in Figure 2.

From the above Table 1 and Figure 2 the HPM approximate solutions with both defuzzification (19) and (20) satisfy the fuzzy numbers properties as in section 2 by taking the triangular fuzzy numbers shape.

Since Eq. (18) without exact analytical solution, to show the accuracy of 5-terms of HPM approximate series solution $\varphi_5(t;r) = \sum_{i=0}^{4} \tilde{y}_i(t;r)$ of Eq. (18) at $t = 0.5$ and $r \in [0,1]$, we need to define the residual error [20, 21] such that

$$[\tilde{E}]_r HPM = \left| \widetilde{\phi_5}'(t;r) - \left[\widetilde{\phi_5}(t;r)\right]^2 - t^2 \right|.$$

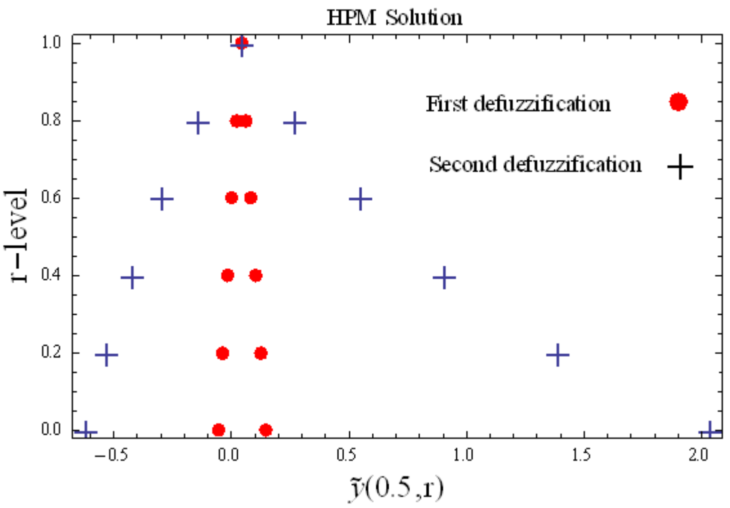

Figure 2: HPM approximate solution of Eq. (18) with two different defuzzification at $t = 0.5$ for all $r \in [0,1]$

Table 2: Accuracy of 5-order of HPM solution of Eq. (18) for all $r \in [0,1]$ at $t = 0.5$ of the first defuzzification (19)

r	$\left[\underline{E}\right]_r HPM$	$\left[\overline{E}\right]_r$ HPM
0	$1.6306788234166092 \times 10^{-6}$	$1.9942054678701020 \times 10^{-6}$
0.2	$1.0343067185503152 \times 10^{-6}$	$1.2797120836904874 \times 10^{-6}$
0.4	$6.3348643697391790 \times 10^{-7}$	$8.0912596364668410 \times 10^{-7}$
0.6	$3.5131604925187070 \times 10^{-7}$	$4.8716489495226330 \times 10^{-7}$
0.8	$1.3490857814213440 \times 10^{-7}$	$2.5042067092195810 \times 10^{-7}$
1	$5.4638608054657920 \times 10^{-8}$	$5.4638608054657920 \times 10^{-8}$

Table 3: Accuracy of 5-order of HPM solution of Eq. (18) for all $r \in [0,1]$ at $t = 0.5$ of the second defuzzification (20)

r	$\left[\underline{E}\right]_r HPM$	$\left[\overline{E}\right]_r$ HPM
0	0.14394838552076383	0.3483126888360939
0.2	0.03385026588328493	0.0684981297806737
0.4	0.00539263472742779	0.0092027480286891
0.6	0.00045757665914492	0.0006649884978752
0.8	0.00001390579453303	0.0000175424930059
1	$5.463860805465792 \times 10^{-8}$	$5.463860805465792 \times 10^{-8}$

From Tables 2 and 3 and Figure 2, we conclude that the small fuzzy region of $\left[\tilde{0}\right]_r$ in the first defuzzification (19) give us more accurate solution than the large region of this fuzzy numbers the in the second defuzzification (20).

5 Conclusions

In this study, HPM was applied to obtain an approximate solution for first order nonlinear FIVP. The accuracy of HPM can be determined for some nonlinear equations without an exact analytical solution. We conclude that the FIVP with small fuzzy number area yields a better solution than from the large area of the dfuzzifications of the initial conditions. We also conclude that the two dfuzzifications can give same solution when $r = 1$. A numerical example involving a first order fuzzy Riccati equation shows the efficiency of the HPM. The obtained results by HPM are satisfy the fuzzy number properties by taking the triangular fuzzy numbers shape.

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