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Stochastic Optimization Methods for Sequential Decision Problems

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Abstract

We study sequential decisions for a firm in a supply chain with one supplier and one company. There are demand risk and supply risk in our problem, and they are from the market and the supplier, respectively. A company may desire to maximize its benefits in the sense of expectation criterion, it can make sequential price decision and quantity decision to minimize its total losses. We establish several new expected value models and derive their optimal decisions. We further analyze the relationship between the optimal decision and supply risk, and conclude that the threshold level of supply risk increases when the firm reduces its quantity decision. Numerical examples are presented to demonstrate our new methods.
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1 Introduction

Production and sale are two most basic aspects for a company, they should handle the relationship between the supplier and the consumer. Even in a simple supply chain problem, a company is usually faced with the demand risk and the supply risk. The demand risk is from the market since the company doesn't know the complete information about the market demand of the product. The supply risk from the supplier is affected by the quality of the production equipment, the degree of proficiency of workers and the third party logistics in the process of transportation accident. These two risks interact and jointly affect the company's profitability. Some researchers have studied the joint price and quantity decision-making problems, especially the simultaneous decision-making problems. Deng [3] analyzed joint pricing-production decisions under different supply chain settings and restrictions. Lim [8] presented a joint optimal pricing and order quantity model, in which the demand is modeled as a function of price and the cost is modeled as a function of quantity. Yano et al. [11] provided a comprehensive review of simultaneous pricing and production decisions. Hu et al. [5] analyzed purchasing decisions under stochastic price and given the approximate solution for order time, order quantity and supplier selection. Dadal et al. [2] studied the production decisions under random production and price uncertainty.

Since the whole supply chain process is dynamic, the flexible decision making problem has been studied in the literature. Lin [9] considered a sequential dynamic pricing model for a seller sells stock. They all showed the dynamic pricing make sellers get more profits. Inderfurth and Clemens [6] analyzed risk sharing contracts can facilitate decision making and improved supply chain performance. Zhang and Brorsen [12] discussed quantity-price strategic decisions for oligopoly. Xiao et al. [10] discussed ordering, wholesale pricing and lead time decisions in a three-stage supply chain consisting of one retailer, one manufacturer and one subcontractor. He [4] studied four cases of sequential decision-making under supply risk and demand risk. In the present paper, we further analyze several new sequential decision-making problems that often happen in our real life. Since stochastic optimization method is a useful tool to analyze practical decision-making problems [1, 7], we build our optimization models by two-stage stochastic programming, in which the decisions are divided into two groups according to the sequential decision-making methods. In some cases, we can provide the analytical solutions to stochastic expectation models. For some hard expectation models, we can also derive the corresponding optimal strategies when stochastic variables follow some special probability distributions

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like uniform distribution. Finally, we analyze the impact of supply risk and market sensitivity on optimal decisions.

The paper is organized as follows. In Section 2, we build several new stochastic expected value models for sequential decision-making problems. We also solve the proposed stochastic optimization models in this section by using two-stage analysis method. Section 3 conducts some numerical experiments to illustrate our new optimization methods. Finally, Section 4 summarizes the main work in this paper.

2 New Stochastic Models and Their Analytical Solutions

In this paper, we assume that a firm faces uncertain demand determined by price and other factors. If the firm decides to order a certain amount of products, then the actual number of the available products is usually less than the required number. We assume that any unsatisfied demand incurs penalty. To describe our sequential decision problem, we adopt the following notations.

Decision variables

p is the price of production (the price decision), and Q is the quantity of production (the quantity decision).

Uncertain parameters

x is the supply uncertainty on support $[0,1]$; y is the demand uncertainty on support $[Y_L, Y_H]$; D is the market demand with $D = y - bp$, and Z is the actual number of available products with $Z = xQ$.

Fixed parameters

b is the sensitivity of market with $b > 0$; c is the production cost per unit input; s is the value of the remaining products per unit; π is the penalty cost of unsatisfied demand per unit; $\bar{x}$ is the expected value of x; $\bar{y}$ is the expected value of y; X is the realization of x, and Y is the realization of y.

2.1 Models Based on Price Decision before Quantity Decision

In this section, we assume that the firm first makes price decision p. This case is suitable for goods whose volatility is very high and the supply lead time is relatively short and reliable. In this situation, the demand information is crucial for the supply chain. We usually take price decision for fashion goods like clothing. The whole supply is a dynamic process, during which the information is often being updated. The company can also obtain more information about x and determine the value of the supply uncertainty through some channels. However, due to some kind of contracts, the firm has to make quantity decision before knowing demand uncertainty. In other words, the demand uncertainty is known after making price decision and quantity decision. The second-stage program is an expected value model. To model this kind of decision-making problems, we build the following expected value model,

$$\begin{array}{ll} \max & \Pi_1 = E_{x,y}[p \cdot \min(y - bp, xQ) + s(xQ - y + bp)^+ - \pi(y - bp - xQ)^+] - cQ \\ \text{s.t.} & p \geq 0, \end{array} \tag{1}$$

where Q is the optimal solution of the following expected value model,

$$\begin{array}{ll} \max & \Pi_2 = E_y[p \cdot \min(y - bp, XQ) + s(XQ - y + bp)^+ - \pi(y - bp - XQ)^+] - cQ \\ \text{s.t.} & Q \geq 0. \end{array} \tag{2}$$

The following theorem describes the optimal decision of model (1), where $g(y)$ is density function of y.

Theorem 1. *Assume that a firm makes the price decision before the quantity decision and cannot postpone the quantity decision, then the optimal Q^* is unique and determined by the following equation*

$$\int_{bp+XQ}^{Y_H} g(y)dy = \frac{c - sX}{(p + \pi - s)X}.$$

Proof. Let's analysis the second-stage programming problem. The profit function in this problem can be rewritten as

$$\begin{aligned} \Pi_2 = &\int_{Y_L}^{bp+XQ} p(y - bp)g(y)dy + \int_{bp+XQ}^{Y_H} p \cdot XQg(y)dy + s\int_{Y_L}^{bp+XQ} (XQ - y + bp)g(y)dy \\ &- \pi\int_{bp+XQ}^{Y_H} (y - bp - XQ)g(y)dy - cQ. \end{aligned} \tag{3}$$

We next discuss the property of Eq. (3), $\frac{d\Pi_2}{dQ} = X(p+\pi-s)\int_{bp+XQ}^{Y_H} g(y)dy + sX - c$. Furthermore, we have $\frac{d^2\Pi_2}{dQ^2} = -X^2(p+\pi-s)g(bp+XQ)$. Obviously $\frac{d^2\Pi_2}{dQ^2} < 0$, so Π_2 is a concave function. According to the properties of concave function Π_2, it reaches its maximum at $\frac{d\Pi_2}{dQ} = 0$. So Q^* is determined by $\frac{d\Pi_2}{dQ} = 0$. □

Furthermore, we assume the demand uncertainty follows a uniform distribution on the interval $[Y_L, Y_H]$ and let $Y_H - Y_L = L$. Then its density function $g(y) = 1/(Y_H - Y_L) = 1/L$. According to Theorem 1, we have $Q^*(p) = \frac{Y_H - bp}{X} - \frac{(c-sX)L}{(p+\pi-s)X^2}$, which is denoted as Q^* for convenience. Here, we set the threshold level as 1/2. Therefore, if the company invests Q units raw materials or orders Q units products, then the actual number of received is not less than $Q/2$. We discuss a special situation that the supply uncertainty x follows a uniform distribution on the interval $[1/2, 1]$. When $xQ^* = y - bp$, we have $x = \frac{cL}{(p+\pi)Y_H - sY_L - (p+\pi-s)y}$. Denote $h(y) = \frac{cL}{(p+\pi)Y_H - sY_L - (p+\pi-s)y}$. If $x > h(y)$, then we have $xQ^* > y - bp$. If $x < h(y)$, then we have $xQ^* < y - bp$. As a consequence, the first-stage profit function can be rewritten as

$$\begin{aligned}\Pi_1 = & \frac{-2pcY_H}{p+\pi-s} ln\frac{s}{p+\pi} + \frac{2pcY_L - 2\pi cL}{p+\pi-s} ln\frac{c}{p+\pi} + \frac{-pcL \cdot ln4 + s^2L - \pi cL \cdot ln4 + c^2L \cdot ln4 - scL}{p+\pi-s} \\ & - \frac{2pcY_H - 2csL}{p+\pi-s} ln\frac{c}{s} - bp^2 + (bc \cdot ln4 + Y_L)p - cY_H \cdot ln4 - \frac{\pi L}{2}.\end{aligned}$$

In this case, the value of the second-order derivative $\frac{d^2\Pi_1}{dp^2}$ depends on the assumed parameter values. The next theorem describes the corresponding optimal strategies.

Theorem 2. *Assume that a firm makes the price decision before the quantity decision and cannot postpone the quantity decision, the supply uncertainty and the demand uncertainty follow uniform distributions on the interval* $[1/2, 1]$ *and* $[Y_L, Y_H]$*, respectively. If* $\frac{d^2\Pi_1}{dp^2} < 0$*, then* p^* *is the solution of the following equation*

$$\frac{2cL}{p+\pi-s} + \frac{2csL}{(p+\pi-s)^2} ln\frac{s}{p+\pi} + \frac{L(s-c)(c \cdot ln4 - s)}{(p+\pi-s)^2} - 2bp + bc \cdot ln4 + Y_L = 0.$$

2.2 Models Based on Quantity Decision before Price Decision

In this section, we study the sequential decision-making problem when operations department moves first and the price decision follows. Some agriculture products, which take several months to grow, are typical examples. To be more precise, the first-stage decision is represented by Q, while the second-stage decision is represented by p. Before the second-stage, the random variables x, y become known. Although the sequence of realized x and y cannot affect the optimal decisions, we still give them an order – x realized after y. In [4], the author just analyzed the cases that y is a constant and y follows a simple Bernoulli distribution. In this section, we assume y is a general random variable, and build the following two-stage expected value model,

$$\begin{array}{ll} \max & \Pi_1 = E_{x,y}[p \cdot \min(y - bp, xQ) + s(xQ - y + bp)^+ - \pi(y - bp - xQ)^+] - cQ \\ \text{s.t.} & Q \geq 0, \end{array} \tag{4}$$

where p is the optimal solution of the following programming model,

$$\begin{array}{ll} \max & \Pi_2 = p \cdot \min(Y - bp, XQ) + s(XQ - Y + bp)^+ - \pi(Y - bp - XQ)^+ - cQ \\ \text{s.t.} & p \geq 0. \end{array} \tag{5}$$

Given realizations of x and y, the second-stage problem is a deterministic mathematical model. The next theorem describes the corresponding optimal strategies.

Theorem 3. *Assume that a firm makes the quantity decision before the price decision and can postpone the price decision after the uncertainties are revealed, then the optimal strategy is* $Q^* = \frac{\bar{x}\bar{y} - bc}{2\bar{x}^2}$.

Proof. In the second-stage problem, we have $Y - bp = XQ$ and $p^* = (Y - XQ)/b$. Substitute p^* into $\Pi_1(p, Q(p))$, we have

$$\Pi_1 = E_{(x,y)}[\frac{y - xQ}{b} \cdot xQ - cQ] = \frac{\bar{y} - \bar{x}Q}{b} \cdot \bar{x}Q - cQ = \frac{-\bar{x}^2}{b}Q^2 + (\frac{\bar{x}\bar{y}}{b} - c)Q.$$

So Π_1 is a quadratic function of Q, we have $Q^* = \frac{\bar{x}\bar{y}-bc}{2\bar{x}^2}$ and $p^* = \frac{Y-X\cdot\frac{\bar{x}\bar{y}-bc}{2\bar{x}^2}}{b}$. Thus, the company eventually get the profit is $\Pi_1 = \frac{\bar{x}^2\bar{y}^2-2bc\bar{x}\bar{y}+b^2c^2}{4b\bar{x}^2}$. □

In the following, the company first makes the quantity decision. Before making the price decision, it is required to do some market surveys to get more market information. The company can use the obtained information to determine the demand uncertainty. According to the contract, the company must develop a price decision before knowing the value of supply uncertainty. In this case, the second-stage problem is an expected value model.

Formally, we model this sequential decision-making problem as the following two-stage expectation model,

$$\begin{array}{ll} \max & \Pi_1 = E_{x,y}[p\cdot\min(y-bp,xQ)+s(xQ-y+bp)^+-\pi(y-bp-xQ)^+]-cQ \\ \text{s.t.} & Q\geq 0, \end{array} \tag{6}$$

where p is the optimal solution of the following expected value model,

$$\begin{array}{ll} \max & \Pi_2 = E_x[p\cdot\min(Y-bp,xQ)+s(xQ-Y+bp)^+-\pi(Y-bp-xQ)^+] \\ \text{s.t.} & p\geq 0. \end{array} \tag{7}$$

Let's first analyze the second-stage problem. When $Y-bp=xQ$, we have $x=(Y-bp)/Q$. If $x\geq(Y-bp)/Q$, then $Y-bp\leq xQ$. If $x<(Y-bp)/Q$, then $Y-bp>xQ$. Because $x\in[0,1]$, we divide our discussion into two cases.

Case I: $(Y-bp)/Q\geq 1$, that is $p\leq(Y-Q)/b$. In this situation, we have $Y-bp\geq xQ$. Thus the second-stage profit function can be rewritten as $\Pi_2(p)=E_x[p\cdot xQ-\pi(Y-bp-xQ)]=(\bar{x}Q+\pi b)p-\pi Y+\pi\bar{x}Q$, it is maximum at $p^*=(Y-Q)/b$. The first-stage profit function can be rewritten as

$$\Pi_1(Q)=E_{x,y}[p\cdot xQ-\pi(Q-xQ)]-cQ=-\frac{\bar{x}}{b}Q^2-(\pi-\pi\bar{x}+c-\frac{\bar{x}\bar{y}}{b})Q.$$

Obviously, $\Pi_1(Q)$ is a concave function, it is maximum at $Q=(\bar{x}\bar{y}-(\pi-\pi\bar{x}+c)b)/2\bar{x}$. The next theorem describes the corresponding optimal decisions.

Theorem 4. *Assume that a firm makes the quantity decision before the price decision, the demand uncertainty is realized before making the price decision, and the supply uncertainty is realized after making the price decision. If the inequality $Y-bp\geq Q$ holds, then the optimal quantity decision is $Q^*=\frac{\bar{x}\bar{y}-(\pi-\pi\bar{x}+c)b}{2\bar{x}}$ and the profit is $\Pi_1=\frac{(\bar{x}\bar{y}-b\pi+b\pi\bar{x}-bc)^2}{4b\bar{x}}$.*

Case II: $(Y-bp)/Q<1$. If $0<x\leq(Y-bp)/Q$, then $Y-bp\geq xQ$. If $(Y-bp)/Q\leq x\leq 1$, then $Y-bp\leq xQ$. In this situation, the profit function $\Pi_2(p|(Q,Y))$ can be rewritten as

$$p\int_0^{\frac{Y-bp}{Q}}xQf(x)dx+p\int_{\frac{Y-bp}{Q}}^1(Y-bp)f(x)dx+s\int_{\frac{Y-bp}{Q}}^1(xQ-Y+bp)f(x)dx-\pi\int_0^{\frac{Y-bp}{Q}}(Y-bp-xQ)f(x)dx.$$

Furthermore, we have $\frac{d^2\Pi_2}{dp^2}<0$, Π_2 is a concave function. Given the quantity decision and the market realization, the following theorem describes the optimal price decision in the second-stage expected value model.

Theorem 5. *Assume that a firm makes the quantity decision before the price decision, the demand uncertainty is realized before making the price decision, and the supply uncertainty is realized after making the price decision. If the inequality $Y-bp<Q$ holds, and p' is solution of the following equation*

$$2bp-\int_0^{\frac{Y-bp}{Q}}(xQ+\pi b-Y+2bp-sb)f(x)dx=Y+sb,$$

then the optimal price decision $p^*=\max(p',\frac{Y-Q}{b})$.

3 Numerical Experiments

Suppose a clothing firm decides to order clothes from a supplier and sell the products in different regions with various market sensitive coefficients. In our problem, we only analyze the influences of the supply uncertainty x and parameter b. The related dada used in our experiments are provided in Table 1.

Table 1: The data used in the numerical experiments

cost per unit c	penalty value π	remaining product value s	Y_L	Y_H
10 ¥	2 ¥	12 ¥	50	100

We first analyze the impact of the supply uncertainty x. We set $b = 2$, and employ MATLAB software to solve our optimization problem. The computational results are collected in Table 2.

Table 2: The optimal solutions with different X

$b = 2$	price decision p^*	quantity decision Q^*	profit Π_1
$X = 0.90$	25.4148	57.8374	193.6905
$X = 0.85$	25.4148	58.7454	193.6905
$X = 0.80$	25.4148	59.4357	193.6905
$X = 0.75$	25.4148	59.7940	193.6905
$X = 0.70$	25.4148	59.6519	193.6905
$X = 0.65$	25.4148	58.7569	193.6905
$X = 0.60$	25.4148	56.7224	193.6905

From Table 2, we find that the realization of supply uncertainty cannot affect optimal price decision and final profit. However, as X decreases, the firm's optimal quantity decision Q^* will change accordingly. When the supply risk increases, the firm have to order more products from the supplier to meet the demand of the market. But the firm cannot always increase the quantity to avoid the loss. There exists a threshold level of supply risk, over which the firm will reduce the quantity decision.

In this following, we discuss impact of the market sensitivity b, and let $X = \bar{x} = 0.75$. We report the computational results about the influence of different b on decision variables and profit in Table 3.

Table 3: The optimal solutions with different b

$X = 0.75$	price decision p^*	quantity decision Q^*	profit Π_1
$b = 0.5$	67.9446	86.5029	1737.479
$b = 1.0$	40.3008	76.6654	755.1614
$b = 1.5$	30.5194	67.9626	387.2008
$b = 2.0$	25.4148	59.7940	193.6905
$b = 2.5$	22.2540	51.8994	76.3050

From Table 3, we observe that the increase of market sensitivity will lead to the decrease of the optimal decision and profits. The greater b is, the more price-sensitive consumers are. So the firm should develop a lower price decision to achieve the lower profit.

4 Conclusions

On the basis of stochastic optimization theory, we studied the sequential price and quantity decision problem, and obtained the following new results.

Firstly, several new stochastic expected value models for sequential decision problems were proposed and analyzed. For models based on price decision before quantity decision, we can obtain their analytical solutions.

Secondly, the expectation models based on quantity decision before price decision are hard optimization problems, and their analytical solutions are usually unavailable in general case. To avoid this difficulty, we assumed random variables follow uniform distributions, and solved the proposed expectation models by using MATLAB software.

Finally, some numerical experiments were performed to demonstrate our new modelling ideas. From the computational results, we observed that when the supply risk increased, the firm's optimal decisions first increased then decreased, in which the threshold level was the maximum acceptable risk.

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