Journal of Uncertain Systems
Vol.9, No.1, pp.62-74, 2015
Online at: www.jus.org.uk

Credibility Hypothesis Testing of Fuzzy Triangular Distributions

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Received 10 April 2013; Revised 24 April 2014

Abstract

This paper introduces the notion of testing hypothesis about parameters involved in credibility distributions and defines a criterion called membership ratio criterion for testing a given null credibility hypothesis against an alternative credibility hypothesis. The proposed membership ratio criterion has been used to develop credibility tests for testing hypotheses about credibility parameters which appear in some special types of triangular credibility distributions. The tests derived are shown to have higher credibility values under alternative credibility hypothesis in a class of credibility tests having a given credibility level under null credibility hypothesis. Some illustrative examples are also included to demonstrate the theory developed in this paper.

Keywords: credibility hypothesis, credibility test, credibility rejection region, power of credibility test, best credibility rejection region, membership ratio criterion

1 Introduction

In real life situations, uncertainties of various types arise. Uncertainty can be broadly classified as Uncertainty caused by randomness and Uncertainty caused by Fuzziness. In the words of Liu [3], *randomness is a basic type of objective uncertainty whereas fuzziness is a basic type of subjective uncertainty*. Probability theory is a very old branch of mathematics meant for random phenomena having lot of applications in diversified fields of study. Credibility theory is a relatively new branch of mathematics, which is similar to Probability theory. It is specifically meant for studying the behavior of fuzzy phenomena. Liu and Liu [4] initiated the first study on Credibility theory and Liu [2] carried out further refinements of the theory. Many researchers have studied applications of Credibility theory in different fields of study. Peng and Liu [5] developed a methodology for modeling parallel machine scheduling problems with fuzzy processing times. Zheng and Liu [16] designed a fuzzy optimization model for fuzzy vehicle routing problem by considering travel times as fuzzy variable. Ke and Liu [1] considered a fuzzy project scheduling problem. Wang, Tang and Zhao [14] designed an inventory model without backordering for the fuzzy variables. Wang, Zhao and Tang [15] considered a fuzzy programming model for vendor selection problem in a supply chain. Sampath [6] and Sampath and Deepa [7] have applied Credibility theory in designing acceptance sampling plans for situations involving both fuzziness and randomness. Sampath and Vaidyanathan [10] and Sampath and Kalaivani [8] have used Credibility theory in Clustering of fuzzy data.

A thorough review of the existing literature on Credibility theory indicates several developments in Credibility theory, which are analogues of those available in Probability theory. For example, fuzzy variables, credibility distributions, expectation, variance, moments and related results corresponding to Credibility Theory are analogues of random variables, probability distributions, moments etc defined under Probability theory. Even though significant amount of works have been done in Credibility theory one can feel the absence of studies on the inferential aspects (parallel to Theory of Estimation and Testing of Hypotheses in Probability distributions) of parameters (constants) involved in credibility distributions. Recently, researchers have turned their attention in this direction for Uncertainty distributions. The problem of estimating parameters in Uncertainty distributions has been studied by Wang and Peng [13] and Wang, Gao and Guo [11]. Another significant work is due to Wang, Gao and Guo [12] in which the idea of testing Uncertainty hypothesis has been introduced. Sampath and Ramya [9] have also considered a distance based test procedure for testing uncertainty hypotheses.

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It is to be noted that so far no work has been done in testing hypotheses about parameters (constants) appearing in credibility distributions. Different credibility distributions meant for various practical situations are available in the literature. For a detailed discussion on them one can refer to [2]. The choice of the credibility distribution and prior knowledge about the parameters which appear in a credibility distribution are absolutely necessary for a practitioner. For example, if one identifies that triangular credibility distribution is appropriate for a given situation, credibility value of the fuzzy variable assuming values in a given set and other characteristics of interest like its expected value, variance etc can be arrived at only if values of the three parameters a, b and c of the triangular credibility distribution are fully known. When a practitioner is presented with different sets of values by domain experts about the parameters, the question of choosing an appropriate set for a given situation arises. To deal with these types of problems, we consider a tool which is similar to the Theory of Testing of Hypothesis in Probability distributions based on the idea of Neyman-Pearson theory and make a study on its properties.

The paper is organized as follows. The second section of this paper gives a brief description on Credibility theory and the third section gives certain definitions needed in the process of developing tests for parameters in credibility distributions. The fourth section is devoted for identifying optimal tests in the case of parameters involved in triangular credibility distributions. Conclusions and directions for future work are given in the fifth section of this paper.

2 Credibility Theory

In this section, we give a brief introduction to Credibility Theory [2, 4] as well as description about the terminology used in Credibility Theory.

Let Θ be a nonempty set, P be a $\sigma-$algebra over Θ. Elements of P are called events. $Cr\{A\}$ indicates the credibility of occurrence of event A which is the number assigned to each event A that satisfies following axioms.

Axiom of Normality: $Cr\{\Theta\}=1$;

Axiom of Monotonicity: $Cr\{A\} \le Cr\{B\}$ whenever $A \subset B$;

Axiom of Self-Duality: $Cr\{A\}+Cr\{A^c\}=1$ for any event A;

Axiom of Maximality: $Cr\left\{\bigcup_i A_i\right\}=\sup_i\left(Cr\{A_i\}\right)$ for any events $\{A_i\}$ with $\sup_i\left(Cr\{A_i\}\right)<0.5$.

Credibility Measure Space: The set function Cr is called a credibility measure if it satisfies the axioms of normality, monotonicity, self-duality and maximality. The triplet (Θ, P, Cr) is known as credibility space.

Fuzzy Variable: Fuzzy variable ξ is a measurable function from the credibility measure space (Θ, P, Cr) to the set of real numbers.

Membership Function: The membership function μ of a fuzzy variable ξ defined on the credibility measure space (Θ, P, Cr) is defined as $\mu(x)=\left(2Cr\{\xi=x\}\right) \wedge 1,\ x \in R.$

Credibility Inversion Theorem: Let ξ be a fuzzy variable defined on the credibility measure space (Θ, P, Cr) with membership function $\mu.$ Then for any set B of real numbers,

$$Cr\{\xi \in B\}=\frac{1}{2}\left(\sup_{x\in B}\mu(x)+1-\sup_{x\notin B}\mu(x)\right).$$

Credibility Distribution: Let ξ be a fuzzy variable defined on the credibility measure space (Θ, P, Cr) with membership function $\mu.$ Then the credibility distribution $\Phi: R \to [0,1]$ is defined as

$$\Phi(x)=Cr\{\theta \in \Theta \mid \xi(\theta) \le x\}, \text{ for all } x \in R.$$

By credibility inversion theorem, the credibility distribution is given by

$$\Phi(x)=\frac{1}{2}\left(\sup_{y\le x}\mu(y)+1-\sup_{y>x}\mu(y)\right), \text{ for all } x \in R.$$

3 Testing Credibility Hypotheses

There are many credibility distributions available in the literature. Some popular and widely studied distributions are equipossible, triangular and trapezoidal credibility distributions. The membership functions associated with them are given below.

Equipossible: An equipossible fuzzy variable is determined by the pair (a,b) of real numbers with the membership function

$$\mu(x)=\begin{cases}1 & \text{if } a\le x\le b\\ 0 & \text{otherwise.}\end{cases}$$

Triangular: A triangular fuzzy variable is fully determined by the triplet (a,b,c) of real numbers with membership function

$$\mu(x)=\begin{cases}\dfrac{x-a}{b-a} & \text{if } a\le x\le b\\ \dfrac{c-x}{c-b} & \text{if } b\le x\le c.\end{cases}$$

Throughout this paper, we use the symbolic expression $\xi \sim Tri(a,b,c)$ to denote the fact that the fuzzy variable ξ has a fuzzy triangular distribution defined on (a,b,c).

Trapezoidal: A trapezoidal fuzzy variable is defined on the quadruplet (a,b,c,d) of real numbers with membership function

$$\mu(x)=\begin{cases}\dfrac{x-a}{b-a} & \text{if } a\le x\le b\\ 1 & \text{if } b\le x\le c\\ \dfrac{d-x}{d-c} & \text{if } c\le x\le d.\end{cases}$$

The credibility distributions identified by the above membership functions contain certain unknown constants. Equipossible membership function has two parameters, namely, a and b, the triangular membership function contains three parameters, namely, a,b and c and the trapezoidal membership function has four parameters, a,b,c and d. In our further discussion we refer these constants as credibility parameters. In this present work, we consider a test criterion for developing a procedure to test whether a given credibility distribution is the one associated with the fuzzy environment under study. To facilitate the understanding of the proposed criterion we present below certain definitions. These definitions are the credibility versions of related ideas from statistical theory of testing of hypotheses.

Support: Support of the credibility distribution of a fuzzy variable is nothing but the collection of values assumed by the fuzzy variable with nonzero credibility. For example, in the case of $\xi \sim Tri(1,4,5)$, the support of fuzzy variable is $\{x \mid 1\le x\le 5\}$.

Credibility Hypothesis: It is a statement about the credibility distribution of a fuzzy variable. For example, $H:\xi \sim Tri(1,4,7)$ is a credibility hypothesis which states the fuzzy variable $\xi \sim Tri(1,4,7)$.

Null Credibility Hypothesis: A credibility hypothesis that is being tested for possible rejection is known as null credibility hypothesis.

Alternative Credibility Hypothesis: A credibility hypothesis, which will be accepted in the event of rejecting a null credibility hypothesis is known as alternative credibility hypothesis.

In this paper, the letters H and K are used to denote the null and alternative credibility hypotheses respectively. The formulation of null and alternative credibility hypotheses is a domain oriented subject which takes into account the opinions expressed by subject experts. For example, consider the menstrual cycle length of young women. It is believed that the menstrual cycle length is lunar cycle length. However, if one suspects the change in life style and living conditions have affected the menstrual cycle length then one can think of suitable credibility hypotheses testing problem. If a gynecologist believes that based on experience, the cycle length has increased and identifies its distribution as triangular credibility distribution $Tri(29,31,33)$, then the null and alternative credibility hypotheses can be taken as $H:\xi \sim Tri(26,28,30)$ and $K:\xi \sim Tri(29,31,33)$, respectively.

Credibility Test: It is a rule that helps the practitioner to either accept or reject a null credibility hypothesis if the experimental values of the underlying fuzzy variables are known.

Credibility Rejection Region: The subset of the support of a fuzzy variable containing the points leading to the rejection of the null credibility hypothesis as identified by a credibility test is known as Credibility Rejection Region and is denoted by C. Mathematically, the credibility rejection region is defined as $C = \{\xi \in \Theta \mid H \text{ is rejected}\}$.

Type I and Type II Errors: Type I error is the action of rejecting the null credibility hypothesis when it is true and Type II error is the action of accepting the null credibility hypothesis when it is false.

Level of Significance: The maximum credibility for committing Type I error by a credibility test is known as level of significance and is denoted by α.

Power of a Credibility Test: The power of the credibility test with rejection region C is defined by $Cr(\xi \in C)$.

Best Credibility Rejection Region: A credibility rejection region C^* is said to be best credibility rejection region of level α if it has maximum power under alternative credibility hypothesis than that of any other credibility rejection region of same level. That is, C^* the best credibility rejection region, if $Cr(\xi \in C^* \mid K) \geq Cr(\xi \in C \mid K)$ where C is any subset of the support of the fuzzy variable such that $Cr(\xi \in C \mid H) \leq \alpha$.

Here $Cr(\xi \in A \mid H)$ and $Cr(\xi \in A \mid K)$ indicate the credibility values of the event $\xi \in A$ corresponding to the credibility distributions defined under null and alternative credibility hypotheses respectively. For example,

$$Cr(\xi \in A \mid H) = \frac{1}{2}\left[\underset{\xi \in A}{Sup}\, \mu_H(x) + 1 - \underset{\xi \notin A}{Sup}\, \mu_H(x)\right]$$

where μ_H is the membership function associated with the credibility distribution specified under null credibility hypothesis. Computation of the above credibility is based on credibility inversion theorem. It is to be noted that credibility value of the fuzzy variable lying in a rejection region under the alternative credibility hypothesis needs to be maximum, because it is associated with a correct action, namely the action of rejecting the null credibility hypothesis when the alternative is true. Further such a credibility value computed under the null credibility hypothesis should be small, because the corresponding event is an incorrect action.

Best Credibility Test: The rule that helps the practitioner to identify best credibility rejection region is known as best credibility test.

Membership Ratio criterion: It is pertinent to note that the definitions stated above are nothing but the credibility theory analogues of the definitions available in Neyman-Pearson approach related to the classical theory of testing of statistical hypothesis. Hence, it is decided to develop a method for determining a best credibility test. Following the lines of Neyman-Pearson we suggest a criterion for testing a null credibility hypothesis against an alternative credibility hypothesis using the membership functions corresponding to the credibility distributions mentioned under the hypotheses as given below.

For testing the null credibility hypothesis, $H : \xi$ has credibility distribution ξ_0 against the alternative credibility hypothesis $K : \xi$ has credibility distribution ξ_1, the membership ratio criterion is "*Reject the null credibility hypothesis if the observed value of the fuzzy variable $\xi \in C$ where $C = \{x \mid \mu_2(x)/\mu_1(x) > k\}$, μ_1 and μ_2 are the membership functions corresponding to the credibility distributions ξ_0 and ξ_1 ; k being a constant selected so that $Cr(\xi \in C \mid H) = \alpha$.*"

The set of values satisfying the inequality used in C as stated above have relatively higher values with respect to the membership function μ_2 defined under the alternative credibility hypothesis when compared to the function μ_1 given under the null credibility hypothesis. Hence, the credibility of the event defined by such observations will be relatively higher under alternative credibility hypothesis when compared to the same under null credibility hypothesis. Therefore, when the observed value of the fuzzy variable is a member of the set defined by the membership ratio criterion, the decision of rejecting the null credibility hypothesis and accepting the alternative is more meaningful. The choice of the constant k is governed by the value of $Cr(\xi \in C \mid H)$.

It is interesting to note that the above criterion have certain characteristics that are not in line with those possessed by Neyman-Pearson lemma. The test identified by the above criterion need not be unique as observed in the later part of this work. Further, the absence of additive property of the credibility measure makes it difficult to give a general proof for establishing the optimal property (Best Credibility Rejection Region) of the test obtained using the above criterion. Hence, it becomes necessary to make studies on the properties of the test derived using the above criterion for specific distributions. Even though the membership ratio criterion can be applied to any credibility distribution we confine ourselves to the triangular credibility distributions. The following section of this paper studies certain aspects related to the existence of best credibility test for testing hypotheses about triangular credibility distribution.

4 Test for Triangular Credibility Distribution

The credibility distribution function of the triangular fuzzy variable $\xi \sim Tri(a,b,c)$ is

$$\Phi(x)=\begin{cases}0 & \text{if } x\le a\\ \dfrac{x-a}{2(b-a)} & \text{if } a\le x\le b\\ \dfrac{x+c-2b}{2(c-b)} & \text{if } b\le x\le c\\ 1 & \text{if } x\ge c.\end{cases}$$

In this section, we consider the problem of developing best credibility rejection region for testing $H:\xi \sim Tri(a_1,b_1,c_1)$ against the alternative credibility hypothesis $K:\xi \sim Tri(a_2,b_2,c_2)$ for specific choices of the parameters. Before considering the question of developing best credibility test for fuzzy triangular distributions, we list below certain important propositions associated with triangular credibility distribution $Tri(a,b,c)$. These propositions are needed in studying the properties of the test obtained using membership ratio criterion.

Proposition 4.1: If $\xi \sim Tri(a,b,c)$ and $\alpha<1/2$, then all intervals of the form $(x,a+2\alpha(b-a))$, $x>a$ will have credibility α.

Proof: Consider the interval (x_0,x_1) where $a<x_0<x_1<b$. The credibility of the event that the fuzzy variable ξ belongs to the interval is given by

$$Cr\left[\xi\in(x_0,x_1)\right]=\frac{1}{2}\left[\underset{x\in(x_0,x_1)}{Sup}\mu(x)+1-\underset{x\notin(x_0,x_1)}{Sup}\mu(x)\right]=\frac{1}{2}\left[\frac{x_1-a}{b-a}+1-1\right]\quad(\because x_1<b).$$

If $Cr\left[\xi\in(x_0,x_1)\right]=\alpha$ then

$$\frac{1}{2}\left[\frac{x_1-a}{b-a}\right]=\alpha.$$

Solving for x_1 we get $x_1=a+2\alpha(b-a)$.

Since the expression for $Cr\left[\xi\in(x_0,x_1)\right]$ is free from x_0 of the stated choice, we conclude that all intervals of the form $(x,a+2\alpha(b-a))$, $x>a$ will have credibility α. ■

It may be noted that $(a,a+2\alpha(b-a))$ is the longest interval with credibility α that lies to the left of b.

Proposition 4.2: If $\xi \sim Tri(a,b,c)$ and $\alpha<1/2$, then all intervals of the form $(a+2\alpha(b-a),x)$, $x<b$ will have the credibility greater than α.

Proof: Let $x_0>a+2\alpha(b-a)$ and $x_1<b$. Then the credibility of fuzzy variable ξ lies in the interval (x_0,x_1) is

$$Cr\left[\xi\in(x_0,x_1)\right]=\frac{1}{2}\left[\underset{x\in(x_0,x_1)}{Sup}\mu(x)+1-\underset{x\notin(x_0,x_1)}{Sup}\mu(x)\right]=\frac{1}{2}\frac{x_1-a}{b-a}.$$

Since $x_1>a+2\alpha(b-a)$, we get $Cr\left[\xi\in(x_0,x_1)\right]>\alpha$.

Since the expression for $Cr\left[\xi\in(x_0,x_1)\right]$ is free from x_0 of the stated choice, we conclude that all intervals of the form $(a+2\alpha(b-a),x)$, $x<b$ will have credibility greater than α. ■

Proposition 4.3: If $\xi \sim Tri(a,b,c)$ and $\alpha<1/2$, then all intervals of the form $(c-2\alpha(c-b),x)$, $x<c$ will have the credibility α.

Proof: For any interval (x_0,x_1) where $b<x_0<x_1<c$,

$$Cr\left[\xi \in (x_0,x_1)\right] = \frac{1}{2}\left[\underset{x\in(x_0,x_1)}{Sup}\mu(x) + 1 - \underset{x\notin(x_0,x_1)}{Sup}\mu(x)\right] = \frac{1}{2}\left[\frac{c-x_0}{c-b} + 1 - 1\right].$$

If $Cr\left[\xi \in (x_0,x_1)\right] = \alpha$ then $x_0 = c - 2\alpha(c-b)$. Since the expression for $Cr\left[\xi \in (x_0,x_1)\right]$ is free from x_1 of the stated choice, we conclude that all intervals of the form $\left(c-2\alpha(c-b), x\right)$, $x < c$ will have credibility α. ■

Note that $\left(c-2\alpha(c-b), c\right)$ is the largest set in the region to the right of b which has credibility α.

Proposition 4.4: All intervals of the form $\left(x, c-2\alpha(c-b)\right), x > b$ where $\alpha < 1/2$, will have the credibility greater than α.

Proof is straightforward and hence omitted. ■

Proposition 4.5: All intervals of the form (x_0, x_1), $x_0 \in (a,b)$ and $x_1 \in (b,c)$ have credibility value greater than $1/2$.

Proof: The credibility of ξ assuming values in (x_0,x_1), where $x_0 \in (a,b)$ and $x_1 \in (b,c)$ is

$$\begin{aligned}
Cr\left[\xi \in (x_0,x_1)\right] &= \frac{1}{2}\left[\underset{x\in(x_0,x_1)}{Sup}\mu(x) + 1 - \underset{x\notin(x_0,x_1)}{Sup}\mu(x)\right] \\
&= \frac{1}{2}\left[1 + 1 - \max\left\{\mu(x_0), \mu(x_1)\right\}\right] \\
&= 1 - \frac{1}{2}\max\left\{\frac{x_0-a}{b-a}, \frac{c-x_1}{c-b}\right\} \\
&= 1 - \frac{1}{2}(<1) \\
&> \frac{1}{2}.
\end{aligned}$$

Hence the proposition is proved. ■

Consider the problem of testing the null credibility hypothesis $H: \xi \sim Tri(a_1,b_1,c_1)$ against the alternative credibility hypothesis $K: \xi \sim Tri(a_2,b_2,c_2)$ where $a_1 < a_2$ by using the membership ratio criterion defined in the previous section. The rejection region can be identified by the membership ratio criterion on making use of the nature of the membership ratio. The nature of the membership ratio depends on the relative positions of the parameters used in the triangular credibility distributions defined under null and alternative credibility hypotheses. Table 1 gives ten possible cases that arise when we take into account the ordering of the parameters involved in the credibility distributions.

Table1: Possible ordering of parameters

Case	Ordering of the parameters
1	$a_1 < b_1 < c_1 < a_2 < b_2 < c_2$
2	$a_1 < b_1 < a_2 < c_1 < b_2 < c_2$
3	$a_1 < b_1 < a_2 < b_2 < c_1 < c_2$
4	$a_1 < b_1 < a_2 < b_2 < c_2 < c_1$
5	$a_1 < a_2 < b_1 < c_1 < b_2 < c_2$
6	$a_1 < a_2 < b_1 < b_2 < c_1 < c_2$
7	$a_1 < a_2 < b_1 < b_2 < c_2 < c_1$
8	$a_1 < a_2 < b_2 < b_1 < c_1 < c_2$
9	$a_1 < a_2 < b_2 < b_1 < c_2 < c_1$
10	$a_1 < a_2 < b_2 < c_2 < b_1 < c_1$

It is interesting to observe that in the cases, 1,2,3,5 and 6 the inequalities $a_1 < a_2, b_1 < b_2$ and $c_1 < c_2$ are satisfied. Whenever the parameters in the two fuzzy variables $Tri(a_1,b_1,c_1)$ and $Tri(a_2,b_2,c_2)$ satisfy these conditions we call one of them as a *right shift* of the other. Formally, we define the *right shift* of a fuzzy triangular variable as follows.

Right Shift: The triangular fuzzy variable $Tri(a_2, b_2, c_2)$ is said to be "*right shift*" of another triangular fuzzy variable $Tri(a_1, b_1, c_1)$ if $a_1 < a_2, b_1 < b_2$ and $c_1 < c_2$.

Similar definition can be given for *left shift* of a triangular fuzzy variable as given below.

Left Shift: The triangular fuzzy variable $Tri(a_2, b_2, c_2)$ is said to be "*left shift*" of another triangular fuzzy variable $Tri(a_1, b_1, c_1)$ if $a_2 < a_1, b_2 < b_1$ and $c_2 < c_1$.

Table 2 gives the values of the ratio $\mu_2(x)/\mu_1(x)$ with respect to right shifted triangular credibility distributions. The first column gives different situations and the second column gives the values of the ratio $\mu_2(x)/\mu_1(x)$. It is easy to verify that in all these five cases, the ratio is non decreasing in *x*.

Table 2: Values of membership ratio for right shifted triangular distributions

Case	$\frac{\mu_2(x)}{\mu_1(x)}$
$a_1 < b_1 < c_1 < a_2 < b_2 < c_2$	$\frac{\mu_2(x)}{\mu_1(x)} = \begin{cases} 0 & \text{if } a_1 < x < c_1 \\ \infty & \text{if } a_2 < x < c_2 \end{cases}$
$a_1 < b_1 < a_2 < c_1 < b_2 < c_2$	$\frac{\mu_2(x)}{\mu_1(x)} = \begin{cases} 0 & \text{if } a_1 < x < a_2 \\ \frac{x - a_2}{c_1 - x} \frac{c_1 - b_1}{b_2 - a_2} & \text{if } a_2 < x < c_1 \\ \infty & \text{if } c_1 < x < c_1 \end{cases}$
$a_1 < b_1 < a_2 < b_2 < c_1 < c_2$	$\frac{\mu_2(x)}{\mu_1(x)} = \begin{cases} 0 & \text{if } a_1 < x < a_2 \\ \frac{x - a_2}{c_1 - x} \frac{c_1 - b_1}{b_2 - a_2} & \text{if } a_2 < x < b_2 \\ \frac{c_2 - x}{c_1 - x} \frac{c_1 - b_1}{c_2 - b_2} & \text{if } b_2 < x < c_1 \\ \infty & \text{if } c_1 < x < c_2 \end{cases}$
$a_1 < a_2 < b_1 < c_1 < b_2 < c_2$	$\frac{\mu_2(x)}{\mu_1(x)} = \begin{cases} 0 & \text{if } a_1 < x < a_2 \\ \frac{x - a_2}{x - a_1} \frac{b_1 - a_1}{b_2 - a_2} & \text{if } a_2 < x < b_1 \\ \frac{x - a_2}{c_1 - x} \frac{c_1 - b_1}{b_2 - a_2} & \text{if } b_1 < x < c_1 \\ \infty & \text{if } c_1 < x < c_2 \end{cases}$
$a_1 < a_2 < b_1 < b_2 < c_1 < c_2$	$\frac{\mu_2(x)}{\mu_1(x)} = \begin{cases} 0 & \text{if } a_1 < x < a_2 \\ \frac{x - a_2}{x - a_1} \frac{b_1 - a_1}{b_2 - a_2} & \text{if } a_2 < x < b_1 \\ \frac{x - a_2}{c_1 - x} \frac{c_1 - b_1}{b_2 - a_2} & \text{if } b_1 < x < b_2 \\ \frac{c_2 - x}{c_1 - x} \frac{c_1 - b_1}{c_2 - b_2} & \text{if } b_2 < x < c_1 \\ \infty & \text{if } c_1 < x < c_2 \end{cases}$

The following theorem gives the credibility rejection region obtained on using membership ratio criterion and proves that the resulting rejection is the best credibility rejection region with significance level α when the alternative credibility hypothesis specifies a triangular credibility distribution which is a *right shift* of the triangular credibility distribution specified under null credibility hypothesis for $\alpha < 1/2$.

Theorem: 4.1 If $Tri(a_2, b_2, c_2)$ is a *right shift* of $Tri(a_1, b_1, c_1)$, then for testing the null credibility hypothesis $H : \xi \sim Tri(a_1, b_1, c_1)$ against the alternative credibility hypothesis $K : \xi \sim Tri(a_2, b_2, c_2)$,

(a) membership ratio criterion credibility rejection region of level α is given by $C^* = \{x \mid x > c_1 - 2\alpha(c_1 - b_1)\}$ and

(b) the credibility rejection region $C^* = \{x \mid x > c_1 - 2\alpha(c_1 - b_1)\}$ is the best credibility rejection region of level α.

Proof: (a) From Table 2, it is observed that whenever the credibility distribution $Tri(a_2,b_2,c_2)$ is a *right shift* of the distribution $Tri(a_1,b_1,c_1)$, the ratio $\mu_2(x)/\mu_1(x)$ is non decreasing in *x*. Hence the set of values obtained using the membership ratio criterion given by

$$\left\{x \mid \frac{\mu_2(x)}{\mu_1(x)} > k\right\}$$

reduces to $C^* = \{x \mid x > c_0\}$ where c_0 is selected such that

$$Cr\left(\xi \in C^* \mid H\right) = \alpha.$$

This implies

$$\frac{1}{2}\left[\underset{x>c_0}{Sup}\,\mu_1(x) + 1 - \underset{x\le c_0}{Sup}\,\mu_1(x)\right] = \alpha.$$

Hence

$$\frac{1}{2}\left[\frac{c_1 - c_0}{c_1 - b_1}\right] = \alpha.$$

Solving for c_0, we get

$$c_0 = c_1 - 2\alpha(c_1 - b_1). \quad (4.1)$$

Thus we have proved (a).

(b) In order to prove that $C^* = \{x \mid x > c_0\}$ with c_0 as defined in (4.1) is the best credibility rejection region of level α, we must show that

$$Cr\left(\xi \in C^* \mid K\right) \ge Cr\left(\xi \in C \mid K\right) \quad (4.2)$$

where C satisfies $Cr\left(\xi \in C \mid H\right) \le \alpha.$

Note that the value of $Cr\left(\xi \in C^* \mid K\right)$ depends on the position of the cutoff point c_0. It can lie either in the interval $\left(a_2, b_2\right)$ or in $\left(b_2, c_2\right)$. The two cases are considered below.

Case 1: $c_0 \in \left(a_2, b_2\right)$

In this case,

$$Cr\left(\xi \in C^* \mid K\right) = Cr\left(\xi > c_0 \mid K\right) = \frac{1}{2}\left[1 + 1 - \frac{c_0 - a_2}{b_2 - a_2}\right]. \quad (4.3)$$

Therefore,

$$Cr\left(\xi \in C^* \mid K\right) = 1 - \frac{1}{2}\left[\frac{c_0 - a_2}{b_2 - a_2}\right]. \quad (4.4)$$

Now we shall prove (4.2) by considering C of different forms as identified in F_1, F_2 and the unions of members of F_1 and/or F_2 where

$$F_1 = \{x \mid x \in (c_0, x_0), x_0 \le c_1\} \quad (4.5)$$

and

$$F_2 = \{x \mid x \in (x_0, a_0), x_0 \ge a_1\}. \quad (4.6)$$

with $c_0 = c_1 - 2\alpha(c_1 - b_1)$ and $a_0 = a_1 + 2\alpha(b_1 - a_1)$.

Consider the case where $C \in F_1 = \{x \mid x \in (c_0, x_0), x_0 \le c_1\}$.

It may be noted that two cases arise regarding the position of x_0 when the credibility value is computed under K, namely, $x_0 \in (a_2, b_2)$ and $x_0 \in (b_2, c_2)$.

Consider the case, where $x_0 \in (a_2, b_2)$ as shown in the Figure 4.1. Here,

$$Cr\left(\xi \in C \mid K\right) = \frac{1}{2}\left[\underset{x\in(c_0,x_0)}{Sup}\,\mu_2(x) + 1 - \underset{x\notin(c_0,x_0)}{Sup}\,\mu_2(x)\right] = \frac{1}{2}\left[\frac{x_0 - a_2}{b_2 - a_2}\right].$$

Therefore,

$$Cr\left(\xi \in C^* \mid K\right) - Cr\left(\xi \in C \mid K\right) = 1 - \frac{x_0 + c_0 - 2a_2}{2(b_2 - a_2)} \geq 0.$$

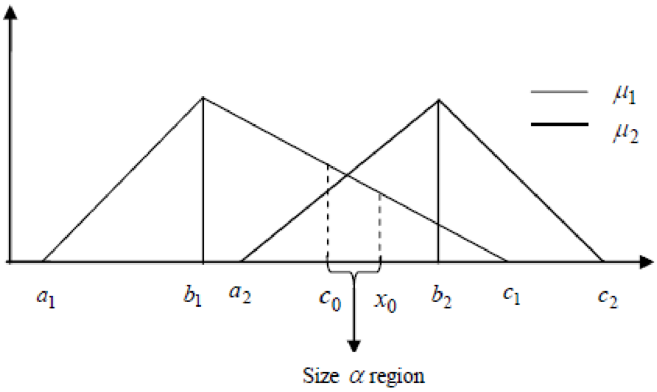

Figure 4.1: $x_0, c_0 \in \left(a_2, b_2\right)$

On the other hand, if $x_0 \in (b_2, c_2)$ as shown in Figure 4.2, we have

$$\begin{aligned} Cr\left(\xi \in C \mid K\right) &= \frac{1}{2}\left[\underset{x \in (c_0, x_0)}{Sup} \mu_2(x) + 1 - \underset{x \notin (c_0, x_0)}{Sup} \mu_2(x)\right] \\ &= 1 - \frac{1}{2}\max\left\{\frac{c_0 - a_2}{b_2 - a_2}, \frac{c_2 - x_0}{c_2 - b_2}\right\} \\ &= \begin{cases} 1 - \dfrac{1}{2}\dfrac{c_0 - a_2}{b_2 - a_2}, & \text{if } \dfrac{c_0 - a_2}{b_2 - a_2} > \dfrac{c_2 - x_0}{c_2 - b_2} \\ 1 - \dfrac{1}{2}\dfrac{c_2 - x_0}{c_2 - b_2}, & \text{if } \dfrac{c_2 - x_0}{c_2 - b_2} > \dfrac{c_0 - a_2}{b_2 - a_2} \end{cases} \end{aligned} \quad (4.7)$$

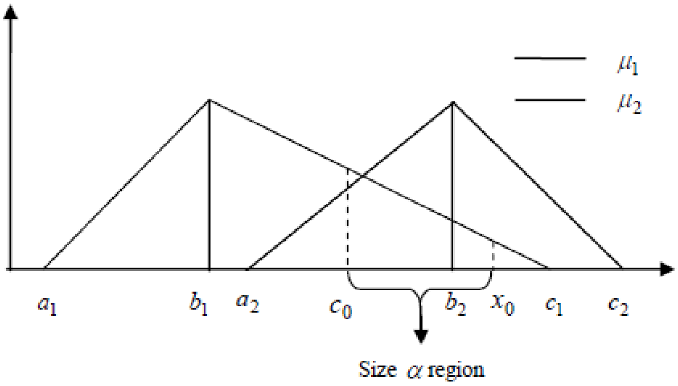

Figure 4.2: $x_0 \in (b_2, c_2),\ \ c_0 \in (a_2, b_2)$

Using (4.4) and (4.7) we get

$$Cr\left(\xi \in C^* \mid K\right) - Cr\left(\xi \in C \mid K\right) = \begin{cases} 0 & \text{if } \dfrac{c_0 - a_2}{b_2 - a_2} > \dfrac{c_2 - x_0}{c_2 - b_2} \\ \dfrac{1}{2}\left(\dfrac{c_2 - x_0}{c_2 - b_2} - \dfrac{c_0 - a_2}{b_2 - a_2}\right) & \text{if } \dfrac{c_2 - x_0}{c_2 - b_2} > \dfrac{c_0 - a_2}{b_2 - a_2}. \end{cases}$$

Thus we have proved irrespective of the position of x_0, for every $C \in F_1$,

$$Cr\left(\xi \in C^* \mid K\right) \geq Cr\left(\xi \in C \mid K\right).$$

Now we shall prove the inequality (4.2) by considering credibility rejection regions $C \in F_2$ where $F_2 = \{x \mid x \in (x_0, a_0), x_0 \geq a_1\}$.

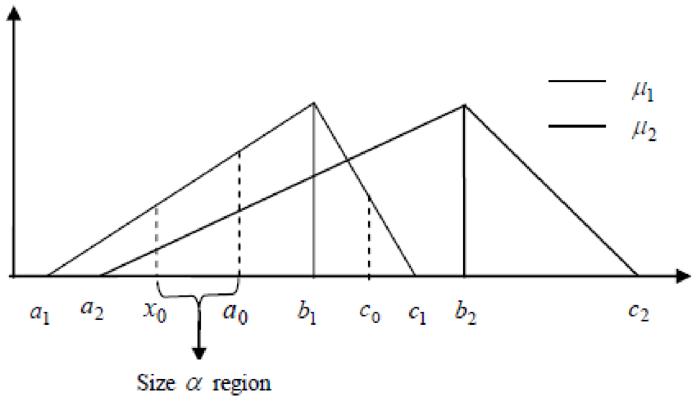

Figure 4.3: $x_0, a_0, c_0 \in (a_2, b_2)$

If $x_0 \in (a_2, b_2)$ as shown in the Figure 4.3 then we have

$$Cr(\xi \in C \mid K) = \frac{1}{2}\left[\underset{x\in(x_0,a_0)}{Sup}\ \mu_2(x) + 1 - \underset{x\notin(x_0,a_0)}{Sup}\ \mu_2(x)\right] = \frac{1}{2}\left[\frac{a_0 - a_2}{b_2 - a_2}\right].$$

Hence

$$Cr(\xi \in C^* \mid K) - Cr(\xi \in C \mid K) = 1 - \frac{1}{2}\left(\frac{c_0 - a_2}{b_2 - a_2} + \frac{a_0 - a_2}{b_2 - a_2}\right).$$

Therefore,

$$Cr(\xi \in C^* \mid K) - Cr(\xi \in C \mid K) \geq 0.$$

The case $x_0 \in (b_2, c_2)$ will not arise because $x_0 < a_0$ and $a_0 \in (a_1, b_1)$. Thus we have proved (4.2) for all $C \in F_2$.

By the maximum property of credibility measure, we observe that when the rejection region C of size α is constructed by taking the union of two or more regions belonging to F_1 and /or F_2, the inequality

$$Cr(\xi \in C^* \mid K) \geq Cr(\xi \in C \mid K)$$

continues to hold good.

Thus we have proved under all possible scenarios existing under case 1,

$$Cr(\xi \in C^* \mid K) \geq Cr(\xi \in C \mid K).$$

Case 2: $c_0 \in (b_2, c_2)$

In this case,

$$Cr(\xi \in C^* \mid K) = Cr(\xi > c_0 \mid K) = \frac{1}{2}\left[\underset{x>c_0}{Sup}\,\mu_2(x) + 1 - \underset{x\leq c_0}{Sup}\,\mu_2(x)\right] = \frac{1}{2}\left[\frac{c_2 - c_0}{c_2 - b_2}\right]. \tag{4.8}$$

Therefore,

$$Cr(\xi \in C^* \mid K) = \frac{1}{2}\left[\frac{c_2 - c_0}{c_2 - b_2}\right].$$

Now we shall prove the inequality (4.2) by considering different possibilities as in the previous case.

Let $C \in F_1$ where $F_1 = \{x \mid x \in (c_0, x_0), x_0 \leq c_1\}$. If $x_0 \in (b_2, c_2)$ as shown in Figure 4.4 then we have

$$Cr(\xi \in C \mid K) = \frac{1}{2}\left[\underset{x\in(c_0,x_0)}{Sup}\ \mu_2(x) + 1 - \underset{x\notin(c_0,x_0)}{Sup}\ \mu_2(x)\right] = \frac{1}{2}\left(\frac{c_2 - c_0}{c_2 - b_2}\right).$$

Hence $Cr(\xi \in C^* \mid K) - Cr(\xi \in C \mid K) = 0$. Thus we have proved (4.2) is satisfied with equality sign. The case $x_0 \in (a_2, b_2)$ does not arise in this situation.

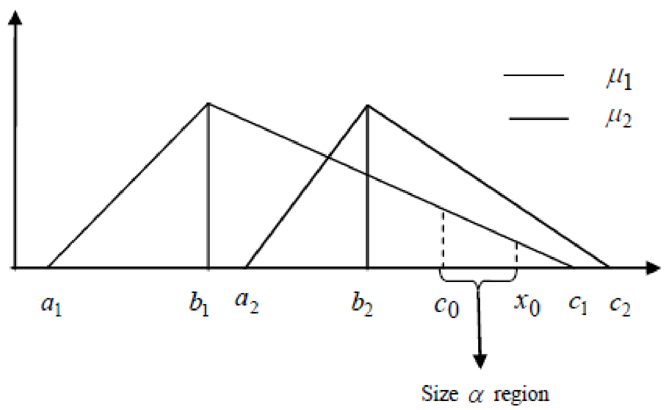

Figure 4.4: $x_0, c_0 \in (b_2, c_2)$

Let $C \in F_2$ where $F_2 = \{x \mid x \in (x_0, a_0), x_0 \geq a_1\}$. If $x_0 \in (a_2, b_2)$ as shown in the Figure 4.5, then

$$Cr(\xi \in C \mid K) = \frac{1}{2}\left[\sup_{x \in (x_0, a_0)} \mu_2(x) + 1 - \sup_{x \notin (x_0, a_0)} \mu_2(x)\right] = \frac{1}{2}\left[\frac{a_0 - a_2}{b_2 - a_2}\right].$$

By Propositions 1 and 3, we have

$$\frac{a_0 - a_1}{b_1 - a_1} = \frac{c_1 - c_0}{c_1 - b_1} = 2\alpha.$$

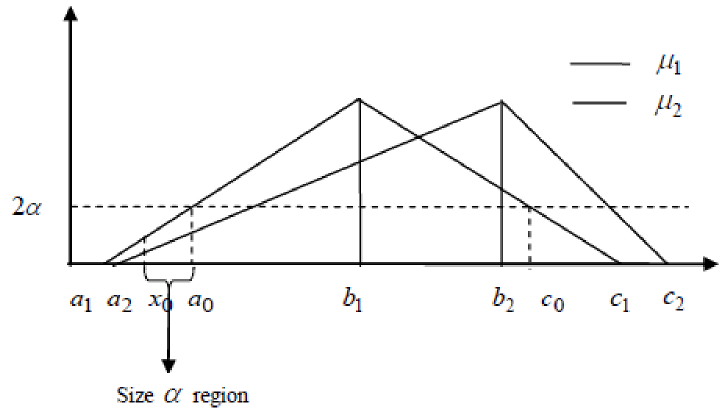

Figure 4.5: $x_0 \in (a_2, b_2), c_0 \in (b_2, c_2)$

Under the condition $a_1 < a_2, b_1 < b_2$ and $c_1 < c_2$,

$$\mu_2(a_0) = \frac{a_0 - a_2}{b_2 - a_2} < \frac{a_0 - a_1}{b_1 - a_1} = 2\alpha \text{ (refer Figure 4.5)}$$

which gives

$$\frac{1}{2}\frac{a_0 - a_2}{b_2 - a_2} < \alpha.$$

Therefore,

$$Cr(\xi \in C \mid K) = \frac{1}{2}\frac{a_0 - a_2}{b_2 - a_2} < \alpha.$$

Using a similar argument, we get

$$Cr(\xi \in C^* \mid K) = \frac{1}{2}\frac{c_2 - c_0}{c_2 - b_2} > \alpha.$$

Hence $Cr\left(\xi \in C^* \mid K\right) - Cr\left(\xi \in A^* \mid K\right) \geq 0.$

As pointed out in the previous case, by the maximality property of Credibility measure when the rejection region C of size α is constructed by taking the union of two or more regions belonging to F_1 and /or F_2, the inequality $Cr\left(\xi \in C^* \mid K\right) \geq Cr\left(\xi \in C \mid K\right)$ continues to hold good.

Thus we have proved (b) of the theorem. ■

Proceeding as in the case of Theorem 4.1, for the case of *left shift* triangular credibility distributions, the best credibility rejection region can be developed easily. The region identified by the membership ratio criterion and its optimal property are stated in the theorem 4.2 without proof.

Theorem: 4.2 If $Tri(a_2,b_2,c_2)$ is a *left shift* of $Tri(a_1,b_1,c_1)$, then for testing the null credibility hypothesis $H:\xi \sim Tri(a_1,b_1,c_1)$ against the alternative credibility hypothesis $K:\xi \sim Tri(a_2,b_2,c_2)$,

(a) membership ratio criterion credibility rejection region of level α is given by $C^* = \left\{x \mid x < a_1 + 2\alpha\left(b_1 - a_1\right)\right\}$ and

(b) the credibility rejection region $C^* = \left\{x \mid x < a_1 + 2\alpha\left(b_1 - a_1\right)\right\}$ is the best credibility rejection region of level α.

Illustration

It is believed that the menstrual cycle length of young women is lunar cycle length. However, it is suspected that the change in life style and living conditions have affected the menstrual cycle length in a group of women. A gynecologist believes that the cycle length of women in the group under study has triangular credibility distributions $Tri(29,31,33)$ instead of triangular credibility distribution $Tri(26,28,30)$ which supports the lunar cycle length belief. In order to choose a proper set of values it is decided to treat this as a credibility hypotheses testing problem where the null and alternative hypotheses are $H:\xi \sim Tri(26,28,30)$ and $K:\xi \sim Tri(29,31,33)$ with significance level 0.05.

Here we have $a_1 = 26, b_1 = 28, c_1 = 30, a_2 = 29, b_2 = 31$ and $c_2 = 33$. For the given data we note that the distribution mentioned under $K:\xi \sim Tri(29,31,33)$ is a *right shift* of $H:\xi \sim Tri(26,28,30)$. Further, $c_1 - 2\alpha\left(c_1 - b_1\right) = 29.80$. Hence by Theorem 4.1, we get the membership ratio criterion test as "*Reject the null credibility hypothesis if the observed value of the fuzzy variable is an element of the set* $\left\{x \mid x > 29.80\right\}$." Hence, the best credibility rejection region for the given testing problem is $C^* = \left\{x \mid x > 29.80\right\}$.

For example, if it is observed that the menstrual cycle length of a young woman is 30 then we reject the null credibility hypothesis and accept the alternative credibility hypothesis with level 0.05. That is, we conclude that the belief of the gynecologist can be accepted with level 0.05.

Comparison of Best Credibility Region with Other Region with Same Levels

In the above example, the power of the test defined by the best credibility critical region under the alternative credibility hypothesis is

$$Cr\left(\xi \in C^* \mid K\right) = \frac{1}{2}\left(1 + 1 - \frac{29.80 - 29}{31 - 29}\right) = 1 - \frac{0.8}{4} = 0.8.$$

Now consider the set $A = \left\{x \mid x < 26.1\right\} \cup \left\{x \mid x > 30\right\}$. Evidently the set A satisfies the condition $Cr\left(\xi \in A \mid H\right) \leq 0.05$ and hence it is a member of the regions of level 0.05. The power under alternative of the set $A = \left\{x \mid x < 26.1\right\} \cup \left\{x \mid x > 30\right\}$ is

$$Cr\left(\xi \in A \mid K\right) = 0 \wedge \frac{1}{2}\left(1 + 1 - \frac{30 - 29}{31 - 29}\right) = 1 - \frac{1}{4} = 0.75$$

which is obviously smaller when compared to $Cr(\xi \in \left\{x \mid x > 29.80\right\} \mid K) = 0.80$. This clearly supports the use of the test defined by the best credibility test.

5 Conclusion

In the previous section of this paper, best credibility rejection regions have been identified for testing credibility hypotheses related to fuzzy triangular distributions where the distribution under the alternative credibility hypothesis is a *right (left) shift* of the triangular credibility distribution under null credibility hypothesis. Further, it is important to note that the membership ratios corresponding to the cases 4, 7, 9 and 10 listed in Table 1 do not possess the monotonicity. The ratios are concave downwards and hence the credibility rejection regions identified by the membership ratio criterion will be of the form $\{x \mid x \in (\lambda_1, \lambda_2)\}$ where λ_1 and λ_2 are constants selected such that the credibility rejection region has size α. In such cases, the credibility rejection region of the given size α can be determined in more than one way. The following are various choices for λ_1 and λ_2 yielding credibility rejection regions of size α.

(i) $\lambda_1 = a_1, \lambda_2 = a_0$

(ii) $\lambda_1 = c_0, \lambda_2 = c_1$

(iii) $\lambda_1 = x_0, \lambda_2 = a_0, x_0 < a_0$

(iv) $\lambda_1 = c_0, \lambda_2 = x_0, x_0 > c_0$

where $a_0 = a_1 + 2\alpha(b_1 - a_1)$ and $c_0 = c_1 - 2\alpha(c_1 - b_1)$ are as identified in Theorem 4.1. Existence of more than one set of values for λ_1 and λ_2 makes the process of identifying the best credibility rejection region a difficult one. Further, in case 8 of Table 1, the ratio has more than one turning point which complicates the process of identifying the rejection region. Hence further studies are needed for developing optimal tests in these cases. The tests developed in this paper are meant only for testing a simple null credibility hypothesis against a simple alternative credibility hypothesis. That is, the underlying hypothesis is about only one distribution. Hence, there is a good scope for future work in developing test procedures when more than one distribution is involved in the credibility hypothesis being considered.

The entire paper is focused only on triangular credibility distribution. The authors are working on other types of credibility distributions like, exponential, normal and trapezoidal as well.

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