Journal of Uncertain Systems
Vol.9, No.3, pp.215-220, 2015
Online at: www.jus.org.uk

Continuous Portfolio Selection Models under Uncertain Circumstances

Qi Zhu*

Department of Financial Engineering, Jiangsu Normal University, Xuzhou 221116, Jiangsu, China

Received 9 February 2015; Revised 24 May 2015

Abstract

This paper introduces two uncertain continuous portfolio selection models under the assumption that securities follow uncertain differential equations. The models are established to be consistent with the goals of investors based on expected value criterion. Solution methods for the models are proposed based on the α-pathes of uncertain differential equation and genetic algorithm. An example is presented to validate the approaches.

Keywords: portfolio selection, uncertainty theory, uncertain differential equation, α-path, genetic algorithm

1 Introduction

Portfolio selection problem is a classical problem in financial economics. The problem concerns on optimization of allocating a wealth to some securities. Since mean-variance theory was introduced by Markowitz in 1952 [18, 19], portfolio selection has been an interesting field of study in finance, and many relative literatures appeared, for example, [21, 10, 20, 6, 9, 25, 1, 5, 11]. In these literatures, there were two types of papers under the assumption that securities are assumed to have random returns: one is on discrete-time portfolio selection and the other is on continuous-time portfolio selection. In continuous-time models, a stock price is assumed to be determined by an Ito's stochastic differential equation.

However, the security market is complex. Liu [16] pointed out that the real stock price may be impossible to follow any Ito's stochastic differential equation. It was suggested that a new uncertain finance theory should be developed based on uncertainty theory and uncertain differential equation. The uncertainty theory was established by Liu [14] in 2007 and refined in 2010 [15]. In 2009, Liu [17] introduced a canonical process as a counterpart of a Wiener process and proposed uncertain differential equation driven by a canonical process. Some results on uncertain differential equations may be seen in literatures [2, 3, 4, 7, 8, 24]. Uncertain differential equation was first introduced into finance by Liu [17] in which an uncertain stock model was proposed and European option price formulas were documented.

Based on the uncertainty theory, an continuous-time uncertain portfolio selection model was studied by Zhu [26] in 2010. While an uncertain portfolio selection model was established by Huang [12] for a discrete case in 2011. In [26], a security is assumed to have an uncertain return which follows an uncertain differential equation, and the expected value of the return is maximized. In addition, Sheng and Zhu [22] dealt with an uncertain portfolio selection model based on optimistic value criterion.

In this paper, we will introduce two uncertain portfolio selection models which differ from the existing ones. In our proposed models, an investor's goal will be satisfied based on expected value criterion in that one is at the final time and the other is at the earliest time. The solution methods integrate α-path of uncertain differential equation for expressing expected values and genetic algorithm.

The structure of the paper is as follows: first, some concepts and results in uncertainty theory will be reviewed. Then, two uncertain portfolio selection models will be established. Next, the solution methods for the models will be presented. Finally, an example will be given to validate the proposed approaches.

*Corresponding author.
Email: zhuqi1@sina.com (Q. Zhu).

2 Preliminary

To begin with, we review some useful concepts and results in uncertainty theory founded by Liu [14]. Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event.

Definition 1 *Set function $\mathcal{M}$ from $\mathcal{L}$ to $[0,1]$ is called an uncertain measure if it satisfies three axioms: (normality) $\mathcal{M}\{\Gamma\} = 1$; (duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ; and (countable subadditivity) $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$.*

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [17]: Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$, where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

An uncertain variable ξ is defined by Liu [14] as a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set R of real numbers such that the set $\{\xi \in B\}$ is in $\mathcal{L}$ for any Borel set B. The uncertainty distribution $\Phi : R \to [0,1]$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for $x \in R$. The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\} dr$$

provided that at least one of the two integrals is finite. The variance of ξ is defined by $V[\xi] = E[(\xi - E[\xi])^2]$.

A normal uncertain variable with expected value e and variance σ^2 has the uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ x \in R$$

which is denoted by $\xi \sim \mathcal{N}(e, \sigma)$.

The uncertain variables $\xi_1, \xi_2, \ldots \xi_m$ are said to be independent [17] if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} (\xi_i \in B_i)\right\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots B_m$ of real numbers. For numbers a and b, $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ if ξ and η are independent uncertain variables.

Liu [13] defined uncertain process as a measurable function from $S \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers where S is a totally ordered set.

Definition 2 ([17]) *An uncertain process C_t is called a canonical process if it satisfies: (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, denoted by $C_{s+t} - C_s \sim \mathcal{N}(0, t)$.*

For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|$. Then the uncertain integral of an uncertain process X_t with respect to C_t is defined by Liu [17] as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. If there exist two uncertain processes μ_t and σ_t such that $Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$ for any $t \geq 0$, then we say Z_t has an uncertain differential $\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t$. An uncertain differential equation driven by a canonical process C_t is defined as

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

where f and g are two given functions. A solution X_t of the uncertain differential equation is equivalent to a solution of the uncertain integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s.$$

Definition 3 ([23]) *Let $0 < \alpha < 1$. An uncertain differential equation (1) is said to be have an α-path X_t^α if it solves the corresponding ordinary differential equation*

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t \tag{2}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Theorem 1 ([23]) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t,$$

respectively. Then the solution X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Furthermore, for any monotone (increasing or decreasing) function J, we have

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha.$$

3 Portfolio Selection Models

Portfolio selection problem studies allocating personal wealth between investment in a risk-free security and investment in a risk asset. Under the assumption that the risk asset earns a random return, Merton [20] studied a portfolio selection model by stochastic optimal control. If we assume that the risk asset earns an uncertain return, this Merton type of model may be solved by uncertain optimal control introduced by Zhu [26].

Let X_t be the wealth of an investor at time t. The investor allocates a fraction w of the wealth in a sure asset and remainder in a risk asset. The sure asset produces a rate of return b. The risk asset is assumed to earn an uncertain return, and yields a mean rate of return μ ($\mu > b$) along with a variance of σ^2 ($\sigma > 0$) per unit time. That is, the risk asset earns a return $\mathrm{d}r_t$ in time interval $(t, t+\mathrm{d}t)$, where $\mathrm{d}r_t = \mu\mathrm{d}t + \sigma\mathrm{d}C_t$, and C_t is a canonical process. Thus

$$\begin{aligned} X_{t+\mathrm{d}t} &= X_t + bwX_t\mathrm{d}t + \mathrm{d}r_t(1-w)X_t \\ &= X_t + bwX_t\mathrm{d}t + (\mu\mathrm{d}t + \sigma\mathrm{d}C_t)(1-w)X_t \\ &= X_t + [bw + \mu(1-w)]X_t\mathrm{d}t + \sigma(1-w)X_t\mathrm{d}C_t. \end{aligned}$$

That is

$$\mathrm{d}X_t = [bw + \mu(1-w)]X_t\mathrm{d}t + \sigma(1-w)X_t\mathrm{d}C_t. \tag{3}$$

The α-path of the uncertain differential equation (3) is

$$\mathrm{d}X_t^\alpha = [bw + \mu(1-w)]X_t^\alpha\mathrm{d}t + \sigma(1-w)X_t^\alpha\Phi^{-1}(\alpha)\mathrm{d}t \tag{4}$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

It follows that the solution of (4) is

$$X_t^\alpha = x_0\exp\{f(w,\alpha)t\} \tag{5}$$

where x_0 is the initial wealth of the investor and

$$f(w,\alpha) = \mu + (b-\mu)w + \sigma(1-w)\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Assume that an investor has the initial wealth x_0 and is interested in assigning a target rate of income p and minimizing the deviation of expected income from the target income at the final time T. Then a portfolio selection model is provided by

$$\begin{cases} \min\limits_{w\in[0,1]} \left|E\left[X_T\right]-x_0(1+p)\right| \\ \text{subject to} \\ \qquad \mathrm{d}X_t=[bwX_t+\mu(1-w)X_t]\mathrm{d}t+\sigma(1-w)X_t\mathrm{d}C_t. \end{cases} \tag{6}$$

Theorem 2 *The problem (6) is equivalent to the following optimization problem:*

$$\min_{w\in[0,1]} g(w)\equiv\left|\exp\{[\mu+(b-\mu)w]T\}\int_0^1\left(\frac{\alpha}{1-\alpha}\right)^{\frac{\sqrt{3}\,\sigma T}{\pi}(1-w)}\mathrm{d}\alpha-(1+p)\right|. \tag{7}$$

Proof: It follows from Theorem 1 and (5) that

$$\begin{aligned} E\left[X_T\right] &= \int_0^1 X_T^\alpha \mathrm{d}\alpha \\ &= x_0\exp\{[\mu+(b-\mu)w]T\}\int_0^1\left(\frac{\alpha}{1-\alpha}\right)^{\frac{\sqrt{3}\,\sigma T}{\pi}(1-w)}\mathrm{d}\alpha. \end{aligned}$$

Then

$$\min_{w\in[0,1]}|E[X_T]-x_0(1+p)|=\min_{w\in[0,1]}x_0\left|\exp\{[\mu+(b-\mu)w]T\}\int_0^1\left(\frac{\alpha}{1-\alpha}\right)^{\frac{\sqrt{3}\,\sigma T}{\pi}(1-w)}\mathrm{d}\alpha-(1-p)\right|$$

which is equivalent to the form (7). The theorem is proved.

If an investor hope to minimize the deviation at the earliest time, then another portfolio selection model is provided by

$$\begin{cases} \min\limits_{t}\min\limits_{w\in[0,1]} \left|E\left[X_t\right]-x_0(1+p)\right| \\ \text{subject to} \\ \qquad \mathrm{d}X_t=[bwX_t+\mu(1-w)X_t]\mathrm{d}t+\sigma(1-w)X_t\mathrm{d}C_t. \end{cases} \tag{8}$$

By the similar method to the proof of Theorem 2, we can get the following conclusion.

Theorem 3 *The problem (8) is equivalent to the following optimization problem:*

$$\min_{t}\min_{w\in[0,1]} h(w,t)\equiv\left|\exp\{[\mu+(b-\mu)w]t\}\int_0^1\left(\frac{\alpha}{1-\alpha}\right)^{\frac{\sqrt{3}\,\sigma t}{\pi}(1-w)}\mathrm{d}\alpha-(1+p)\right|. \tag{9}$$

Remark 1 It is seen that the optimal fraction of portfolio selection problem (6) and optimal fraction and earliest time of portfolio selection problem (8) are independent of the investor's wealth by Theorem 2 and Theorem 3, respectively.

4 Solution Methods

It can be seen from (7) and (9) that we need to calculate a type of improper integral:

$$\int_0^1\left(\frac{\alpha}{1-\alpha}\right)^{\tau}\mathrm{d}\alpha,\quad \tau\in(0,1).$$

This type of improper integral can be approximated by compound Simpson formula as follows:

$$\int_0^1 \left(\frac{\alpha}{1-\alpha}\right)^\tau \mathrm{d}\alpha \approx \frac{h}{6}\sum_{i=0}^{n-1}\left[\left(\frac{\alpha_i}{1-\alpha_i}\right)^\tau + 4\left(\frac{\alpha_i+h/2}{1-\alpha_i-h/2}\right)^\tau + \left(\frac{\alpha_{i+1}}{1-\alpha_{i+1}}\right)^\tau\right] \tag{10}$$

for sufficient large number n, and $h = (1-\varepsilon)/n$ with small $\varepsilon > 0$, $\alpha_i = i \times (1-\varepsilon)/n$ $(i = 0, 1, 2, \ldots, n)$.

Thus the solution of problem (7) can be approximatively derived from the solution of the following problem:

$$\begin{aligned}\min_{w\in[0,1]} g(w) \quad \equiv \quad & \frac{h}{6}\Bigg|\exp\{[\mu+(b-\mu)w]T\}\left\{\sum_{i=0}^{n-1}\left(\frac{\alpha_i}{1-\alpha_i}\right)^\tau + 4\left(\frac{\alpha_i+h/2}{1-\alpha_i-h/2}\right)^\tau\right. \\ & \left. + \left(\frac{\alpha_{i+1}}{1-\alpha_{i+1}}\right)^\tau\right\} - (1+p)\Bigg|\end{aligned} \tag{11}$$

where $\tau = \sqrt{3}\,\sigma T(1-w)/\pi$. And the solution of problem (9) can be approximatively derived from the solution of the following problem:

$$\begin{aligned}\min_t \min_{w\in[0,1]} h(w,t) \quad \equiv \quad & \frac{h}{6}\Bigg|\exp\{[\mu+(b-\mu)w]t\}\left\{\sum_{i=0}^{n-1}\left(\frac{\alpha_i}{1-\alpha_i}\right)^\tau\right. \\ & \left. +4\left(\frac{\alpha_i+h/2}{1-\alpha_i-h/2}\right)^\tau + \left(\frac{\alpha_{i+1}}{1-\alpha_{i+1}}\right)^\tau\right\} - (1+p)\Bigg|\end{aligned} \tag{12}$$

where $\tau = \sqrt{3}\,\sigma t(1-w)/\pi$.

Now we will employ the genetic algorithm (GA) to solve the problems (11) and (12).

5 Example

Assume that $b = 0.021$, $\mu = 0.035$, $\sigma = 0.06$, $T = 1$. An investor wishes to earn a profit 3% of his initial wealth, i.e., $p = 0.03$. Let $n = 128$ and $\varepsilon = 0.00001$. For portfolio selection problem (6), the optimal fraction is $w = 0.373$ derived from the solution of model (11) by GA. That is to say, 37.3% of the total wealth is invested for the sure asset and the remainder for the risk asset so that the investor achieves his goal at final time $T = 1$. For portfolio selection problem (8), the optimal fraction is $w = 0.0852$ and optimal time is $t = 0.8677$ derived from the solutions of model (12) by GA. That is to say, 8.52% of the total wealth is invested for the sure asset and the remainder for the risk asset so that the investor achieves his goal at time $t = 0.8677$.

6 Conclusions

Two uncertain portfolio selection models were established based on expected value criterion. The expected values in the models are expressed by the integrals of α-pathes of uncertain differential equations while the relative integrals are approximated by the compound Simpson formula. Genetic algorithm was employed to find the optimal solutions. A numerical example showed the efficiency of the proposed models and methods.

References

[1] Abdelaziz, F.B., Aouni, B., and R.E. Fayedh, Multi-objective stochastic programming for portfolio selection, *European Journal of Operational Research*, vol.177, pp.1811–1823, 2007.

[2] Bărbăcioru, I.C., Uncertainty functional differential equations for finance, *Surveys in Mathematics and Its Applications*, vol.5, pp.275–284, 2010.

[3] Chen, X., and D.A. Ralescu, Liu process and uncertain calculus, *Journal of Uncertainty Analysis and Applications*, vol.1, article 3, 2013.

[4] Chen, X., and B. Liu, Existence and uniqueness theorem for uncertain differential equations, *Fuzzy Optimization and Decision Making*, vol.9, no.1, pp.69–81, 2010.

[5] Corazza, M., and D. Favaretto, On the existence of solutions to the quadratic mixed-integer meanvariance portfolio selection problem, *European Journal of Operational Research*, vol.176, pp.1947–1960, 2007.

[6] Dumas, B., and E. Liucinao, An exact solution to a dynamic portfolio choice problem under transaction costs, *Journal of Finance*, vol.46, pp.577–595, 1991.

[7] Ge, X., and Y. Zhu, Existence and uniqueness theorem for uncertain delay differential equations, *Journal of Computational Information Systems*, vol.8, no.20, pp.8341–8347, 2012.

[8] Ge, X., and Y. Zhu, A necessary condition of optimality for uncertain optimal control problem, *Fuzzy Optimization and Decision Making*, vol.12, no.1, pp.41–51, 2013.

[9] Grauer, R.R., and N.H. Hakansson, On the use of mean-variance and quadratic approximations in implementing dynamic investment strategies: a comparison of returns and investment policies, *Management Science*, vol.39, pp.856–871, 1993.

[10] Hakansson, N.H., Multi-period mean-variance analysis: toward a general theory of portfolio choice, *Journal of Finance*, vol.26, pp.857–884, 1971.

[11] Hirschberger, M., Qi, Y., and R.E. Steuer, Randomly generatting portfolio-selection covariance matrices with specified distributional characteristics, *European Journal of Operational Research*, vol.177, pp.1610–1625, 2007.

[12] Huang, X., Mean-risk model for uncertain portfolio selection, *Fuzzy Optimization and Decision Making*, vol.10, no.1, pp.71–89, 2011.

[13] Liu, B., Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems*, vol.2, no.1, pp.3–16, 2008.

[14] Liu, B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[15] Liu, B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.

[16] Liu, B., Toward uncertain finance theory, *Journal of Uncertainty Analysis Applications*, vol.1, article 1, 2013.

[17] Liu, B., Some research problems in uncertainy theory, *Journal of Uncertain Systems*, vol.3, no.1, pp.3–10, 2009.

[18] Markowitz, H., Portfolio selection, *Journal of Finance*, vol.7, pp.77–91, 1952.

[19] Markowitz, H., *Portfolio Selection: Efficient Diversification of Investments*, Wiley, New York, 1959.

[20] Merton, R.C., Optimal consumption and portfolio rules in a continuous time model, *Journal of Economic Theory*, vol.3, pp.373–413, 1971.

[21] Mossin, J., Optimal multiperiod portfolio policies, *Journal of Business*, vol.41, pp.215–229, 1968.

[22] Sheng, L., and Y. Zhu, Optimistic value model of uncertain optimal control, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.21, suppl.1, pp.75–83, 2013.

[23] Yao, K., and X. Chen, A numerical method for solving uncertain differential equations, *Journal of Intelligent & Fuzzy Systems*, vol.25, no.3, pp.825–832, 2013.

[24] Yao, K., Gao, J., and Y. Gao, Some stability theorems of uncertain differential equation, *Fuzzy Optimization Decision Making*, vol.12, no.1, pp.3–13, 2013.

[25] Zhou, X., and D. Li, Continuous-time mean-variance portfolio selection: a stochastic LQ framework, *Applied Mathematics and Optimization*, vol.42, pp.19–33, 2000.

[26] Zhu, Y., Uncertain optimal control with application to a portfolio selection model, *Cybernetics and Systems: An International Journal*, vol.41, no.7, pp.535–547, 2010.