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Nexus over an Ordinal

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Abstract

In this paper the notion the nexuses over an ordinal is defined and some related results are investigated. In particular, all prime and maximal subnexuses of a nexus over an ordinal are characterized. Furthermore, the notion of the fractions of a nexus N over an ordinal is introduced and finally, we show that the fraction $S^{-1}N$ is a bounded distributive lattice which has only one maximal ideal, where S is a meet closed subset of the nexus N over an ordinal.

Keywords: nexus, ordinal, prime subnexus, maximal subnexus, fraction of a nexus

1 Introduction

The basic idea of a nexus has been further developed as a mathematical object for general use (see [1, 2, 3, 13]). The aim of this recent study is to evolve a mathematical object that allows complex processes on groups of mathematical objects to be formulated with ease of elegance. This notion is very useful for the study of space structures (see [7, 8, 9, 10, 11, 12]). In this paper the notion nexus over an ordinal is defined and some related results are obtained. Nexus over an ordinal is generalized of nexus.

This paper is structured as follows. After the introduction, in section 2, we recall some basic notions and results on ordinals and inf semilattices. In Section 3, the notion of the nexuses over an ordinal is introduced and we have studied order on a nexus over an ordinal. In Section 4, the notion of the prime subnexuses of a nexus over an ordinal is introduced. Next some important properties of prime subnexuses of a nexus over an ordinal will be studied. In Section 5, maximal subnexuses of a nexus over an ordinal will be studied. Also by an example we show that Theorem 1.14 in [1] and Theorem 3.6 in [13], is incorrect (see Example 5.6) and a corrected version of the Theorem 1.14 in [1] and the Theorem 3.6 in [13], is considered Proposition 5.7. Finally, in Section 6, the notion of the fractions of a nexus over an ordinal is introduced. Next some important properties of the fractions of a nexus over an ordinal will be studied. Also by an example we show that Theorem 2.26 (i) in [1] is incorrect (see Example 6.14) and a corrected version of the Theorem 2.26 (i) in [1] is considered Proposition 6.15. The continuation of this article can be followed in [5].

2 Preliminaries

An ordered set A that for every $x, y \in A$ either $x \leq y$ or $y \leq x$ is said to be *linearly ordered* or *totally ordered*. An ordered set A is said to be *well ordered* if and only if whenever B is any nonempty subset of A, then B has a least element. Every well ordered set is linearly ordered.

Let $(X, \leq)$ be a well ordered set, $a \in X$. By the segment X_a of X determined by a we mean the set $X_a = \{x \in X | x < a\}$. An *ordinal number* is a well ordered set α where for all $x \in \alpha$, $\alpha_x = x$. The collection of all ordinal numbers is a proper class and we denote by $\mathfrak{O}$. Let α be an ordinal. If $a \in \alpha$, then α_a is an ordinal. Also, If $Y \subseteq \alpha$ is an ordinal, then $Y = \alpha_a$, for some $a \in \alpha$. If α, β are ordinals, then $\alpha \cap \beta$ is an ordinal. Every well ordered set is isomorphic to a unique ordinal.

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It is common in contemporary set theory to reserve lower-case Greek letters $\alpha, \beta, \dots$ to denote ordinals. It is also customary to denote the order relation between ordinals by $\alpha < \beta$ instead of the two equivalent forms $\alpha \subset \beta$, $\alpha \in \beta$, though the latter is also quite common. If α is an ordinal, then by definition we will have $\alpha = \{\beta \in \mathfrak{O} | \beta < \alpha\}$. That is, an ordinal is the set of all smaller ordinals. In general, if α is an ordinal, the next ordinal will be $\alpha \cup \{\alpha\}$. It is customary to denote the first ordinal after α by $\alpha + 1$, the (*ordinal*) *successor* of α. Thus $\alpha + 1 = \alpha \cup \{\alpha\}$. If $\beta = \alpha + 1$, then we define $\beta - 1 = \alpha$. An ordinal number greater than 0 that is not the successor of any other ordinal number is said to be a *limit ordinal*. An ordinal that is the successor of some ordinal is called a *successor ordinal* or *non-limit ordinal*.

If $\alpha, \beta \in \mathfrak{O}$, then either $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$. If A is a set of ordinals, then $\bigcup A$ is an ordinal.

An inf semilattice is a poset S in which any two elements a, b have an inf, denoted by $a \wedge b$ or simply by ab. Equivalently, an inf semilattice is a poset in which every nonempty finite subset has an inf.

An inf semilattice homomorphism is a function $f : N \to M$ between inf semilattices N and M such that $f(x \wedge y) = f(x) \wedge f(y)$ for all x and y in N. Each inf semilattice homomorphism is isotone, that is, $x \leq y$ implies that $f(x) \leq f(y)$.

Let A be a poset. For $X \subseteq A$ and $x \in A$ we write:

1. $\downarrow X = \{a \in A : a \leq x \text{ for some } x \in X\}$.
2. $\uparrow X = \{a \in A : a \geq x \text{ for some } x \in X\}$.
3. $\downarrow x = \downarrow \{x\}$.
4. $\uparrow x = \uparrow \{x\}$.

For undefined terms and notations, see [4, 6].

3 Order on Nexus over an Ordinal

In this section the notion of nexus over an ordinal is defined and we have studied order on a nexus over an ordinal.

Definition 3.1. Let $\gamma, \delta \in \mathfrak{O}$, $\gamma \geq 1$ and $\delta \geq 1$. An *address* over γ is a function $a : \delta \to \gamma$, such that $a(i) = 0$ implies that $a(j) = 0$ for all $j \geq i$. We denote by $A(\gamma)$, the set of all address over γ.

Let $a : \delta \to \gamma$ be an address over γ. If for every $i \in \delta$, $a(i) = 0$, then it is called the *empty address* and denoted by $()$. If a is a nonempty address, then there exists a unique element $\beta \in \delta + 1$ such that for every $i \in \beta$, $a(i) \neq 0$ and for every $\beta \leq i \in \delta$, $a(i) = 0$. We denote this address by $(a_i)_{i \in \beta}$, where $a_i = a(i)$ for every $i \in \beta$.

Let $a : \delta \to \gamma$ and $b : \beta \to \eta$ be addresses and $\delta \leq \beta$. We say $a = b$, if for every $i \in \delta$, $a_i = b_i$ and for every $i \in \beta \setminus \delta$, $b_i = 0$. In other words, there exists a unique element $\beta \in \mathfrak{O}$ such that $a = (a_i)_{i \in \beta} = b$.

Definition 3.2. The *level* of $a \in A(\gamma)$ is said to be:

1. 0 , if $a = ()$.
2. β, if $() \neq a = (a_i)_{i \in \beta}$.

The level of a is denoted by $l(a)$.

Definition 3.3. Let a, b be two elements of $A(\gamma)$. Then we say $a \leq b$ if $l(a) = 0$ or one of the following cases satisfies for $a = (a_i)_{i \in \beta}$ and $b = (b_i)_{i \in \delta}$:

1. If $\beta = 1$, then $a_0 \leq b_0$.
2. If $\beta \geq 2$ is a non-limit ordinal, then $a|_{\beta-1} = b|_{\beta-1}$ and $a_{\beta-1} \leq b_{\beta-1}$.
3. If β is a limit ordinal, then $a = b|_{\beta}$.

Proposition 3.4. *$(A(\gamma), \leq)$ is an inf semilattice.*

Proof. Clearly $\leq$ is reflexive. Let $a, b \in A(\gamma)$, $a \leq b$ and $b \leq a$. Thus $l(a) \leq l(b)$ and $l(b) \leq l(a)$ and this implies that $l(a) = l(b) = \beta$, for some $\beta \in \mathfrak{O}$. Now, suppose that $a = (a_i)_{i\in\beta}$, $b = (b_i)_{i\in\beta}$ and $\beta \neq 0$. If β is a limit ordinal, then $a = b|_\beta$ and $a|_\beta = b$, which follows that $a = b$. If β is a non-limit ordinal, then $a|_{\beta-1} = b|_{\beta-1}$, $a_{\beta-1} \leq b_{\beta-1}$ and $b_{\beta-1} \leq a_{\beta-1}$, which follows that $a = b$. Thus $\leq$ on $A(\gamma)$ is antisymmetric.

Now, let $a \leq b$ and $b \leq c$. If $l(a) = 0$, it is clear that $a \leq c$. Let $l(a) = \beta \neq 0$, $a = (a_i)_{i\in\beta}$, $b = (b_i)_{i\in\delta}$, and $c = (c_i)_{i\in\eta}$. Then $\beta \leq \delta \leq \eta$.

Case 1: Let $\beta = 1$. Then $\delta \geq 1$ and $a_0 \leq b_0$. If $\delta = 1$, then $b_0 \leq c_0$. Hence $a_0 \leq c_0$, that is, $a \leq c$. If $\delta > 1$, then there exists $1 \leq \sigma \in \delta$ such that $b|_\sigma = c|_\sigma$, which follows that $a_0 \leq b_0 = c_0$, that is, $a \leq c$.

Case 2: If $1 < \beta$, δ are limit ordinals, then $a = b|_\beta$ and $b = c|_\delta$, which follows that $a = c|_\beta$. Hence $a \leq c$.

Case 3: Assume that $1 < \beta$ and δ are limit and non limit ordinal, respectively. Then $\beta \leq \delta - 1$, $a = b|_\beta$, $b|_{\delta-1} = c|_{\delta-1}$ and $b_{\delta-1} \leq c_{\delta-1}$. Hence $a = c|_\beta$, which follows that $a \leq c$.

Case 4: Assume that $1 < \beta$ and δ are non limit and limit ordinal, respectively. Then $\beta + 1 \leq \delta$, $a|_{\beta-1} = b|_{\beta-1}$, $a_{\beta-1} \leq b_{\beta-1}$ and $b = c|_\delta$. Hence $a|_{\beta-1} = c|_{\beta-1}$ and $a_{\beta-1} \leq b_{\beta-1} = c_{\beta-1}$, which follows that $a \leq c$.

Case 5: If $1 < \beta$, δ are non limit ordinals, then $a|_{\beta-1} = b|_{\beta-1}$, $a_{\beta-1} \leq b_{\beta-1}$, $b|_{\delta-1} = c|_{\delta-1}$, and $b_{\delta-1} \leq c_{\delta-1}$. Hence $a|_{\beta-1} = c|_{\beta-1}$ and $a_{\beta-1} \leq b_{\beta-1} \leq c_{\beta-1}$, which follows that $a \leq c$.

Therefore, $\leq$ on $A(\gamma)$ is transitive.

Let $a = (a_i)_{i\in\beta}$, $b = (b_i)_{i\in\delta}$ be two elements of $A(\gamma)$. If $a = ()$ or $b = ()$, then $a \wedge b = ()$. Now, suppose that $a \neq ()$ and $b \neq ()$. We put $T = \{i \in \beta \wedge \delta : a_i \neq b_i\}$. Let η be the smallest ordinal such that $a_\eta \neq b_\eta$. We define $c : \eta + 1 \to \gamma$ by

$$c_i = \begin{cases} a_i & \text{if } i \in \eta \\ a_\eta \wedge b_\eta & \text{if } i = \eta. \end{cases}$$

By some manipulations we can see $a \wedge b = c$. □

Corollary 3.5. *Let $a, b \in A(\gamma)$. Then $a \wedge b = ()$ if and only if $a = ()$ or $b = ()$.*

Proof. By the end of the proof of Proposition 3.4, it is evident. □

Proposition 3.6. *$(A(\gamma), \leq)$ is a lattice if and only if $\gamma = 1$ or $\gamma = 2$.*

Proof. If $\gamma = 1$, then $A(\gamma) = \{()\}$, which follows that $(A(\gamma), \leq)$ is a lattice. Let $\gamma = 2$. Then $() \neq (a_i)_{i\in\beta} \in A(\gamma)$ if and only if for every $i \in \beta$, $a_i = 1$. Therefore, if a, b are two nonempty elements of $A(\gamma)$, then $a \leq b$ or $b \leq a$, which follows that $(A(\gamma), \leq)$ is a lattice.

Let $\gamma \geq 3$, $a = (2, 2)$ and $b = (1, 1)$. If $c \in A(\gamma)$ such that $a \leq c$ and $b \leq c$, then $a_1 = c_1 = b_1$, which is a contradiction. Therefore, $(A(\gamma), \leq)$ is a lattice if and only if $\gamma = 1$ or $\gamma = 2$. □

Let $() \neq a = (a_i)_{i\in\beta}$ be an element of $A(\gamma)$. For every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, we put $a^{(\delta,j)} : \delta + 1 \to \gamma$ such that for every $i \in \delta + 1$,

$$a_i^{(\delta,j)} = \begin{cases} a_i & \text{if } i \in \delta \\ j & \text{if } i = \delta. \end{cases}$$

Definition 3.7. A *nexus* N over γ is a set of addresses with the following properties:

1. $\emptyset \neq N \subseteq A(\gamma)$.
2. If $() \neq a = (a_i)_{i\in\beta} \in N$, then for every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, $a^{(\delta,j)} \in N$.

Proposition 3.8. *Let N be the set of addresses over γ. Then N is a nexus over γ if and only if $\emptyset \neq N \subseteq A(\gamma)$ and for every $(a, b) \in N \times A(\gamma)$, $b \leq a$ implies that $b \in N$.*

Proof. It is evident. □

Definition 3.9. Let N be a nexus over γ and $\emptyset \neq M \subseteq N$. M is called a *subnexus* of N, if M itself is a nexus over γ. The set of all subnexuses of N is denoted by $Sub(N)$. It is clear that $\{()\}$, N are trivial subnexuses of the nexus N.

Proposition 3.10. *If N is a nexus over γ and $\{M_i\}_{i\in I} \subseteq Sub(N)$, then $\bigcup_{i\in I} M_i \in Sub(N)$ and $\bigcap_{i\in I} M_i \in Sub(N)$.*

Proof. It is evident. □

Definition 3.11. Let N be a nexus over γ and $X \subseteq N$. The smallest subnexus of N containing X is called the *subnexus of N generated by X* and denoted by $< X >$. If $|X| = 1$, then $< X >$ is called a cyclic subnexus of N. It is clear that $< \emptyset >= \{()\}$ and $< N >= N$.

Remark 3.12. Let $\emptyset \neq N \subseteq A(\gamma)$. N is a nexus over γ if and only if

$$N =\downarrow N = \bigcup_{a \in N} \downarrow a.$$

Proposition 3.13. *Let N be a nexus over γ and $\emptyset \neq X \subseteq N$. Then*

$$< X >=\downarrow X = \bigcup_{x \in X} \downarrow x.$$

Proof. By Proposition 3.8, $\bigcup_{x\in X} \downarrow x$ is a subnexus of N and $X \subseteq \bigcup_{x\in X} \downarrow x$. Hence $< X >\subseteq \bigcup_{x\in X} \downarrow x$. On the other hand, if M is a subnexus of N such that $X \subseteq M$, then by Proposition 3.8, $\bigcup_{x\in X} \downarrow x \subseteq M$. So $< X >= \bigcup_{x\in X} \downarrow x$. □

Proposition 3.14. *Let N be a nexus over γ. $(N, \leq)$ is an inf semilattice.*

Proof. Argument similar to the proof of Proposition 3.4. □

Proposition 3.15. *If N is a cyclic nexus over γ, then every two elements of N are comparable and so $(N, \leq)$ is a bounded distribiutive lattice.*

Proof. Let $N =< a >$ and $a = (a_i)_{i\in\beta}$. By Proposition 3.13, $N =\downarrow a$. Let $() \neq b = (b_i)_{i\in\delta}$ and $() \neq c = (c_i)_{i\in\eta}$ be two elements of N. Without loss of generality, suppose that $\delta \leq \eta$. By the definition of $\leq$ on nexus, we have $\delta \leq \eta \leq \beta$ and:

Case 1: If $\eta = 1$, then $\delta = 1$. Since $b_0, c_0 \in \mathfrak{D}$, we conclude that $b_0 \leq c_0$ or $c_0 \leq b_0$, it follwos that $b \leq c$ or $c \leq b$.

Case 2: Let $\delta = 1 < \eta$. Then $b_0 \leq a_0$ and there exists $1 \leq \sigma \in \eta$ such that $c|_\sigma = a|_\sigma$. Hence $b_0 \leq c_0$, which follows that $b \leq c$.

Case 3: If $1 < \delta$, η are limit ordinals, then $b = a|_\delta$ and $c = a|_\eta$. So $b = c|_\delta$ and $b \leq c$.

Case 4: If $1 < \delta$ and $1 < \eta$ are non limit ordinals, then $b|_{\delta-1} = a|_{\delta-1}$, $b_{\delta-1} \leq a_{\delta-1}$, $c|_{\eta-1} = a|_{\eta-1}$ and $c_{\eta-1} \leq a_{\eta-1}$. Hence $b|_{\delta-1} = c|_{\delta-1}$ and $b_{\delta-1} \leq c_{\delta-1}$, that is, $b \leq c$.

Case 5: Assume that $1 < \delta$ and $1 < \eta$ are limit and non limit ordinal, respectively. Then $b = a|_\delta$, $c|_{\eta-1} = a|_{\eta-1}$ and $c_{\eta-1} \leq a_{\eta-1}$. Since $\delta \leq \eta - 1$, we conclude that $b = c|_\delta$, that is, $b \leq c$.

Case 6: Assume that $1 < \delta$ and $1 < \eta$ are non limit and limit ordinal, respectively. Then $b|_{\delta-1} = a|_{\delta-1}$, $b_{\delta-1} \leq a_{\delta-1}$ and $c = a|_\eta$. Hence $b|_{\delta-1} = c|_{\delta-1}$, $b_{\delta-1} \leq c_{\delta-1}$, that is, $b \leq c$.

Therefore, every two elements of N are comparable and so, $(N, \leq)$ is a bounded distribiutive lattice. □

Let T and S be two nonempty subsets of a nexus N over γ. We define the set $T \wedge S = \{t \wedge s |\ t \in T \text{ and } s \in S\}$.

Proposition 3.16. *Let N be a nexus over γ and $a, b \in N$.*

(1) $< a > \wedge < b >=< a \wedge b >$.

(2) *If a and b are not comparable addresses, then $\uparrow a \cap \uparrow b = \emptyset$.*

(3) *If a and b are not comparable addresses, then $\uparrow a \wedge \uparrow b = \{a \wedge b\}$.*

Proof. (1) Since $\downarrow a \wedge \downarrow b =\downarrow (a \wedge b)$, we have $< a > \wedge < b >=< a \wedge b >$.

(2) If $c \in\uparrow a \cap \uparrow b$, then $a, b \in< c >$ and by Proposition 3.15, $a \leq b$ and $b \leq a$, which is a contradiction.

(3) It is clear that $() \neq a = (a_i)_{i\in\beta}$ and $() \neq b = (b_i)_{i\in\delta}$. We put $T = \{i \in \beta \wedge \delta : a_i \neq b_i\}$. Let η be the smallest ordinal such that $a_\eta \neq b_\eta$. We define $c : \eta + 1 \to \gamma$ with

$$c_i = \begin{cases} a_i & \text{if } i \in \eta \\ a_\eta \wedge b_\eta & \text{if } i = \eta. \end{cases}$$

Hence $a \wedge b = c$. It is clear that $a \wedge b \in\uparrow a \wedge \uparrow b$. Now, suppose that $\beta \leq \delta$ and $d \in\uparrow a \wedge \uparrow b$. Then there exists $x = (x_i)_{i\in\mu} \geq a$ and $y = (y_i)_{i\in\sigma} \geq b$ such that $d = x \wedge y$.

Case 1: If β and δ are limit ordinals, then $a = x|_\beta$ and $b = y|_\delta$, which follows that η is the smallest ordinal such that $x_\eta \neq y_\eta$. Since $\eta \in \beta \cap \delta$, we conclude that $x_\eta \wedge y_\eta = a_\eta \wedge b_\eta$. Therefore, $d = x \wedge y = a \wedge b$.

Case 2: Assume that β and δ are limit and non-limit ordinal, respectively. Hence $a = x|_\beta$, $b|_{\delta-1} = y|_{\delta-1}$ and $b_{\delta-1} \leq y_{\delta-1}$. Also, η is the smallest ordinal such that $x_\eta \neq y_\eta$. Since β is a limit ordinal and $\eta \in \beta \leq \delta$, we conclude that $\eta \in \delta - 1$, which follows that $x_\eta \wedge y_\eta = a_\eta \wedge b_\eta$. Therefore, $d = x \wedge y = a \wedge b$.

Case 3: Assume that β and δ are non-limit and limit ordinal, respectively. Hence $a|_{\beta-1} = x|_{\beta-1}$, $a_{\beta-1} \leq x_{\beta-1}$ and $b = y|_\delta$. If $\eta < \beta - 1$, then since η is the smallest ordinal such that $x_\eta \neq y_\eta$, we conclude that $x_\eta \wedge y_\eta = a_\eta \wedge b_\eta$. Therefore, $d = x \wedge y = a \wedge b$.

Now, assume that $\eta = \beta - 1$. If $a_\eta \leq b_\eta$, then $a \leq b$, which is a contradiction. Hence $y_\eta = b_\eta \leq a_\eta \leq x_\eta$ and we infer that $x_\eta \wedge y_\eta = y_\eta = b_\eta = a_\eta \wedge b_\eta$. Therefore, $d = x \wedge y = a \wedge b$.

Case 4: Assume that β and δ are non-limits. Hence $a|_{\beta-1} = x|_{\beta-1}$, $a_{\beta-1} \leq x_{\beta-1}$, $b|_{\delta-1} = y|_{\delta-1}$ and $b_{\delta-1} \leq y_{\delta-1}$. If $\eta < \beta - 1$, then since η is the smallest ordinal such that $x_\eta \neq y_\eta$, we conclude that $x_\eta \wedge y_\eta = a_\eta \wedge b_\eta$. Therefore, $d = x \wedge y = a \wedge b$.

Now, assume that $\eta = \beta - 1$. If $a_\eta \leq b_\eta$, then $a \leq b$, which is a contradiction. Hence $b_\eta \leq a_\eta$. If $\beta = \delta$, then $b \leq a$, which is a contradiction. So $\beta < \delta$ and we infer that $y_\eta = b_\eta \leq a_\eta \leq x_\eta$. Hence, $x_\eta \wedge y_\eta = y_\eta = b_\eta = a_\eta \wedge b_\eta$. Therefore, $d = x \wedge y = a \wedge b$. □

Definition 3.17. Let $a = (a_i)_{i\in\beta}$ be an address of nexus N over γ. The set $\{b \in N |\ a = b|_\beta \text{ and } a \neq b\}$ is called the *remus* of a and is denoted by r_a. Let S be a non-empty subset of N, then $r_S = \bigcup_{a\in S} r_a$.

Definition 3.18. A subset X of a nexus N over γ is called *closed under finite meet operation*, if $a \wedge b \in S$, for all $a, b \in S$.

Proposition 3.19. *Let N be a nexus over γ and $a = (a_i)_{i\in\beta} \in N$. Then*

(1) $r_{()} = N \setminus \{()\}$.

(2) $r_a \subset\uparrow a$.

(3) r_a *is closed under finite meet operation.*

(4) $r_a = \emptyset$ *if and only if a is maximal element of N if and only if $\uparrow a = \{a\}$.*

(5) *If $r_a \neq \emptyset$ and $b : \beta + 1 \to \gamma$ with*

$$b_i = \begin{cases} a_i & \text{if } i \in \beta \\ 1 & \text{if } i = \beta, \end{cases}$$

then $\bigwedge r_a = b \in r_a$.

(6) *If for every $k \in \beta \setminus \{0\}$, $a^{(k)} : \beta + 1 \to \gamma$ with*

$$a^{(k)}(i) = \begin{cases} a_i & \text{if } i \in \beta \\ k & \text{if } i = \beta, \end{cases}$$

then $r_{r_a} = r_a \setminus \{a^{(k)} | k \in \beta \setminus \{0\}\}$.

Proof. By definition of r_a, the proof is trivial. □

Proposition 3.20. *Let N be a nexus over γ and $a, b \in N$. If $a < b$ and $l(a) = l(b)$, then*

(1) $r_a \cap r_b = \emptyset$.

(2) $r_a \wedge r_b = \{a\}$.

Proof. Suppose that $a = (a_i)_{i\in\beta}$ and $b = (b_i)_{i\in\beta}$. If β is limit ordinal, then $a = b|_\beta = b$, which is a contradiction. Hence β is non-limit ordinal, $a|_{\beta-1} = b|_{\beta-1}$ and $a_{\beta-1} < b_{\beta-1}$.

(1) If $d \in r_a \cap r_b$, then $a = d|_\beta = b$, which is a contradiction.

(2) Let $d \in r_a \wedge r_b$. Then there exists $x \in r_a$ and $y \in r_b$ such that $d = x \wedge y$. Hence $a = x|_\beta$ and $b = y|_\beta$. So $\beta - 1$ is the smallest ordinal such that $x_{\beta-1} \neq y_{\beta-1}$. Since $x_{\beta-1} \wedge y_{\beta-1} = a_{\beta-1}$, we conclude that $d = x \wedge y = a$. Therefore, $r_a \wedge r_b \subseteq \{a\}$. If $x = (x_i)_{i\in\beta+1}$ and $y = (y_i)_{i\in\beta+1}$ such that $a = x|_\beta$, $b = y|_\beta$ and $x_\beta = 1 = y_\beta$, then $x \in r_a$, $y \in r_b$ and $a = x \wedge y \in r_a \wedge r_b$. Therefore, $r_a \wedge r_b = \{a\}$. □

Proposition 3.21. *Let N be a nexus over γ and $a \in N$. If $r_a \neq \emptyset$ and $f : N \to N$ with*

$$f(b) = \begin{cases} a & \text{if } b \in r_a \\ b & \text{if } b \notin r_a, \end{cases}$$

then f is a homomorphism.

Proof. We show that $f(x \wedge y) = f(x) \wedge f(y)$, for every $x, y \in N$. If $a = ()$, then by Proposition 3.20, $r_{()} = N \setminus \{()\}$. It is clear that $f(x \wedge y) = f(x) \wedge f(y)$, for every $x, y \in N$. Let $() \neq a = (a_i)_{i\in\beta}$. If $x, y \notin r_a$, then $x \wedge y \notin r_a$, which follows that $f(x \wedge y) = x \wedge y = f(x) \wedge f(y)$. If $x, y \in r_a$, then $x \wedge y \in r_a$, which follows that $f(x \wedge y) = a = f(x) \wedge f(y)$. Let $x = (x_i)_{i\in\delta} \in r_a$ and $y = (y_i)_{i\in\eta} \notin r_a$. Hence $a = x|_\beta$, $a < x$, and $y = a$ or $a \neq y|_\beta$.

Case 1: If $y = a$, then $x \wedge y = y \notin r_a$ and it implies that $f(x \wedge y) = b = f(x) \wedge f(y)$.

Case 2: Let $a \neq y|_\beta$ and t be the least element of $\{i \in \beta : y_i \neq a_i\}$. If $t \in \beta \setminus \eta$, then $y_t = 0$ and $x \wedge y = y \notin r_a$. Hence $f(x \wedge y) = b = f(x) \wedge f(y)$. Let $t \in \beta \cap \eta$ and $c : t + 1 \to \gamma$ with

$$c_i = \begin{cases} x_i & \text{if } i \in t \\ x_i \wedge y_i & \text{if } i = t. \end{cases}$$

then $x \wedge y = c$. If $c \in r_a$, then $\beta < t + 1$, which is a contradiction. Hence $x \wedge y \notin r_a$ and $f(x \wedge y) = b = f(x) \wedge f(y)$. □

Function of Proposition 3.21 is called pruning and denoted by f_{r_a}.

Proposition 3.22. *Let N be a nexus over γ and $a \in N$. If $f : N \to N$ with*

$$f(b) = \begin{cases} a & \text{if } b \in \uparrow a \\ b & \text{if } b \notin \uparrow a, \end{cases}$$

then f is a homomorphism.

Proof. It is trivial. □

The function introduced in Proposition 3.22 is usually denoted by $f_{\uparrow a}$.

4 Prime Subnexuses of a Nexus over an Ordinal

In this section the notion of prime subnexus of a nexus is defined and all prime subnexuses of a nexus over an ordinal are characterized.

Definition 4.1. A proper subnexus P of a nexus N over γ is said to be a *prime subnexus* of N if $a \wedge b \in P$ implies that $a \in P$ or $b \in P$, for any $a, b \in N$.

If $N \neq \{()\}$, then by Corollary 3.5, $\{()\}$ is a prime subnexus of N.

Lemma 4.2. *Let M and T are subnexuses of a nexus N over γ.*

(1) *$M \wedge T$ is a subnexus of N.*

(2) *If for some $a, b \in N$, $a \wedge b \in M$, then $< M \cup \{a\} > \wedge < M \cup \{b\} >= M$.*

Proof. (1) Since by Remark 3.12, $M \wedge T = (\bigcup_{m\in M} \downarrow m) \wedge (\bigcup_{t\in T} \downarrow t) = \bigcup_{(m,t)\in M\times T} \downarrow (m \wedge t)$, we conclude from the Remark 3.12 that $M \wedge T$ is a subnexus of N.

(2) Since by Proposition 3.8, $\downarrow (a \wedge b) \subseteq M$, we conclude from the Proposition 3.13 that $< M \cup \{a\} > \wedge < M \cup \{b\} >= (\bigcup_{x\in M\cup\{a\}} \downarrow x) \wedge (\bigcup_{y\in M\cup\{b\}} \downarrow y) = (\bigcup_{m\in M} \downarrow m) \cup \downarrow (a \wedge b) = \bigcup_{m\in M} \downarrow m = M$. □

Proposition 4.3. *Let P be a proper subnexus of N over γ. Then the following are equivalent:*

(1) *P is a prime subnexus of N.*

(2) *$K_1 \wedge K_2 \subseteq P$ implies that $K_1 \subseteq P$ or $K_2 \subseteq P$, for any subnexuses K_1 and K_2 of N.*

(3) *$< a > \wedge < b >\subseteq P$ implies that $a \in P$ or $b \in P$, for any $a, b \in N$.*

(4) *$K_1 \wedge K_2 = P$ implies that $K_1 = P$ or $K_2 = P$, for any subnexuses K_1 and K_2 of N.*

Proof. (1) $\Rightarrow$ (2) Let for some subnexuses K_1 and K_2 of N, $K_1 \wedge K_2 \subseteq P$, $K_1 \nsubseteq P$ and $K_2 \nsubseteq P$. Then there exists $a \in K_1 \setminus P$ and $b \in K_2 \setminus P$. Since $a \wedge b \in K_1 \wedge K_2 \subseteq P$ and P is a prime subnexus of N, we conclude that $a \in P$ or $b \in P$, and this is a contradiction.

(2) $\Rightarrow$ (3) It is evident.

(3) $\Rightarrow$ (4) Let for some subnexuses K_1 and K_2 of N, $K_1 \wedge K_2 = P$, $K_1 \neq P$ and $K_2 \neq P$. If $p \in P$, then $p = a \wedge b$, for some $(a, b) \in K_1 \times K_2$. Hence $p \leq a$ and $p \leq b$, which follows that $P \subseteq K_1 \cap K_2$. Therefore, there exists $a \in K_1 \setminus P$ and $b \in K_2 \setminus P$. Since $< a > \wedge < b > \subseteq K_1 \wedge K_2 = P$, we conclude that $a \in P$ or $b \in P$, which is a contradiction.

(4) $\Rightarrow$ (1) Suppose that $a \wedge b \in P$, for some $a, b \in N$. By Lemma 4.2, $< P \cup \{a\} > \wedge < P \cup \{b\} >= P$. Hence $< P \cup \{a\} >= P$ or $< P \cup \{b\} >= P$, which follows that $a \in P$ or $b \in P$. Therefore, P is a prime subnexus of N. $\square$

Proposition 4.4. *Let P be a proper subnexus of a nexus N over γ. Then P is a prime subnexus of N if and only if $N \setminus P$ is closed under finite meet operation.*

Proof. Let P be a prime subnexus of N and $a, b \in N \setminus P$. If $a \wedge b \notin N \setminus P$, then since P is a prime subnexus of N, we conclude that $a \notin N \setminus P$ or $b \notin N \setminus P$, which is a contradiction. Hence $N \setminus P$ is closed under finite meet operation.

Let $N \setminus P$ be closed under finite meet operation and $a \wedge b \in P$. Hence $a \notin N \setminus P$ or $b \notin N \setminus P$, which follows that P is a prime subnexus of N. $\square$

Lemma 4.5. *Let N be a nexus over γ. For every $\emptyset \neq X \subseteq N$, $N \setminus \uparrow X$ and $N \setminus r_X$ are subnexuses of N.*

Proof. Let $a \in N \setminus \uparrow X$, $b \in N$ and $b \leq a$. If $b \in \uparrow X$, then there exists $x \in X$ such that $x \leq b$. By Proposition 3.14, $x \leq a$, that is, $a \in \uparrow X$, which is a contradiction. Therefore, by Proposition 3.8, $N \setminus \uparrow X$ is a subnexus of N.

The proof for $N \setminus r_X$ is similar. $\square$

Corollary 4.6. *Let N be a nexus over γ. For every $() \neq a \in N$, $N \setminus \uparrow a$ is a prime subnexus of N.*

Proof. Since for every $a \in N$, $\uparrow a$ is closed under finite meet operation, we conclude from the Proposition 4.4 and Lemma 4.5 that $N \setminus \uparrow a$ is a prime subnexus of N, for every $a \in N$. $\square$

Lemma 4.7. *Let P be a prime subnexus of nexus N over γ and $b \in N \setminus P$. Then $\downarrow b \setminus P$ has a least element.*

Proof. Let η be a least element of $\{l(x) : x \in \downarrow b \setminus P\}$. Then

Case 1: Let η be a limit ordinal. Then there exists $a = (a_i)_{i \in \eta} \in \downarrow b \setminus P$. Since $a \in \downarrow b$, we have $a = b|_\eta$. Now, assume that $x = (x_i)_{i \in \delta} \in \downarrow b \setminus P$. Hence $\eta \leq \delta$. If δ is a limit ordinal, then $x = b|_\delta$, which follows that $a = x|_\eta$. Therefore, $a \leq x$. If δ is a non-limit ordinal, then $x|_{\delta-1} = b|_{\delta-1}$ and $x_{\delta-1} \leq b_{\delta-1}$. Since $\eta \leq \delta - 1$, we conclude that $a = x|_\eta$, which follows that $a \leq x$. Hence a is a least element of $\downarrow b \setminus P$.

Case 2: Let η be a non-limit ordinal and $\eta = \sigma + 1$. Suppose that t_σ is the least element of

$$\{t \in \mathfrak{O} | \text{there exists an element of } x \in \downarrow b \setminus P \text{ such that } l(x) = \eta \text{ and } x_\sigma = t\}.$$

We define $a : \eta \to \gamma$ with

$$a_i = \begin{cases} b_i & \text{if } i \in \sigma \\ t_\sigma & \text{if } i = \sigma. \end{cases}$$

By definition t_σ, there exists $x = (x_i)_{i \in \eta} \in \downarrow b \setminus P$ such that $x_\sigma = t_\sigma$. Since $x \leq b$, we conclude that $x|_\sigma = b|_\sigma = a|_\sigma$ and $a_\sigma = t_\sigma = x_\sigma$. Hence $a = x \in \downarrow b \setminus P$.

Now, assume that $x = (x_i)_{i \in \delta} \in \downarrow b \setminus P$. We show that $a \leq x$. By Proposition 3.15, $a \leq x$ or $x \leq a$. If $x \leq a$, then $l(x) \leq l(a)$. So by definition η, $l(a) = l(x)$. Since $x|_\sigma = a|_\sigma$ and $x_\sigma \leq a_\sigma \leq x_\sigma$, we conclude that $a = x$. Hence a is a least element of $\downarrow b \setminus P$. $\square$

Proposition 4.8. *Every prime subnexus of a nexus N over γ is of the form of $N \setminus \uparrow a$, for some $a \in N$.*

Proof. Let P be a prime subnexus of N and $b \in N \setminus P$. By Lemma 4.7, $\downarrow b \setminus P$ has a least element, say a. We show that $P = N \setminus \uparrow a$. Let $d \in \uparrow a$. If $d \in P$, then $a \in p$, which is a contradiction. Hence $\uparrow a \subseteq N \setminus P$. Now, suppose that $d \in N \setminus P$. If $d \notin \uparrow a$, then $a \nleq d$ and so $a \wedge d \neq a$. Since $a \wedge d < a$ and $a \wedge d \in \downarrow b$, we conclude that $a \wedge d \in P$, which follows that $a \in P$ or $d \in P$, which is a contradiction. Hence $N \setminus P \subseteq \uparrow a$, and this completes the proof. $\square$

Corollary 4.9. *Let N be a nexus over γ. For every $a \in N$, if $r_a \neq \emptyset$, then $N \setminus r_a$ is a prime subnexus of N.*

Proof. Since for every $a \in N$, r_a is closed under finite meet operation, we conclude from the Proposition 4.4 and Lemma 4.5 that $N \setminus \uparrow a$ is a prime subnexus of N, for every $a \in N$. □

Corollary 4.10. *Let N be a nexus over γ. For every $a \in N$, if $r_a \neq \emptyset$, then there exists $b \in N$ such that $r_a = \uparrow b$.*

Proof. Let $a \in N$, then by Corollary 4.9, $N \setminus r_a$ is a prime subnexus of N. So by Proposition 4.8, there exists $b \in N$ such that $N \setminus r_a = N \setminus \uparrow b$, which follows that $r_a = \uparrow b$. □

Proposition 4.11. *Let N be a nexus over γ and $\emptyset \neq X \subseteq N$. If X is closed under finite meet operation, then $N \setminus \uparrow X$ is a prime subnexus of N.*

Proof. By Lemma 4.5, $N \setminus \uparrow X$ is a subnexus of N. Now, suppose that $a \wedge b \in N \setminus \uparrow X$ and $a \notin N \setminus \uparrow X$, for some $a, b \in N$. If $b \notin N \setminus \uparrow X$, then there exists $x_1, x_2 \in X$ such that $x_1 \leq a$ and $x_2 \leq b$. Since X is closed under finite meet operation and $x_1 \wedge x_2 \leq a \wedge b$, we conclude that $a \wedge b \in \uparrow X$, which is a contradiction. Hence $N \setminus \uparrow X$ is a prime subnexus of N. □

Corollary 4.12. *Let N be a nexus over γ and $\emptyset \neq X \subseteq N$. If X is closed under finite meet operation, then there exists $a \in X$ such that $\uparrow a = \uparrow X$ and $a = \bigwedge X$.*

Proof. By Proposition 4.11, $N \setminus \uparrow X$ is a prime subnexus of N. So by Proposition 4.8, there exists $a \in N$ such that $N \setminus \uparrow a = N \setminus \uparrow X$, which follows that $\uparrow a = \uparrow X$. It is clear that $a \in X$ and $a = \bigwedge X$. □

Lemma 4.13. *Let N be a nexus over γ and let $P_1 = N \setminus \uparrow a$ and $P_2 = N \setminus \uparrow b$ be prime subnexuses of N. Then,*

(1) *If a and b are two comparable addresses, then $P_1 \cap P_2$ is a prime subnexus of N.*

(2) *If a and b are not comparable addresses, then $P_1 \cap P_2$ is not a prime subnexus of N and $N = P_1 \cup P_2$.*

Proof. (1) Without loss of generally, suppose that $a \leq b$. Then $N \setminus \uparrow a \subseteq N \setminus \uparrow b$. Hence by Corollary 4.6, $P_1 \cap P_2 = P_1$ is a prime subnexus of N.

(2) Since a and b are not comparable addresses, $a \wedge b < a$ and $a \wedge b < b$. Hence $a \wedge b \in P_1 \cap P_2$, $a \notin P_1$ and $b \notin P_2$. Therefore, $P_1 \cap P_2$ is not a prime subnexus of N.

Let $c \in N$ and $c \notin P_1$. Then $c \in \uparrow a$ and $a \leq c$. If $c \notin P_2$, then $b \leq c$. Hence by Proposition 3.15, a and b are two comparable addresses, which is a contradiction. Thus $c \in P_2$. Therefore, $N = P_1 \cup P_2$. □

Proposition 4.14. *Let N be a nexus over γ. The following assertions are equivalent:*

(1) *Nexus N is linearly ordered.*

(2) *Every proper subnexus of N is prime.*

Proof. (1) $\Rightarrow$ (2) Let N be linearly ordered and $N \neq P \in Sub(N)$. If $a \wedge b \in P$, since $a \leq B$ or $b \leq a$, then $a \in P$ or $b \in P$. Hence P is a prime subnexus of N.

(2) $\Rightarrow$ (1) Let every proper subnexus of N be prime and $a, b \in N$. We put $P_1 = N \setminus \uparrow a$ and $P_2 = N \setminus \uparrow b$. If a and b are not comparable addresses, then by Lemma 4.13, the proper subnexus $P_1 \cap P_2$ is not a prime subnexus of N, which is a contradiction. Thus a and b are comparable addresses, that is, N is linearly ordered. □

Corollary 4.15. *Let N be a finite nexus over γ. The following assertions are equivalent:*

(1) *The nexus N is cyclic.*

(2) *Every proper subnexus of N is prime.*

Proof. By Propositions 3.15 and 4.14, it is clear. □

By the following example, we prove that the condition being infinite on N does not imply N is cyclic if and only if every proper subnexus of N is prime.

Example 4.16. Let ω be the first countable limit ordinal. Hence if $n \in \omega$, then n is a finite ordinal. For every $n \in \omega$, we define $O^n : n+1 \to 2$ by $O^n(i) = 1$, for all $i \in n+1$.

Let $N = \{()\} \cup \{O^n | n \in \omega\}$. The following assertions hold:

1. N is an infinite nexus over 2.
2. N is a linearly ordered.
3. Every proper subnexus of N is prime.
4. The nexus N does not have any maximal subnexus.
5. N is not a cyclic nexus.

Proposition 4.17. *Let S be a nonempty subset of nexus N over γ and $() \neq a = \bigwedge S \in N$. If for every $s \in S$, $P_s = N \setminus \uparrow s$, then $P = N \setminus \uparrow a$ is the largest prime subnexus of N such that $P \subseteq \bigcap_{s \in S} P_s$.*

Proof. If $s \in S$ and $x \in \uparrow s$, then $\bigwedge S \leq s \leq x$, that is, $x \in \uparrow a$. Hence $\bigcup_{s \in S} \uparrow s \subseteq \uparrow a$ and

$$P = N \setminus \uparrow a \subseteq N \setminus \bigcup_{s \in S} \uparrow s = \bigcap_{s \in S} N \setminus \uparrow s = \bigcap_{s \in S} P_s.$$

Let Q be a prime subnexus of N such that $Q \subseteq \bigcap_{s \in S} P_s$. By Proposition 4.8, there exists $b \in N$ such that $Q = N \setminus \uparrow b$, which follows that $\bigcup_{s \in S} \uparrow s \subseteq \uparrow b$. Hence $b \leq a$ and it implies that $\uparrow a \subseteq \uparrow b$. Therefore, $Q \subseteq P$. □

Proposition 4.18. *Let S be a nonempty subset of nexus N over γ and $a = \bigwedge S \in N$. The following assertions hold:*

(1) *If for some $b \in N$, $S \subseteq \uparrow b$, then $\uparrow a \subseteq \uparrow b$.*

(2) *If $() \neq a$, then $P = N \setminus \uparrow a$ is the largest prime subnexus of N such that $S \cap P = \emptyset$.*

Proof. It is evident. □

Proposition 4.19. *Let N and M be two nexuses and $f : N \to M$ be a homomorphism.*

(1) *If P is a subnexus of M, then $f^{-1}(P)$ is a subnexus of N.*

(2) *If P is a prime subnexus of M and $f^{-1}(P) \neq N$, then $f^{-1}(P)$ is a prime subnexus of N.*

Proof. (1) Let $a, b \in N$ such that $a \leq b \in f^{-1}(P)$. Then $f(a) \leq f(b) \in P$, which follows that $a \in f^{-1}(P)$. Hence $f^{-1}(P) = \bigcup_{x \in f^{-1}(P)} \downarrow x$ and by Proposition 3.8, $f^{-1}(P)$ is a subnexus of N.

(2) Let $a, b \in N$ such that $a \wedge b \in f^{-1}(P)$. Then $f(a \wedge b) = f(a) \wedge f(b) \in P$, which follows that $f(a) \in P$ or $f(b) \in P$, that is, $a \in f^{-1}(P)$ or $b \in f^{-1}(P)$. By the statement (1), $f^{-1}(P)$ is a prime subnexus of N. □

Proposition 4.20. *Every prime subnexus of a nexus N over γ is an inverse image of a set under $f_{\uparrow a}$, for some $a \in N$.*

Proof. Let P be a prime subnexus of N. By Proposition 4.8, there exists $a \in N$ such that $P = N \setminus \uparrow a$. Hence by Proposition 3.22,

$$\begin{aligned} x \in f_{\uparrow a}^{-1}(P) &\Leftrightarrow f_{\uparrow a}(x) \in P \\ &\Leftrightarrow x = f_{\uparrow a}(x) \notin \uparrow a \\ &\Leftrightarrow x \in P. \end{aligned}$$

□

Proposition 4.21. *Let N and M be two nexuses, $() \neq b \in M$ and $f : N \to M$ be a homomorphism. Then $f^{-1}(\uparrow b) = \uparrow f^{-1}(b)$.*

Proof. By Corollary 4.6, $M \setminus \uparrow b$ is a prime subnexus of M. Since by Proposition 4.19, $f^{-1}(M \setminus \uparrow b)$ is a prime subnexus of N, we conclude that there exists $a \in N$ such that $N \setminus f^{-1}(\uparrow b) = f^{-1}(M \setminus \uparrow b) = N \setminus \uparrow a$, which follows that $f^{-1}(\uparrow b) = \uparrow a$. Hence $b \leq f(a)$ and for every $d \in f^{-1}(b)$, $f(a) \leq f(d) = b$ and we infer that $f(a) = b$. Since $f^{-1}(b) \subseteq \uparrow a$, we have for every $d \in f^{-1}(b)$, $\uparrow d \subseteq \uparrow a$, which means that $\uparrow f^{-1}(b) = \bigcup_{d \in f^{-1}(b)} \uparrow d \subseteq \uparrow a$. On the other hand, since $a \in f^{-1}(b)$, we conclude that $\uparrow a \subseteq \uparrow f^{-1}(b)$. Therefore, $f^{-1}(\uparrow b) = \uparrow f^{-1}(b)$. □

5 Maximal Subnexuses of a Nexus over an Ordinal

In this section the notion of a maximal subnexus of a nexus is defined and all maximal subnexuses of a nexus over an ordinal are characterized.

Definition 5.1. A *maximal subnexus* a nexus N over γ is a subnexus M, not equal to N, such that there are no subnexus in between M and N.

Lemma 5.2. *If M is a maximal subnexus of a nexus N over γ, then a is a maximal element of N, for every $a \in N \setminus M$.*

Proof. Let $a \in N \setminus M$. If there exists $b \in N$ such that $a < b$, then $b \notin M$ and $b \notin\downarrow a$. Hence $M \subset M\cup < a >\subset N$, which is a contradiction. □

Proposition 5.3. *Let M be a subnexus of a nexus N over γ. The following assertions are equivalent:*

(1) *M is a maximal subnexus of N.*

(2) *There exists a maximal element $a \in N$ such that $M = N \setminus \{a\}$.*

Proof. (1) ⇒ (2) By hypothesis there exists $a \in N \setminus M$. Let $b \in N \setminus M$ and $a \neq b$. It is clear that $b \in M\cup \downarrow a = N$. Hence $b \leq a$ and since by Lemma 5.2, b is a maximal element of N, we conclude that $a = b$, which is a contradiction. Therefore, $M = N \setminus \{a\}$.

(2) ⇒ (1) It is trivial. □

Let N be a nexus over γ. By Proposition 5.3, the number of maximal subnexus of N is equal to the number of maximal addresses of N.

Proposition 5.4. *Every maximal subnexus of a nexus N over γ is prime.*

Proof. Let M be a maximal subnexus of N. By Proposition 5.3, there exists a maximal element $a \in N$ such that $M = N\setminus\{a\}$. Since $\uparrow a = \{a\}$, we conclude from the Corollary 4.6 that M is a prime subnexus of N. □

Proposition 5.5. *Let N be a nexus over γ. The following assertions are equivalent:*

(1) *Every prime subnexus of N is maximal.*

(2) $N = \{(), (1)\}$.

Proof. (1) ⇒ (2) By Propositions 4.8 and 5.3, if $() \neq a \in N$, then $\uparrow a = \{a\}$, that is, a is a maximal element of N and since $(1) \leq a$, we conclude that $N = \{(), (1)\}$.

(2) ⇒ (1) It is trivial. □

Example 5.6. For every $n \in \omega$, we define $t^n : n + 1 \to 3$ by

$$t^n(i) = \begin{cases} 2 & \text{if } i = 0 \\ 1 & \text{if } i \neq 0. \end{cases}$$

and $O^n : n + 1 \to 3$ by $O^n(i) = 1$, for all $i \in n + 1$.

Let $M = \{()\} \cup \{O^n|n \in \omega\} \cup \{t^n|n = 0, 1\}$. The following assertions hold:

1. M is a nexus over 3.
2. $\{()\} \cup \{O^n|n \in \omega\}$ is a subnexus of M.
3. t^1 is the unqiue maximal element of M.
4. The subnexus $M \setminus \{t^1\}$ is the unqiue maximal subnexus of M.
5. For every $1 \leq n \in \omega$, $O^n \not\leq t^1$.
6. M is not a cyclic nexus.

Hence Theorem 1.14 in [1] and Theorem 3.6 in [13], are incorrect.

Now correcting Theorem 1.14 in [1] and Theorem 3.6 in [13], can bring in the following.

Proposition 5.7. *Let N be a nexus over γ. The following assertions are equivalent:*

(1) *N is a cyclic nexus.*

(2) *N has just one maximal subnexus and for every $a \in N$, there exists a maximal element $b \in N$ such that $a \leq b$.*

Proof. (1) $\Rightarrow$ (2) By Propositions 3.13, there exists $a \in N$ such that $N = \downarrow a$. Since a is the unqiue maximal element of N, we conclude from the Proposition 5.3 that $N \setminus \{a\}$ is the unqiue maximal subnexus of N. It is clear that for every $x \in N$, $x \leq a$.

(2) $\Rightarrow$ (1) Let M be unqiue maximal subnexus of N. By Proposition 5.3, $M = N \setminus \{a\}$, where a is the unqiue maximal element of N. If $b \in N$, then by hypothesis, $b \leq a$. Therefore, $N = \downarrow a$ is a cyclic nexus. □

6 The Fraction of a Nexus over an Ordinal

In this section the fractions of a nexus N over an ordinal is defined and denoted by $S^{-1}N$, where S is a meet closed subset of N. It is shown that this structure is isomorphic with a cyclic subnexus of N; and it is a bounded distributive lattice which has only one maximal ideal. Finally all ideals of $S^{-1}N$ are characterized.

Definition 6.1. A *meet closed subset* of nexus N over γ is a nonempty subset S of N such that $() \notin S$ and $a \wedge b \in S$, for all $a, b \in S$.

Let S be a meet closed subset of nexus N over γ. Introduce the following relation $\sim_S$ on $N \times S$:

$$(a, s) \sim_S (b, t) \Leftrightarrow \ \exists u \in S \text{ such that } (a \wedge t) \wedge u = (s \wedge b) \wedge u;$$

it will be proved shortly that $\sim_S$ is an equivalence relation. Write a/s for the class of (a, s). The set of all equivalence classes $\sim_S$ on $N \times S$ is denoted by $S^{-1}N$ and it is called the *fraction* of N with respect to S.

Definition 6.2. Let S be a meet closed subset of nexus N over γ. Let a/s, b/t be two elements of $S^{-1}N$. Then we say $a/s \leq b/t$, if there exists $u \in S$ such that $s \wedge b \wedge a \wedge u = s \wedge t \wedge a \wedge u$.

Proposition 6.3. *Let S be a meet closed subset of a nexus N over γ. Then $(S^{-1}N, \leq)$ is a lattice. In particular,*

$$\frac{a}{s} \wedge \frac{b}{t} = \frac{a \wedge b}{s \wedge t} \quad \text{and} \quad \frac{a}{s} \vee \frac{b}{t} = \frac{\max\{a \wedge s \wedge t, b \wedge s \wedge t\}}{s \wedge t}$$

for every $a/s, b/t \in S^{-1}N$. Also, for every $s \in S$, $\bot = ()/s$ and $\top = s/s$.

Proof. It is clear that $\leq$ on $S^{-1}N$ is reflexive. Let $a/s \leq b/t$ and $b/t \leq a/s$, for some $a/s, b/t \in S^{-1}N$. Then there exists $r, v \in S$ such that $s \wedge b \wedge a \wedge r = s \wedge t \wedge a \wedge r$ and $t \wedge a \wedge b \wedge v = t \wedge s \wedge b \wedge v$. Hence $(a \wedge t) \wedge (s \wedge t \wedge r \wedge v) = a \wedge b \wedge s \wedge r \wedge t \wedge v = (b \wedge s) \wedge (s \wedge t \wedge r \wedge v)$. Since S be a meet closed subset of N, we conclude that $s \wedge t \wedge r \wedge v \in S$, which follows that $a/s = b/t$. Thus $\leq$ on $S^{-1}N$ is antisymmetric.

Let $a/s \leq b/t$ and $b/t \leq c/r$, for some $a/s, b/t, c/r \in S^{-1}N$. Then there exists $v, w \in S$ such that $s \wedge b \wedge a \wedge v = s \wedge t \wedge a \wedge v$ and $t \wedge c \wedge b \wedge w = t \wedge r \wedge b \wedge w$. Hence

$$\begin{aligned}
(s \wedge r \wedge a) \wedge (t \wedge v \wedge w) &= (s \wedge t \wedge a \wedge v) \wedge (r \wedge w \wedge t) \\
&= (s \wedge b \wedge a \wedge v) \wedge (r \wedge w \wedge t) \\
&= (t \wedge r \wedge b \wedge w) \wedge (a \wedge s \wedge v) \\
&= (t \wedge c \wedge b \wedge w) \wedge (a \wedge s \wedge v) \\
&= (s \wedge b \wedge a \wedge v) \wedge (t \wedge c \wedge w) \\
&= (s \wedge t \wedge a \wedge v) \wedge (t \wedge c \wedge w) \\
&= (s \wedge c \wedge a) \wedge (t \wedge v \wedge w)
\end{aligned}$$

Since S is a meet closed subset of N, we conclude that $t \wedge v \wedge w \in S$, which follows that $a/s \leq c/r$. Thus $\leq$ on $S^{-1}N$ is transitive and $(S^{-1}N, \leq)$ is a partial order set.

Let $a/s, b/t \in S^{-1}N$. Since $(s\wedge t)\wedge a\wedge(a\wedge b) = (s\wedge t)\wedge s\wedge(a\wedge b)$ and $(s\wedge t)\wedge b\wedge(a\wedge b) = (s\wedge t)\wedge t\wedge(a\wedge b)$, we conclude that

$$\frac{a\wedge b}{s\wedge t} \leq \frac{a}{s} \quad \text{and} \quad \frac{a\wedge b}{s\wedge t} \leq \frac{b}{t}.$$

Now, let $c/r \in S^{-1}N$ such that $c/r \leq a/s$ and $c/r \leq b/t$. Then there exists $v, w \in S$ such that $r\wedge a\wedge c\wedge v = r\wedge s\wedge c\wedge v$ and $r\wedge b\wedge c\wedge w = r\wedge t\wedge c\wedge w$. Hence

$$\begin{array}{rcl}(r\wedge a\wedge b\wedge c)\wedge(v\wedge w) & = & (r\wedge a\wedge c\wedge v)\wedge(r\wedge b\wedge c\wedge w)\\ & = & (r\wedge s\wedge c\wedge v)\wedge(r\wedge t\wedge c\wedge w)\\ & = & (r\wedge s\wedge t\wedge c)\wedge(v\wedge w).\end{array}$$

Since S is a meet closed subset of N, we conclude that $v\wedge w \in S$, which follows that

$$\frac{c}{r} \leq \frac{a\wedge b}{s\wedge t}.$$

Therefore,

$$\frac{a}{s}\wedge\frac{b}{t} = \frac{a\wedge b}{s\wedge t}.$$

Let $a/s, b/t \in S^{-1}N$. Since $a\wedge t\wedge s, b\wedge t\wedge s \in \downarrow(t\wedge s)$, we conclude from Proposition 3.15 that $a\wedge t\wedge s$ and $b\wedge t\wedge s$ are comparable. Let $a\wedge t\wedge s \leq b\wedge t\wedge s$, that is, $\max\{a\wedge t\wedge s, b\wedge t\wedge s\} = b\wedge t\wedge s$.

Since $s\wedge(b\wedge s\wedge t)\wedge s = (a\wedge s\wedge t)\wedge(b\wedge s\wedge t) = s\wedge(s\wedge t)\wedge a$, $s\wedge(b\wedge s\wedge t)\wedge b = s\wedge(s\wedge t)\wedge b$ and $s\wedge t\in S$, we conclude that

$$\frac{a}{s} \leq \frac{b\wedge t\wedge s}{s\wedge t} \quad \text{and} \quad \frac{b}{t} \leq \frac{b\wedge t\wedge s}{s\wedge t}.$$

Now, let $c/r \in S^{-1}N$ such that $a/s \leq c/r$ and $b/t \leq c/r$. Then there exists $v, w \in S$ such that $s\wedge c\wedge a\wedge v = s\wedge r\wedge a\wedge v$ and $t\wedge c\wedge b\wedge w = t\wedge r\wedge b\wedge w$. By attention to $a\wedge t\wedge s \leq b\wedge t\wedge s$, we have

$$\begin{array}{rcl}(t\wedge s)\wedge c\wedge(b\wedge t\wedge s)\wedge(v\wedge w) & = & b\wedge c\wedge s\wedge v\wedge t\wedge w\\ & = & [(s\wedge c\wedge a\wedge v)\wedge(t\wedge w)]\vee\\ & & [(t\wedge c\wedge b\wedge w)\wedge(v\wedge s)]\\ & = & [(s\wedge r\wedge a\wedge v)\wedge(t\wedge w)]\vee\\ & & [(t\wedge r\wedge b\wedge w)\wedge(v\wedge s)]\\ & = & t\wedge r\wedge b\wedge w\wedge v\wedge s\\ & = & (t\wedge s)\wedge r\wedge(b\wedge t\wedge s)\wedge(v\wedge w).\end{array}$$

Thus

$$\frac{b\wedge t\wedge s}{s\wedge t} \leq \frac{c}{r}$$

and hence

$$\frac{a}{s}\vee\frac{b}{t} = \frac{\max\{a\wedge s\wedge t, b\wedge s\wedge t\}}{s\wedge t}.$$

□

Proposition 6.4. *Let S be a meet closed subset of a nexus N over γ. For every $a \in N$ and $s, t \in S$, $a/s = a/t$.*

Proof. Since $(a\wedge t)\wedge(s\wedge t) = (a\wedge s)\wedge(s\wedge t)$ and $t\wedge s\in S$, we conclude that $a/s = a/t$. □

Lemma 6.5. *Let S be a meet closed subset of a nexus N over γ and $m = \bigwedge S$. For every $a, b \in N$ and $s, t \in S$,*

(1) *$(a, m) \sim_S (b, m)$ if and only if $(a, m) \sim_{\{m\}} (b, m)$.*

(2) *If $a/s \leq b/t$ in $S^{-1}N$, then $a/m \leq b/m$ in $\{m\}^{-1}N$.*

Proof. (1) We first suppose that $(a, m) \sim_S (b, m)$. Then there exists $t \in S$ such that $a\wedge m = a\wedge m\wedge t = b\wedge m\wedge t = b\wedge m$, which follows that $(a, m) \sim_{\{m\}} (b, m)$.

Conversely, let $(a, m) \sim_{\{m\}} (b, m)$. Then $a\wedge m = b\wedge m$. Since by Corollary 4.12, $m \in S$, we conclude that $(a, m) \sim_S (b, m)$.

(2) There exists $r \in S$ such that $s\wedge b\wedge a\wedge r = s\wedge t\wedge a\wedge r$. Then $m\wedge b\wedge a = s\wedge b\wedge a\wedge r\wedge m = s\wedge t\wedge a\wedge r\wedge m = m\wedge a$, that is, $a/m \leq b/m$ in $\{m\}^{-1}N$. □

Proposition 6.6. *Let S be a meet closed subset of a nexus N over γ. If $m = \bigwedge S$, then $S^{-1}N \cong \{m\}^{-1}N$ as lattices.*

Proof. We define $\varphi : S^{-1}N \to \{m\}^{-1}N$ with $\varphi(a/s) = a/m$. Then by Lemma 6.5, φ is well-defined and it also preserves the order. If $\varphi(a/s) = \varphi(b/t)$, then $a/m = b/m$. Since by Corollary 4.12, $m \in S$ and $a \wedge t \wedge m = a \wedge m = b \wedge m = b \wedge s \wedge m$, we conclude that $a/s = b/t$, which follows that φ is one-to-one. Also, by Corollary 4.12, $m \in S$, so φ is onto.

Let $a/s, b/t \in S^{-1}N$. If $a \wedge t \wedge s \leq b \wedge t \wedge s$, then $a \wedge m \leq b \wedge m$ and

$$\varphi(\frac{a}{s} \vee \frac{b}{t}) = \varphi(\frac{b \wedge s \wedge t}{s \wedge t}) = \frac{b \wedge s \wedge t}{m} = \frac{b \wedge m}{m} = \frac{a}{m} \vee \frac{b}{m} = \varphi(\frac{a}{s}) \vee \varphi(\frac{b}{t}).$$

Also,

$$\varphi(\frac{a}{s} \wedge \frac{b}{t}) = \varphi(\frac{a \wedge b}{s \wedge t}) = \frac{a \wedge b}{m} = \frac{a}{m} \wedge \frac{b}{m} = \varphi(\frac{a}{s}) \wedge \varphi(\frac{b}{t}).$$

It is clear that

$$\varphi(\frac{()}{s}) = \frac{()}{m} \text{ and } \varphi(\frac{s}{s}) = \frac{s}{m} = \frac{m}{m}.$$

□

Proposition 6.7. *Let S be a meet closed subset of a nexus N over γ. If $m = \bigwedge S$, then $\{m\}^{-1}N \cong \downarrow m$ as lattices.*

Proof. We define $\varphi : \{m\}^{-1}N \to \downarrow m$ with $\varphi(a/m) = a \wedge m$. For every $a, b \in N$,

$$\frac{a}{m} = \frac{b}{m} \Leftrightarrow a \wedge m = b \wedge m \Leftrightarrow \varphi(\frac{a}{m}) = \varphi(\frac{b}{m}).$$

Hence φ is well-defined and one-to-one. It is clear that φ is onto. Let $a/m, b/m \in \{m\}^{-1}N$. By Proposition 3.15, we can assume that $a \wedge m \leq b \wedge m$. Hence

$$\varphi(\frac{a}{m} \vee \frac{b}{m}) = \varphi(\frac{b \wedge m}{m}) = b \wedge m = (a \wedge m) \vee (b \wedge m) = \varphi(\frac{a}{m}) \vee \varphi(\frac{b}{m}).$$

Also,

$$\varphi(\frac{a}{m} \wedge \frac{b}{m}) = \varphi(\frac{a \wedge b}{m}) = a \wedge b \wedge m = \varphi(\frac{a}{m}) \wedge \varphi(\frac{b}{m}).$$

It is clear that

$$\varphi(\frac{()}{m}) = () \wedge m = () \text{ and } \varphi(\frac{m}{m}) = m \wedge m = m.$$

□

Corollary 6.8. *Let S be a meet closed subset of a nexus N over γ. $(S^{-1}N, \leq)$ is isomorphic with a cyclic subnexus of $(N, \leq)$.*

Proof. By Propositions 6.6 and 6.7, it is clear. □

Corollary 6.9. *Let S be a meet closed subset of a nexus N over γ. Then $(S^{-1}N, \leq)$ is a bounded distributive lattice.*

Proof. By Propositions 6.6 and 6.7, it is clear. □

Corollary 6.10. *Let S_1 and S_2 be meet closed subsets of a nexus N over γ. If $\bigwedge S_1 = \bigwedge S_2$, then $S_1^{-1}N \cong S_2^{-1}N$.*

Proof. By Propositions 6.6 and 6.7, it is clear. □

Example 6.11. It is clear that $N = A(\omega)$ is a nexus. Hence $S_1 = \uparrow (2,1)$ and $S_2 = \uparrow (3)$ are meet closed subsets of nexus N over ω. Then $\bigwedge S_1 = (2,1) \neq (3) = \bigwedge S_2$ and $S_1^{-1}N \cong < (2,1) > \cong < (3) > \cong S_2^{-1}N$. Therefore, the converse of Corollary 6.10 is obviously false.

Proposition 6.12. *Let S be a meet closed subset of a nexus N over γ and $m = \bigwedge S$. Then $\varphi : Id(S^{-1}N) \to Sub(\downarrow m)$ with $\varphi(I) = f(I)$ is a lattice isomophism, where $f : S^{-1}N \to \downarrow m$ with $f(a/s) = a \wedge m$.*

Proof. Let $I \in Id(S^{-1}N)$. Then for every $s \in S$, $()/s \in I$ and $f(()/s) = () \in f(I)$, that is, $f(I) \neq \emptyset$. Now, let $x \in f(I)$ and $y \in \downarrow m$ such that $y \leq x$. Then there exists $a/s \in I$ such that $x = f(a/s) = a \wedge m$. Since $y = y \wedge m \leq x \wedge m = a \wedge m = x$ in $\downarrow m$, we conclude that $y/m \leq a/m \leq a/s \in I$ in $S^{-1}N$, which follows that $y/m, a/m \in I$. Hence $y = f(y/m) \in f(I)$ and by Proposition 3.8, $f(I) \in Sub(\downarrow m)$. Thus φ is closed. Let $I, J \in Id(S^{-1}N)$. If $I = J$, then $\varphi(I) = f(I) = f(J) = \varphi(J)$, that is, φ is well-defined. Let $\varphi(I) = f(I) = f(J) = \varphi(J)$ and $a/s \in I$. Then there exists $b/t \in J$ such that $a \wedge t \wedge m = a \wedge m = f(a/s) = f(b/t) = b \wedge m = b \wedge s \wedge m$. Since by Corollary 4.12, $m \in S$, we conclude that $a/s = b/t \in J$. Hence $I = J$ and φ is one-to-one. If $K \in Sub(\downarrow m)$, then $f^{-1}(K) \in Id(S^{-1}N)$ and $f(f^{-1}(K)) = K$. Hence φ is onto. It is clear that for every $I, J \in Id(S^{-1}N)$, $\varphi(I \wedge J) = \varphi(I) \wedge \varphi(J)$ and $\varphi(I \vee J) = \varphi(I) \vee \varphi(J)$. Therefore, φ is a lattice isomophism. □

Proposition 6.13. *Let S be a meet closed subset of a nexus N over γ. Then $(S^{-1}N, \leq)$ has a unique maximal ideal.*

Proof. By Propositions 5.7 and 6.12, it is clear. □

Example 6.14. Let for every $n \in \omega$, O^n be as in example 5.6 and $O : \omega \to 3$ by $O(i) = 1$, for all $i \in \omega$. Let $N = \{()\} \cup \{O^n | n \in \omega\} \cup \{O\}$, $I = \{()\} \cup \{O^n | n \in \omega\}$ and $S = \{O\}$. The following assertions hold:

1. N is a nexus over 3.
2. I is a subnexus of N.
3. I is not a cyclic nexus of N.
4. $S^{-1}I = \{\frac{()}{O}\} \cup \{\frac{O^n}{O} | n \in \omega\}$ is an infinite ideal of $S^{-1}N$ and $S^{-1}I \neq S^{-1}N$.
5. If J is a proper cyclic subnexus of N, then $S^{-1}J$ is a finite ideal of $S^{-1}N$.

Hence Theorem 2.26 (i) in [1] is incorrect.

Now correcting Theorem 2.26 (i) in [1], can bring in the following.

Proposition 6.15. *Let S be a meet closed subset of nexus N over γ.*

(1) *Every ideal of $S^{-1}N$ is of the form of $S^{-1}I$, where I is a subnexus of N.*

(2) *If K is a finite ideal of $S^{-1}N$, then there exists a cyclic subnexus I of N such that $K = S^{-1}I$.*

Proof. (1) Let K be an ideal of $S^{-1}N$ and $I = \{a \in N | \ a/s \in K \text{ for some } s \in S\}$. It is clear that I is a subnexus of N and $K = S^{-1}I$.

(2) Let K be a finite ideal of $S^{-1}N$ and $m = \bigwedge S$. Since by Propositions 3.15, 6.6 and 6.7, every two elements of $S^{-1}N$ are comparable, we conclude that K has a maximal element, say a/s. Hence $K = \downarrow a/s$. We put $I = \downarrow a$ and we claim that $K = S^{-1}I$. Let $b/t \in K$, then there exists $r \in S$, $t \wedge a \wedge b \wedge r = t \wedge s \wedge b \wedge r$, which follows that $(a \wedge b \wedge t) \wedge (t \wedge s \wedge r) = (b \wedge s) \wedge (t \wedge s \wedge r)$. Therefore, $b/t = (a \wedge b)/s \in S^{-1}I$. Now, let $b \in I$ and $t \in S$. Then $t \wedge a \wedge b \wedge m = b \wedge m = t \wedge s \wedge b \wedge m$, which follows that $b/t \leq a/s \in K$. Since K is an ideal of $S^{-1}N$, we conclude that $b/t \in K$, and this completes the proof. □

Proposition 6.16. *Let S be a meet closed subset of nexus N over γ. If I and J are two subnexuses of N, then $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$.*

Proof. It is clear. □

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