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Generalized $\mathcal{N}$-Ideals of Subtraction Algebras

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Abstract

In this paper, we introduce the notions of $(\in, \in \vee q)$-$\mathcal{N}$-ideals and (γ, δ)-$\mathcal{N}$-ideals of subtraction algebras, which are generalization of $\mathcal{N}$-ideals. Some characterization for these generalized $\mathcal{N}$-ideals are derived. Also, we introduce the notions of $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideals and $\mathcal{N}$-ideals with thresholds of subtraction algebras. The characterization of $\mathcal{N}$-ideals with thresholds by their closed (f,t)-*cut* ideals over the subtraction algebras X is also established.

Keywords: subtraction algebra, $\mathcal{N}$-structure, $\mathcal{N}$-ideal, generalization of $\mathcal{N}$-ideal, $\mathcal{N}$-ideal with threshold

1 Introduction

The systems of the form Φ, where $(\Phi; \circ, \backslash)$, considered by Schein [19], is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka [23] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Jun et al. [12] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [11], Jun and Kim established the ideal generated by a set, and discussed related results.

The theory of fuzzy sets which was introduced by Zadeh [22] is applied to many mathematical branches. Rosenfeld [18] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups. Further, Lee et al. [16] introduced the notion of fuzzy ideals in subtraction algebras and discussed characterization of fuzzy ideals. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [17] played a vital role to generate some different types of fuzzy subgroups. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced in earlier paper [4, 5] by using the combined notions of belongingness and quasi-coincidence of fuzzy point and fuzzy set. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfelds fuzzy subgroup. This concept has been studied further in [2, 3, 7, 8, 6, 15, 18, 20]. In [21], Yuan et al. introduced the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfelds fuzzy subgroup and Bhakat and Dass fuzzy subgroup. In [13], Jun and Song discussed some fundamental aspects of $(\in, \in \vee q)$-fuzzy interior ideals. They showed that $(\in, \in \vee q)$-fuzzy interior ideals are generalization of the existing concepts fuzzy interior ideals. Now, it is natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures.

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to 0, 1$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point 1 into the interval $[0, 1]$. Because no negative

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meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [10, 14] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. Further, he applied this $\mathcal{N}$-structure theory in subtraction algebra and BCK/BCI-algebra and studied their related properties.

Our aim in this paper is to introduce and study a new type of $\mathcal{N}$-ideals of a subtraction algebra, called $(\in, \in \vee q)$-$\mathcal{N}$-ideals. Also, we have proved that an $\mathcal{N}$-structure (X, f) over a subtraction algebra X is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X if and only if $C(f;t)(\neq \emptyset)$ is an ideal of subtraction algebra X for all $t \in [-0.5, 0)$. These showed that $(\in, \in \vee q)$-$\mathcal{N}$-ideals are generalization of the existing concepts of $\mathcal{N}$-ideals. More over, the concepts of (γ, δ)-$\mathcal{N}$-ideal, $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal and $\mathcal{N}$-ideal with thresholds are introduced and some interesting properties are investigated.

2 Preliminary

In this section, we cite the fundamental definitions that will be used in the sequel:

Definition 2.1. ([12]) A non-empty set X together with a binary operation "–" is said to be a *subtraction algebra* if it satisfies the following:
(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$, for all $x, y, z \in X$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on $X : a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$; then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c)). \end{aligned}$$

In a subtraction algebra, the following are true (see [13]):

(a1) $(x - y) - y = x - y$,

(a2) $x - 0 = x$ and $0 - x = 0$,

(a3) $(x - y) - x = 0$,

(a4) $x - (x - y) \leq y$,

(a5) $(x - y) - (y - x) = x - y$,

(a6) $x - (x - (x - y)) = x - y$,

(a7) $(x - y) - (z - y) \leq x - z$,

(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$,

(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$,

(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$,

(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$,

(a12) $(x - y) - z = (x - z) - (y - z)$.

Definition 2.2. ([12]) A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:
(b1) $a - x \in A$ for all $a \in A$ and $x \in X$,
(b2) for all $a, b \in A$, whenever $a \vee b$ exists in X, then $a \vee b \in A$.

Proposition 2.3. ([12]) *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies:*
(b3) $0 \in A$,
(b4) $(\forall x \in X)$ $(\forall y \in A)$ $(x - y \in A \Rightarrow x \in A)$.

Proposition 2.4. ([12]) *Let X be a subtraction algebra and $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

Definition 2.5. ([9]) Let X be a subtraction algebra and Y be a nonempty subset of X. Then Y is called a *subalgebra* of X if $x - y \in Y$, whenever $x, y \in Y$.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions, from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

Remarks 1:

- Let f be an $\mathcal{N}$-functions on X and $(X,\ f)$ be an $\mathcal{N}$-structure over a subtraction algebra X.
- For any $\mathcal{N}$-function f on X and $t \in [-1, 0)$, the set $C(f; t) := \{x \in X \mid f(x) \leq t\}$ is called a *closed* (f, t)-*cut* of (X, f).

Definition 2.6. ([10]) By *an ideal (resp. subalgebra)* of X based on $\mathcal{N}$-function f (briefly, $\mathcal{N}$*-ideal (resp. $\mathcal{N}$-subalgebra)* of X), we mean an $\mathcal{N}$-structure (X, f) in which every nonempty closed $(f, t) - cut$ of (X, f) is an ideal (resp. *subalgebra*) of X, for all $t \in [-1, 0)$.

Theorem 2.7. ([10]) *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if it satisfies the following assertions:*
$(NI1)$ $(x, y \in X)$ $(f(x - y) \leq f(x))$,
$(NI2)$ $(x, y \in X)$ $(\exists (x \vee y) \Rightarrow f(x \vee y) \leq max\{f(x), f(y)\})$.

Theorem 2.8. *Let X be a subtraction algebra and $(X,\ f)$ be an $\mathcal{N}$-ideal of X. Then the closed (f, t)-cut $C(f; t)(\neq \emptyset)$ is an ideal of X, for all $t \in [-1, 0)$ if and only if $C(f; t)$ is an $\mathcal{N}$-ideal of X.*

Proof. Let $C(f; t)$ be an $\mathcal{N}$-ideal of X. If $x \in C(f; t), y \in X$ and $t \in [-1, 0)$, then $f(x) \leq t$. Thus $f(x - y) \leq f(x) \leq t$, which implies $x - y \in C(f; t)$. Also, if $x, y \in C(f; t)$ and $t \in [-1, 0)$ then $f(x) \leq t$ and $f(y) \leq t$. Then there exists $x \vee y \in X$ such that $f(x \vee y) \leq \max\{f(x), f(y)\} = \max\{t, t\} = t$, which implies $x \vee y \in C(f; t)$. Hence $C(f; t)$ is an ideal of X.

Conversely, let $C(f; t)$ be an ideal of X. Let if possible, $f(x - y) > f(x)$, for all $x, y \in X$. Choose $t_0 \in [-1, 0)$ such that $f(x - y) > t_0 > f(x)$, which implies $x - y \notin C(f; t_0)$, for all $x \in C(f; t_0)$ and $y \in X$, which is a contradiction. Thus $f(x - y) \leq f(x)$. Also, let if possible, there exists $x \vee y$ such that $f(x \vee y) > \max\{f(x), f(y)\}$, for all $x, y \in X$. Choose $t_1 \in [-1, 0)$ such that $f(x \vee y) > t_1 > f(x) = f(y)$, which implies $x \vee y \notin C(f; t_1)$, for all $x \in C(f; t_1)$ and $y \in C(f; t_1)$, which is a contradiction. Thus $f(x \vee y) \leq \max\{f(x), f(y)\}$. Hence $(X,\ f)$ is an $\mathcal{N}$-ideal of X.

□

3 $(\in, \in \vee\, q)$-$\mathcal{N}$-Ideals of Subtraction Algebras

In what follows, let X denote a subtraction algebra, f be an $\mathcal{N}$-functions on X and $(X,\ f)$ be an $\mathcal{N}$-structure over X unless otherwise specified.

Definition 3.1. An $\mathcal{N}$-function f over a set $(X,\ f)$ of the form

$$f\,(y) = \begin{cases} t \neq 0 & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x. \end{cases}$$

is said to be an $\mathcal{N}$*-point* with support x and value t, is denoted by x_t.

$-$	**0**	**a**	**b**
0	0	0	0
a	a	0	a
b	b	b	0

Table 1: Binary operation 1

Remarks 2:

- An $\mathcal{N}$-point x_t is said to be a *belongs to* in a set $(X,\ f)$, written as $x_t \in f$ if $f(x) \leq t$.
- An $\mathcal{N}$-point x_t is said to be a *quasi-coincident with* in a set $(X,\ f)$, written as $x_t q f$ if $f(x) + t < -1$.
- If $x_t \in f$ or $x_t q f$, then we write $x_t \in \vee q f$.
- If $x_t \in f$ and $x_t q f$, then we write $x_t \in \wedge q f$.
- An $\mathcal{N}$-point x_t is said to be a *not belongs to* in a set $(X,\ f)$, written as $x_t \overline{\in} f$ if $f(x) \nleq t$.
- An $\mathcal{N}$-point x_t is said to be a *not quasi-coincident with* in a set $(X,\ f)$, written as $x_t \overline{q} f$ if $f(x) + t \nless -1$.
- If $x_t \overline{\in} f$ or $x_t \overline{q} f$, then we write $x_t \overline{\in} \vee \overline{q} f$.
- If $x_t \overline{\in} f$ and $x_t \overline{q} f$, then we write $x_t \overline{\in} \wedge \overline{q} f$.
- The symbol $\overline{\in \vee q}$ means neither $\in$ nor q hold.

Definition 3.2. An $\mathcal{N}$-structure $(X,\ f)$ is called an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X if for all $t, r \in [-1, 0)$ and $x, y \in X$,
(GNI1) $(x_t \in f)\ \ ((x-y)_t \in \vee q f)$,
(GNI2) $(x_t, y_r \in f)\ (\ \exists x \vee y \in X \ \Rightarrow\ (x \vee y)_{\min\{t,\, r\}} \in \vee q f)$.

Example 3.3. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be an $\mathcal{N}$-structure defined by:

$$f(0) = -0.7, f(a) = -0.6 \text{ and } f(b) = -0.9.$$

It is easy to check that (X, f) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.

Theorem 3.4. *An $\mathcal{N}$-structure $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X if and only if it satisfies the following assertions:*
(GNI3) $(x, y \in X)\ (f(x-y) \leq max\{f(x), -0.5\})$,
(GNI4) $(x, y \in X)\ (\exists (x \vee y)\ \Rightarrow\ f(x \vee y) \leq max\{f(x), f(y), -0.5\})$.

Proof. Let $x, y \in X$. We consider the following cases: (a) $f(x) > -0.5$, (b) $f(x) \leq -0.5$.

Case (a): Assume that $f(x-y) > \max\{f(x), -0.5\})$, which implies that $(f(x-y) > f(x)$. Choose $t \in [-1, 0)$ such that $f(x-y) > t > f(x)$. Then $x_t, y_t \in f$, but $(x-y)_t \overline{\in \vee q f}$ which is a contradiction.

Case (b): Assume that $f(x-y) > -0.5$. Then $x_{-0.5}, y_{-0.5} \in f$, but $(x-y)_{-0.5} \overline{\in \vee q f}$, which is a contradiction. Thus $f(x-y) \leq \max\{f(x), -0.5\}$.

Now, let $x, y \in x$, there exists $x \vee y \in X$. we consider following cases: (a) $\max\{f(x), f(y)\}) > -0.5$, (b) $\max\{f(x), f(y)\}) \leq -0.5$.

Case(a): Assume that $f(x \vee y) > \max\{f(x), f(y), -0.5\})$, which implies that $f(x \vee y) > \max\{f(x), f(y)\})$. Choose $t \in [-1, 0)$ such that $(f(x-y) > t > f(x)$. Then $x_t \in f$, but $(x-y)_t \overline{\in \vee q f}$, which is a contradiction.

Case(b): Assume that $f(x \vee y) > -0.5$. Then $x_{-0.5} \in f$, but $(x \vee y)_{-0.5} \overline{\in \vee q f}$, which is a contradiction. Thus $f(x \vee y) \leq \max\{f(x), f(y) - 0.5\}$.

Hence the conditions (GNI3) and (GNI4) holds.

Conversely, $(X,\ f)$ satisfies (GNI3) and (GNI4). Let $x, y \in X$ and $t \in [-1, 0)$ be such that $x_t \in f$. Then $f(x) \le t$. Now, we have $f(x-y) \le \max\{f(x), -0.5\} \le \max\{t, -0.5\})$. If $t < -0.5$, then $f(x-y) \le -0.5$, which implies that $f(x-y) + t < -1$. If $t \ge -0.5$, then $f(x-y) \le t$. Thus $(x-y)_t \in \vee q f$.

Finally, let $x, y \in X$ and $t, r \in [-1, 0)$ be such that $x_t, y_r \in f$. Then $f(x) \le t, f(y) \le r$. Now, there exist $x \vee y \in X$, we have $f(x \vee y) \le \max\{f(x), f(y), -0.5\} \le \max\{t, r, -0.5\})$. If $\max\{t, r\} < -0.5$, then $f(x \vee y) \le -0.5$, which implies that $(f(x \vee y) + \max\{t, r\}) < -1$. If $\max\{t, r\} \ge -0.5$, then $f(x \vee y) \le \max\{t, r\}$. Thus $(x \vee y)_{\min\{t,\, r\}} \in \vee q f$. Hence $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.

□

Proposition 3.5. *Every $\mathcal{N}$-ideal is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal.*

However, the following example shows that the converse of Proposition 3.5 is not necessarily true.

Example 3.6. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be a $\mathcal{N}$-structure defined by:

$$f(0) = -0.7, f(a) = -0.8 \text{ and } f(b) = -0.6.$$

It is easy to check that (X, f) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X, but (X, f) is not an $\mathcal{N}$-ideal of X. Since $f(a-a) = f(0) = -0.7 \nleq -0.8 = f(a)$.

Definition 3.7. An $\mathcal{N}$-function χ on B is called an $\mathcal{N}$-*charateristic function* if

$$\chi_B(x) = \begin{cases} -1, & if\ x \in B, \\ 0, & otherwise. \end{cases}$$

Corollary 3.8. *Let χ_A be an $\mathcal{N}$-characteristic function of an ideal A of X. Then the $\bar{A} = (X, \chi_A)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.*

Theorem 3.9. *Every $\mathcal{N}$-structure $(X,\ f)$ of X is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal if and only if it satisfies*
(GNI5) $(\forall a, b \in X)(f(x - ((x-a) - b)) \le max\ \{f(a), f(b), -0.5\})$.

Proof. Let $(X,\ f)$, be a $\mathcal{N}$-ideal of X satisfying (GNI5). Using (a2) and (S3), we have

$$x - y = (x-y) - (((x-y) - x) - x),$$

for all $x, y \in X$. We have

$$\begin{aligned} f(x-y) &= f((x-y) - (((x-y) - x) - x)) \\ &\le \max\ \{f(x), f(x), -0.5\} \\ &= \max\ \{f(x), -0.5\}. \end{aligned}$$

Suppose $x \in X$ is an upper bound for a and b, for all $a, b \in X$ that is $x := a \vee b$, then by Proposition 2.4 and (S3), we have

$$a \vee b = x - ((x-a) - b).$$

We have

$$f(a \vee b) = f(x - ((x-a) - b)) \le \max\ \{f(a), f(b), -0.5\}.$$

Thus (GNI2) and (GNI2) are valid. Hence $(X,\ f)$ is an $(\in, \in \vee q)$- $\mathcal{N}$-ideal of X.

Conversely, $(X,\ f)$ is a $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. If $x \in X$ is an upper bound for a and b, for all $a, b \in X$ that is $x := a \vee b$, then we have

$$a \vee b = x - ((x-a) - b).$$

Thus

$$f(a \vee b) = f(x - ((x-a) - b)) \le \max\ \{f(a), f(b), -0.5\}.$$

□

Proposition 3.10. *Every $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X satisfies the following inequality*
(GNI6) $(\forall x \in X)$ $(f(0) \leq max\ \{f(x), -0.5\})$.

Proof. By taking $y := x$ in (GNI3), we have

$$f(0) = f(x - x) \leq \max\ \{f(x), -0.5\}.$$

□

Example 3.11 shows that the converse of Proposition 3.10 need not be true.

Example 3.11. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be a $\mathcal{N}$-structure defined by:

$$f(0) = -0.7, f(a) = -0.4 \text{ and } f(b) = -0.6.$$

It is clear that, $(f(0) \leq \max\ \{f(x), -0.5\})$, for all $x \in X$, but (X, f) is not an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. Since

$$f(0 - a) = f(a) = -0.4 \not\leq -0.5 = \max\{-0.7, -0.5\} = \max\{f(0), -0.5\}.$$

Corollary 3.12. *Every $(\in, \in \vee q)$-$\mathcal{N}$-ideal $(X,\ f)$ satisfies*
(GNI7) $(\forall x, y \in X)$ $(x \leq y \Rightarrow f(x) \leq max\ \{f(y), -0.5\}$.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$ and so

$$f(x) = f(x - 0) = f(x - ((x - y) - y)) \leq \max\{f(y), f(y), -0.5\} = \max\ \{f(y), -0.5\}.$$

by using (a2), (GNI5) and (GNI6). This completes the proof.

□

Theorem 3.13. *An $\mathcal{N}$-structure $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X if and only if it satisfies the (GNI6), (GNI7) and*
(GNI8) $(\forall x, y, z \in X)(f(x - z) \leq max\{f((x - y) - z), f(y), -0.5\})$.

Proof. Assume that $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X and satisfies (GNI8). Then the conditions (GNI6) and (GNI7) are holds by Proposition 3.10. By (GNI5), we have

$$f(x - ((x - a) - y)) \leq \max\{f(a), f(y), -0.5\}. \tag{1}$$

By setting

$$a = (x - y) - z \tag{2}$$

for all $z \in X$. Now

$$\begin{aligned} ((x - a) - y) &= (x - ((x - y) - z)) - y, \\ &= (x - y) - (((x - y) - z) - y), \text{by (a12)} \\ &= (x - y) - (((x - y) - y) - (z - y)), \text{by (a12)} \\ &= (x - y) - ((x - y) - (z - y)), \text{by (a1)} \\ &= (x - y) - ((x - z) - y), \text{by (a12)} \\ &= (x - y) - ((x - y) - z), \text{by (S3)} \\ &\leq z, \text{by (a4)}. \end{aligned}$$

Thus, by using (a9), for all $x \in X$

$$x - z \leq x - ((x - a) - y).$$

This implies, by using (GNI7),

$$f(x - z) \leq \max\{f(x - ((x - a) - y)), -0.5\}. \tag{3}$$

Using (2), (3) in (1), we have

$$f(x-z) \leq \max\{f(x-((x-((x-y)-z)-y))), -0.5\} \leq \max\{f((x-y)-z), f(y), -0.5\}$$

for all $x, y, z \in X$.

Conversely, assume that $\mathcal{N}$-structure $(X,\ f)$, satisfied (GNI6), (GNI7), and (GNI8). By using (a2), (a3), (GNI5), we have

$$\begin{aligned} f(x-y) &= f((x-y)-(((x-y)-x)-x)) \\ &\leq \max\{f(x), f(x), -0.5\} \\ &= \max\{f(x), -0.5\}. \end{aligned}$$

Now, suppose $x \vee y$ exists for $x, y \in X$. Also, we have $f(x \vee y) \leq \max\{f(x), -0.5\}$, such that

$$f(x \vee y) \leq \max\{f(x), f(y), -0.5\}.$$

Thus $\mathcal{N}$-structure $(X,\ f)$ satisfies (GNI1) and (GNI2). Hence $(X,\ f)$ is an $(\in, \in \vee\, q)$-$\mathcal{N}$-ideal of X.

□

Theorem 3.14. *An $\mathcal{N}$-structure $(X,\ f)$ is an $(\in, \in \vee\, q)$-$\mathcal{N}$-ideal of X if and only if it satisfies the conditions (GNI6) and*
(GNI9) $(\forall x, y \in X)\ (f(x) \leq max\{f(x-y), f(y), -0.5\})$.

Proof. Assume that $(X,\ f)$ is a $(\in, \in \vee\, q)$-$\mathcal{N}$-ideal of X. Then the condition (GNI6) is valid by Proposition 3.10 and the conditions (GNI9) is valid by taking $z = 0$ in (GNI8).

Conversely, $(X,\ f)$ satisfying (GNI6) and (GNI9). Then

$$\begin{aligned} (x-((x-a)-b))-b &= (x-b)-(((x-a)-b)-b), \text{by (a12)} \\ &= (x-b)-((x-a)-b), \text{by (a1)} \\ &\leq x-(x-a), \text{by (a7)} \\ &\leq a, \text{by (a4)}. \end{aligned}$$

That is $((x-((x-a)-b))-b)-a = 0$, for all $x, a, b \in X$, it follows from (GNI6) and (GNI9) that

$$\begin{aligned} f(x-((x-a)-b)) &\leq \max\{f((x-((x-a)-b))-b), f(b), -0.5\} \\ &\leq \max\{\max\{f(((x-((x-a)-b))-b)-a), f(a)\}, f(b), -0.5\} \\ &= \max\{\max\{f(0), f(a)\}, f(b), -0.5\}\} \\ &= \max\{f(a), f(b), -0.5\}. \end{aligned}$$

By Theorem 3.9, $(X,\ f)$ is an $(\in, \in \vee\, q)$-$\mathcal{N}$-ideal of X. This completes the proof.

□

Proposition 3.15. *Every $(\in, \in \vee\, q)$-$\mathcal{N}$-ideal $(X,\ f)$ of X satisfies the following assertion:*
(GNI10) *$(f(x-y) \leq max\{f((x-y)-y)), -0.5\} \Leftrightarrow (f((x-z)-(y-z)) \leq max\{f((x-y)-z)), -0.5\}$ for all $x, y, z \in X$.*

Proof. Assume that $f(x-y) \leq f((x-y)-y)$, for all $x, y \in X$. Also

$$((x-(y-z))-z)-z = ((x-z)-(y-z))-z \leq (x-y)-z,$$

then, by Corollary 3.12,

$$f(((x-(y-z))-z)-z) \leq \max\{f((x-y)-z), -0.5\}.$$

Thus

$$\begin{aligned} f((x-z)-(y-z)) &= f((x-(y-z))-z)-z), \text{by (a1 and a12)} \\ &\leq \max\{f(((x-(y-z))-z)-z), -0.5\}, \text{by (GNI10)} \\ &\leq \max\{f((x-y)-z), -0.5\}. \end{aligned}$$

Conversely, suppose $f((x-z)-(y-z)\leq \max\{f((x-y)-z),-0.5\}$ for all $x,y,z\in X$. Hence, we have

$$f(x-y)=f((x-y)-0)=f((x-y)-(y-y))\leq \max\{f((x-y)-y),-0.5\}.$$

This completes the proof.

□

Theorem 3.16. *An $\mathcal{N}$-structure $(X,\ f)$ is an $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X if and only if it satisfies:*
(GNI11) $(\forall a,b,x\in X)(x-a\leq b\Rightarrow f(x)\leq max\{f(a),f(b),-0.5\})$.

Proof. Assume that $(X,\ f)$ is a $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X. Let $a,b,x\in X$ be such that $x-a\leq b$. Then $(x-a)-b=0$, and so

$$\begin{aligned}
f(x) &\leq \max\{f(x-a),f(a),-0.5\}, \text{ by (GNI9)}\\
&\leq \max\{\max\{f((x-a)-b),f(b)\},f(a),-0.5\}, \text{ by (GNI9)}\\
&= \max\{\max\{f(0),f(b)\},f(a),-0.5\}\\
&= \max\{f(a),f(b),-0.5\}.
\end{aligned}$$

Conversely, let $(X,\ f)$ be an $\mathcal{N}$-structure satisfying the conditions (GNI11). Since $0-x\leq x$, for all $x\in X$, it follows from (GNI11) that

$$f(0)\leq \max\{f(x),f(x),-0.5\}= \max\{f(x),-0.5\}$$

for all $x\in X$. Note that $x-(x-y)\leq y$ for all $x,y\in X$. Using (GNI11), we have $f(x)\leq \max\{f(x-y),f(y),-0.5\}$, for all $x,y\in X$. Hence $(X,\ f)$ is an $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X, by Theorem 3.16. This completes the proof.

□

Theorem 3.17. *An $\mathcal{N}$-structure $(X,\ f)$ is an $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X if and only if it satisfies:*
(GNI12) $f(x)\leq max\{\{f(a_i)|i=1,2,...,n\},-0.5\}$ *for all* $x,a_1,a_2,...,a_n\in X$ *with* $(\cdots((x-a_1)-a_2)-\cdots)-a_n=0$.

Proof. Assume that $(X,\ f)$ is a $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X. If $x-a=0$ for any $x,a\in X$, then $f(x)\leq \max\{f(a),-0.5\}$ by Corollary 3.12. Let $a,b,x\in X$ be such that $(x-a)-b=0$. Then, by Theorem 3.16

$$f(x)\leq \max\{f(a),f(b),-0.5\}.$$

Now, let $x,a_1,a_2,...,a_n\in X$ be such that $(\cdots((x-a_1)-a_2)-\cdots)-a_n=0$. By induction on n, we conclude that $f(x)\leq \max\{\{f(a_i)|i=1,2,...,n\},-0.5\}$.

Conversely, for an $\mathcal{N}$-structure $(X,\ f)$ the condition (GNI12) is valid, for all $x,a_1,a_2,...,a_n\in X$ with $(\cdots((x-a_1)-a_2)-\cdots)-a_n=0$. Using (GNI11), we have $f(x)\leq \max\{f(y),f(z),-0.5\}$. Since $(0-x)-x=0$ for all $x\in X$, it follows from (GNI11) that $f(0)\leq \min\{f(x),f(x),-0.5\}=\min\{f(x),-0.5\}$. Hence $(X,\ f)$ is an $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X, by Theorem 3.16.

□

Remarks 3: For any element $w\in X$, define

$$(X_w,f):=\{y\in X|\, f(y)\leq \max\{f(w),-0.5\}\}.$$

Clearly, $w\in(X_w,f)$, and so (X_w,f) is a non-empty subset of X.

Theorem 3.18. *Let w be an element of X. If $(X,\ f)$ is an $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X. Then (X_w,f) is an ideal of X.*

Proof. Obviously, $0\in(X_w,f)$, by (GNI6). Let $x\in(X_w,f)$ and $y\in X$ be such that $x-y\in(X_w,f)$. Then

$$f(x-y)\leq \max\{f(w),-0.5\}.$$

Since $(X,\ f)$ is an $(\in,\in\vee q)$-$\mathcal{N}$-ideal of X, it follows from (GNI9) that

$$f(x)\leq \max\{f(x-y),f(y),-0.5\}\leq \max\{f(w),-0.5\},$$

so that $x\in(X_w,f)$. Hence (X_w,f) is an ideal of X.

□

However, the following example shows that the converse of Theorem 3.18 is not necessarily true.

Example 3.19. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be an $\mathcal{N}$-structure defined by:

$$f(0) = -0.4, f(a) = -0.6 \text{ and } f(b) = -0.9.$$

The $(X_0, f) = \{0, a, b\}$ is an ideal of X but (X, f) is not an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. Since

$$f(b - b) = f(0) = -0.4 \nleq -0.5 = \max\{f(b), -0.5\}.$$

Theorem 3.20. *Let w be an element of X. If $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. Then*
(i) If (X_w, f) is an ideal of X, then $(X,\ f)$ satisfies the following condition:
(GNI13) $(\forall x, y, z \in X)\,(f(x) \geq max\{f(y - z), f(z), -0.5\} \Rightarrow f(x) \geq max\{f(y), -0.5\})$.
(ii) If $(X,\ f)$ satisfies (GNI6) and (GNI13), then (X_w, f) is an ideal of X.

Proof. (i) Assume that (X_w, f) is an ideal of X, for each $w \in X$. Let $x, y, z \in X$ be such that

$$f(x) \geq \max\{f(y - z), f(z), -0.5\}.$$

Then $y - z \in (X_w, f)$ and $z \in (X_w, f)$. Since (X_w, f) is an ideal of X, it follows that $y \in (X_w, f)$, that is, $f(x) \geq max\{f(y), -0.5\}$.

(ii) Suppose that $(X,\ f)$ satisfies (GNI6) and (GNI13). For each $w \in X$, let $x, y \in X$ be such that $x - y \in (X_w, f)$ and $y \in (X_w, f)$. Then $f(x - y) \leq \max\{f(w), -0.5\}$ and $f(y) \leq \max\{f(w), -0.5\}$, which implies that $\max\{f(w), -0.5\} \geq \max\{f(x-y), f(y), -0.5\}$. Using (GNI9), we have $\max\{f(w), -0.5\} \geq f(x)$. Hence $x \in (X_w, f)$. Obviously $0 \in (X_w, f)$. Therefore (X_w, f) is an ideal of X.

□

Remarks 4:

- For any $\mathcal{N}$-function f on X, we denote

$$\perp := -1 - \inf\{f(x)|x \in X\}.$$

- For any $\alpha \in [\perp, 0]$, we denote $f^\alpha(x) = f(x) + \alpha$, for all $x \in X$.

Obviously, $f^\alpha(x)$ is a mapping from X to $[-1, 0]$, that is, $f^\alpha(x)$ is an $\mathcal{N}$-function on X, we say that (X, f^α) is a α-translation of $(X,\ f)$.

Theorem 3.21. *For every $\alpha \in [\perp, 0]$, the α-translation (X, f^α) of an $(\in, \in \vee q)$-$\mathcal{N}$-ideal $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} f^\alpha(x - y) &= f(x - y) + \alpha \\ &\leq \max\{f(x), -0.5\} + \alpha \\ &= \max\{f(x) + \alpha, -0.5\} \\ &= \max\{f^\alpha(x), -0.5\}. \end{aligned}$$

Also, there exist $x \vee y \in X$ such that

$$\begin{aligned} f^\alpha(x \vee y) &= f(x \vee y) + \alpha \\ &\leq \max\{f(x), f(y), -0.5\} + \alpha \\ &\leq \max\{f(x) + \alpha, f(y) + \alpha, -0.5\} \\ &\leq \max\{f^\alpha(x), f^\alpha(y), -0.5\} \end{aligned}$$

Hence (X, f^α) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.

□

$-$	**0**	**a**	**b**	**c**	**d**
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Table 2: Binary operation 2

Theorem 3.22. *For every $\alpha \in [\perp, 0]$, the α-translation (X, f^α) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal (X, f), then (X, f) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} f(x-y)+\alpha &= f^\alpha(x-y) \\ &\leq \max\{f(x)+\alpha, -0.5\} \\ &\leq \max\{f(x), -0.5\}+\alpha. \end{aligned}$$

Therefore

$$f(x-y) \leq \max\{f(x), -0.5\}.$$

Also, there exist $x \vee y \in X$ such that

$$\begin{aligned} f(x \vee y)+\alpha &= f^\alpha(x \vee y) \\ &\leq \max\{f^\alpha(x), f^\alpha(y), -0.5\} \\ &\leq \max\{f(x)+\alpha, f(y)+\alpha, -0.5\} \\ &\leq \max\{f(x), f(y), -0.5\}+\alpha \end{aligned}$$

Thus

$$f(x \vee y) \leq \max\{f(x), f(y), -0.5\}.$$

Hence (X, f) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.

□

Remarks 5: Let (X, f) and (X, g) be two $\mathcal{N}$-structures. Then (X, f) is said to be a $\mathcal{N}$-*retrenchment* of (X, g) if $g(x) \geq \max\{f(x), -0.5\}$.

Definition 3.23. Let (X, f) be an $\mathcal{N}$-structure. An $\mathcal{N}$-structure (X, g) is called a *created* $(\in, \in \vee q)$-$\mathcal{N}$-*ideal* of (X, f) if it satisfies
(i) (X, g) is an $\mathcal{N}$-retrenchment of (X, f).
(ii) If (X, f) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X, then (X, g) is a $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.

Theorem 3.24. *Let (X, f) be an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. For every $\alpha \in [\perp, 0]$ the α-translation (X, f^α) of (X, f) is an $\mathcal{N}$-retrenchment $(\in, \in \vee q)$-$\mathcal{N}$-ideal of (X, f_N, g_N).*

Proof. Obviously, (X, f^α) is an $\mathcal{N}$-retrenchment of (X, f). Using Theorem 3.20, we conclude that (X, f^α) is an $\mathcal{N}$-retrenchment $(\in, \in \vee q)$-$\mathcal{N}$-ideal of (X, f).

□

The converse of Theorem 3.24 is not true as seen in the following example.

Example 3.25. Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table which is given in Table 2. Let (X, f) be an $\mathcal{N}$-structure defined by:

$$f(0) = -0.9, f(a) = -0.8, f(b) = -0.9, f(c) = -0.8 \text{ and } f(d) = -0.6.$$

It is easy to check that $(X,\ f)$ is a $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.

Also, define $\mathcal{N}$-structure $(X,\ g)$ as follows:

$$g(0) = -0.92, g(a) = -0.83, g(b) = -0.92, g(c) = -0.83 \text{ and } g(d) = -0.62.$$

Then $(X,\ g)$ is a created $(\in, \in \vee q)$-$\mathcal{N}$-ideal of $(X,\ f)$, which is not an α-translation of $(X,\ f)$, for $\alpha \in [\perp, 0]$.

Remarks 6:

- The created $(\in, \in \vee q)$-$\mathcal{N}$-ideal $(X,\ f)$ of X will be denoted by $(X, [f])$.
- The created $(\in, \in \vee q)$-$\mathcal{N}$-ideal of $(X,\ f)$ is the greatest $(\in, \in \vee q)$-$\mathcal{N}$-ideal in X which is an $\mathcal{N}$-retrenchment of $(X,\ f)$.

Now, we discuss how to make a created $(\in, \in \vee q)$-$\mathcal{N}$-ideal of an $\mathcal{N}$-structure.

Theorem 3.26. *For any $\mathcal{N}$-structure the created $(\in, \in \vee q)$-$\mathcal{N}$-ideal $(X, [f])$ of $(X,\ f)$ is described as follows:*

$$[f](x) = inf\,\{max\{\{f(a_i)|i = 1, 2, ..., n\}, -0.5\}|(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0)\}\ .$$

Proof. Let $(X,\ g)$, be an $\mathcal{N}$-structure in which g is defined by

$$g(x) = \inf\ \{\max\{\{f(a_i)|i = 1, 2, ..., n\}, -0.5\}|(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0)\}\ .$$

Let $x, a, b \in X$ be such that

$$(x - a) - b = 0. \tag{4}$$

For any $\epsilon > 0$, there exist $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in X$ such that

$$(\cdots((a - a_1) - a_2) - \cdots) - a_n = 0, (\cdots((b - b_1) - b_2) - \cdots) - b_m = 0. \tag{5}$$

Also, we have

$$g(x) > \max\{\{f(a_i)|i = 1, 2, ..., n\}, -0.5\} - \epsilon \text{ and } g(x) > \max\{\{f(b_i)|i = 1, 2, ..., m\}, -0.5\} - \epsilon.$$

$$g(x) > \max\{\{f(a_i)|i = 1, 2, ..., n\} - \epsilon, -0.5\} \text{ and } g(x) > \max\{\{f(b_i)|i = 1, 2, ..., m\} - \epsilon, -0.5\}. \tag{6}$$

Using (4) in (5), we have

$$(\cdots((((\cdots((x - a_1) - a_2) - \cdots) - a_n) - b_1) - b_2) - \cdots) - b_m = 0.$$

Using (6) in the definition of g, we have

$$\begin{aligned} g(x) &\leq \max\{f(a_1), f(a_2), \cdots, f(a_n), f(b_1), f(b_2), \cdots, f(b_m), -0.5\} \\ &< \max\{g(a) + \epsilon, g(b) + \epsilon, -0.5\} \\ &= \max\{g(a), g(b), -0.5\} + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that

$$g(x) \leq \max\{g(a), g(b), -0.5\},$$

by Theorem 3.16, we have $(X,\ g)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. Now since $x - x = 0$ for all $x \in X$, we obtain $f(x) \geq \max\{g(x), -0.5\}$ for all $x \in X$, and so $(X,\ g)$ is an $\mathcal{N}$-retrenchment of $(X,\ f)$. Let $(X,\ r)$ be an $(\in, \in \vee q)$-$\mathcal{N}$-ideal in X which is an $\mathcal{N}$-retrenchment of $(X,\ f)$. For any $x \in X$, we have

$$\begin{aligned} g(x) &= \inf\{\max\{\{f(a_i)|i = 1, 2, ..., n\}, -0.5\}|(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0)\} \\ &\geq \inf\{\max\{\{r(a_i)|i = 1, 2, ..., n\}, -0.5\}|(\cdots((x - a_1) - a_2) - \cdots) - a_n = 0)\} \\ &= \inf\{\max\{r(x), -0.5\}\} \\ &= \max\{\inf\{r(x)\}, -0.5\} \\ &= \max\{r(x), -0.5\}. \end{aligned}$$

Thus $(X,\ r)$ is an $\mathcal{N}$-retrenchment of $(X,\ g)$. Therefore $(X,\ g)$ is a created $(\in, \in \vee q)$-$\mathcal{N}$-ideal of $(X,\ f)$. Since $(X, [f])$ is greatest, we have $g = [f]$. This completes the proof.

□

−	**0**	**a**	**b**	**c**
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 3: Binary operation 3

Theorem 3.27. *An $\mathcal{N}$-structure $(X,\ f)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X if and only if the set $C(f;t)(\neq \emptyset)$ is an ideal of X, for all $t \in [-0.5, 0)$.*

Proof. Let $C(f;t)(\neq \emptyset)$ be an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X, for all $t \in [-0.5, 0)$. Let $x, y \in C(f;t)$. Then $f(x) \leq t$ and $f(y) \leq t$. Now, we have

$$f(x-y) \leq \max\{f(x), -0.5\} = \max\{t, -0.5\} = t,$$

which implies $x - y \in C(f;t)$. Let $x, y \in C(f;t)$. Then there exists $x \vee y \in X$, we have $f(x \vee y) \leq \max\{f(x), f(y), -0.5\} = \max\{t, t, -0.5\} = t$, which implies $x - y \in C(f;t)$. Hence $(X,\ f)$ is an ideal of X. Conversely, let $(X,\ f)$ be an ideal of X. We need to show that $C(f;t)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. Let $t_0 \in [-0.5, 0)$ and $x, y \in X$, Then, we can write

$$f(x) \leq \max\{f(x), -0.5\} = t_0 \text{ and } f(y) \leq \max\{f(y), -0.5\} = t_0.$$

Thus $x_{t_0}, y_{t_0} \in C(f;t_0)$. Since $C(f;t_0)$ is an ideal of X. Thus $x - y \in C(f;t_0)$ and there exists $x \vee y \in X$ such that $x \vee y \in C(f;t_0)$, which implies $f(x-y) = t_0 \leq \max\{f(x), -0.5\}$ and $f(x \vee y) = t_0 \leq \max\{f(x), f(y), -0.5\}$. Hence $C(f;t)$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X by Theorem 3.4.

□

4 (γ, δ)-$\mathcal{N}$-Ideals of Subtraction Algebras

In what follows let γ and δ denote any one of $\in$, q, $\in \vee q$, or $\in \wedge q$ otherwise specified. To say that $x_t \overline{\gamma} f$ means that $x_t \gamma f$ does not hold.

Definition 4.1. An $\mathcal{N}$-structure $(X,\ f)$ is called an (γ, δ)-$\mathcal{N}$-ideal of X if for all $t, r \in [-1, 0)$ and $x, y \in X$,
(GNI16) $(x_t \gamma f)$ $((x-y)_t \delta f)$,
(GNI17) $(x_t, y_r \gamma f)$ ($\exists\, x \vee y \in X \Rightarrow (x \vee y)_{\min\{t,\, r\}} \delta f)$.

Example 4.2. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the Cayley table which is given in Table 3. Let (X, f) be an $\mathcal{N}$-structure defined by:

$$f(0) = f(a) = -0.3 \text{ and } f(b) = f(c) = -0.9.$$

It is easy to check that (X, f) is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. But

(1) $(X,\ f)$ is not an $(\in, \in)$-$\mathcal{N}$-ideal of X, since $a_{-0.62} \in f$ and $a_{-0.66} \in f$, but $(a-a)_{\min(-0.62,-0.66)} = 0_{-0.62}\, \overline{\in}\, f$.

(2) $(X,\ f)$ is not an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X, since $b_{-0.34}\, qf$ and $c_{-0.86}\, qf$, but $(c-b)_{\min(-0.34,-0.86)} = a_{-0.86}\, \overline{\in \vee q}\, f$.

(3) $(X,\ f)$ is not an $(\in \vee q, \in \vee q)$-$\mathcal{N}$-ideal of X, since $a_{-0.56} \in \vee q f$ and $c_{-0.86} \in \vee q f$, but $(c-a)_{\min(-0.56,-0.86)} = b_{-0.86}\, \overline{\in \vee q}\, f$.

Theorem 4.3. *Every* $(\in\vee q, \in\vee q)$*-*$\mathcal{N}$*-ideal of* X *is an* $(\in, \in\vee q)$*-*$\mathcal{N}$*-ideal.*

Proof. Let (X, f) be an $(\in\vee q, \in\vee q)$-$\mathcal{N}$-ideal of X. Let $x, y \in X$ and $t \in [-1, 0)$ be such that $x_t \in f$. Then $x_t \in\vee q f$, which imply that $(x-y)_t \in\vee q f$. Let $x, y \in X$ and there exists $x \vee y \in X$ be such that $x_t \in f$ and $y_r \in f$, where $t, r \in [-1, 0)$. Then $x_t \in\vee q f$ and $y_r \in\vee q f$, which imply that $(x\vee y)_{\min\{t,r\}} \in\vee q f$. Hence (X, f) is an $(\in, \in\vee q)$-$\mathcal{N}$-ideal of X.

□

Theorem 4.4. *Every* $(\in, \in)$*-*$\mathcal{N}$*-ideal of* (X, f) *is an* $(\in, \in\vee q)$*-*$\mathcal{N}$*-ideal .*

Example 4.5. Example 4.2 shows that the converse of Theorems 4.3 and 4.4 need not be true.

Theorem 4.6. *Let* (X, f) *be an* $\mathcal{N}$*-structure over* X*. Then the left diagram shows the relationship between* (γ, δ)*-*$\mathcal{N}$*-ideals of* (X, f)*, where* γ*,* δ *are one of* $\in$ *and* q*. Also we have the right diagram.*

$$\begin{tikzcd}
& (\gamma,\gamma\vee\delta)\text{-}\mathcal{N}\text{-ideal} & \\
(\gamma,\delta)\text{-}\mathcal{N}\text{-ideal} \arrow[ur] & & (\gamma,\gamma)\text{-}\mathcal{N}\text{-ideal} \arrow[ul] \\
& (\gamma,\gamma\wedge\delta)\text{-}\mathcal{N}\text{-ideal} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\in\vee q,\in\vee q)\text{-}\mathcal{N}\text{-ideal} & \\
(\in\vee q,q)\text{-}\mathcal{N}\text{-ideal} \arrow[ur] & & (\in\vee q,\in)\text{-}\mathcal{N}\text{-ideal} \arrow[ul] \\
& (\in\vee q,\in\wedge q)\text{-}\mathcal{N}\text{-ideal} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Theorem 4.7. *If* (X, f) *is a non-zero* (γ, δ)*-*$\mathcal{N}$*-ideal over* X*, then* $f(0) < 0$*.*

Proof. Assume that $f(0) = 0$. Since (X, f) is non-zero, there exists $x \in X$ such that $f(x) = t < 0$. If $\gamma = \in$ or $\gamma = \in\vee q$ then $x_t \gamma f$, but $(x-x)_{\min\{t,t\}} = 0_t \overline{\delta} f$, which is a contradiction. If $\gamma = q$, then $x_t \gamma f$ because $f(x) - 1 = t - 1 < -1$. But $(x-x)_{\min\{-1,-1\}} = 0_{-1} \overline{\delta} f$, which is a contradiction. Hence $f(0) < 0$.

□

For an $\mathcal{N}$-structure over X, we denote $(X_0, f) = \{x \in X \mid f(x) < 0\}$.

Theorem 4.8. *If* (X, f) *is a nonzero* $(\in, \in)$*-*$\mathcal{N}$*-ideal over* X*, then the set* (X_0, f) *is an ideal over* X*.*

Proof. Let $x, y \in (X_0, f)$. Then $f(x) < 0$ and $f(y) < 0$. Suppose that $f(x-y) = 0$. Note that $x_{f(x)} \in f$ and $y_{f(y)} \in f$, but $(x-y)_{\min\{f(x),f(y)\}} \overline{\in} f$, because $f(x-y) = 0 > \min\{f(x), f(y)\}$, which is a contradiction thus $f(x-y) < 0$, which shows that $x - y \in (X_0, f)$. Also, suppose that $f(x\vee y) = 0$. Then there exists $x \vee y \in X$ such that $(x\vee y)_{\min\{f(x),f(y)\}} \overline{\in} f$, because $f(x\vee y) = 0 > \min\{f(x), f(y)\}$. This is a contradiction, which implies $f(x \vee y) < 0$. Thus $x \vee y \in (X_0, f)$. Hence (X_0, f) is an ideal over X.

□

Theorem 4.9. *If* (X, f) *is a nonzero* $(\in, q)$*-*$\mathcal{N}$*-ideal over* X*, then the set* (X_0, f) *is an ideal over* X*.*

Proof. Let $x, y \in (X_0, f)$. Then $f(x) < 0$ and $f(y) < 0$. Suppose that $f(x-y) = 0$, then

$$f(x-y) + \min\{f(x), f(y)\} = \min\{f(x), f(y)\} \geq -1.$$

Hence $(x-y)_{\min\{f(x),f(y)\}} \overline{q} f$, which is a contradiction, since $x_{f(x)} \in f$ and $y_{f(y)} \in f$. Thus $f(x-y) < 0$, and so $x - y \in (X_0, f)$. Also, if there exists $x \vee y \in X$ such that $f(x\vee y) = 0$. Then

$$f(x\vee y) + \min\{f(x), f(y)\} = \min\{f(x), f(y)\} \geq -1.$$

Hence $(x\vee y)_{\min\{f(x),f(y)\}} \overline{q} f$, which is a contradiction, since $x_{f(x)} \in f$ and $y_{f(y)} \in f$. Thus $f(x\vee y) < 0$, and so $x \vee y \in (X_0, f)$. Hence (X_0, f) is an ideal over X.

□

Theorem 4.10. *If* (X, f) *is a nonzero* $(q, \in)$*-*$\mathcal{N}$*-ideal over* X*, then the set* (X_0, f) *is an ideal of* X*.*

Proof. Let $x, y \in (X_0, f)$. Then $f(x) < 0$ and $f(y) < 0$. Thus $f(x) - 1 < -1$ and $f(y) - 1 < -1$, which imply that $x_1 qf$ and $y_1 qf$. Suppose that $f(x-y) = 0$, then $f(x-y) > -1 = \min\{-1, -1\}$. Hence $(x-y)_{\min\{-1,-1\}}\overline{q}f$, which is a contradiction. Thus $f(x-y) < 0$, and so $x - y \in (X_0, f)$. Also, if there exists $x \vee y \in X$ such that $f(x \vee y) = 0$. Then $f(x \vee y) > -1 = \min\{-1, -1\}$. Hence $(x \vee y)_{\min\{-1,-1\}}\overline{q}f$, which is a contradiction. Thus $f(x-y) < 0$, and so $x \vee y \in (X_0, f)$. Hence (X_0, f) is an ideal over X.

□

Theorem 4.11. *If (X, f) is a nonzero (q, q)-$\mathcal{N}$-ideal over X, then the set (X_0, f) is an ideal of X.*

Proof. Let $x, y \in (X_0, f)$. Then $f(x) < 0$ and $f(y) < 0$. Thus $f(x) - 1 < -1$ and $f(y) - 1 < -1$, which imply that $x_1 qf$ and $y_1 qf$. Suppose that $f(x-y) = 0$, then $f(x-y) + \min\{-1, -1\} = -1$ and so $(x-y)_{\min\{-1,-1\}}\overline{q}f$, which is a contradiction. Thus $x - y \in (X_0, f)$. Also, if there exists $x \vee y \in X$ such that $f(x \vee y) = 0$. Then $f(x \vee y) + \min\{-1, -1\} = -1$. Hence $(x \vee y)_{\min\{-1,-1\}}\overline{q}f$, which is a contradiction. Thus $f(x-y) < 0$, and so $x \vee y \in (X_0, f)$. Hence (X_0, f) is an ideal of X.

□

Theorem 4.12. *If (X, f) is one of the following:*

- *a nonzero $(\in, \in \vee q)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(\in, \in \wedge q)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(\in \wedge q, \in)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(\in \wedge q, q)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(\in \wedge q, q)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(\in \wedge q, \in \vee q)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(q, \in \vee q)$-$\mathcal{N}$-ideal over X,*
- *a nonzero $(q, \in \wedge q)$-$\mathcal{N}$-ideal over X,*

then the set (X_0, f) is an ideal over X.

Theorem 4.13. *Every nonzero (q, q)-$\mathcal{N}$-ideal over X is constant on (X_0, f).*

Proof. Let (X, f) be an (q, q)-$\mathcal{N}$-ideal over X. Assume that (X, f) is not constant. Then there exists $y \in X_0$ such that $t_y = f(y) \neq f(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. Suppose $t_y < t_0$ and choose $t_1, t_2 \in [-1, 0)$ such that $-(1 + t_0) < t_1 < -(1 + t_y) < t_2$. Then $f(0) + t_1 = t_0 + t_1 < -1$ and $f(y) + t_2 = t_y + t_2 < -1$, and so $0_{t_1} qf$ and $y_{t_2} qf$. Since

$$f(y - 0) + \min\{t_1, t_2\} = f(y) + t_1 = t_y + t_1 > -1,$$

we have $(y-0)_{\min\{t_1,t_2\}}\overline{q}f$, which is a contradiction. Next, assume that $t_y > t_0$. Then $f(y) - (1 + t_0) = t_y - 1 - t_0 < -1$, and so $y_{-(1+t_0)} qf$. Since

$$f(y - y) - (1 + t_0) = f(0) - (1 + t_0) = t_0 - 1 - t_0 < -1,$$

this implies that $(y-y)_{\min\{-1-t_0,-1-t_0\}} qf$, which is a contradiction. Hence (X, f) is a constant of (X_0, f).
□

Theorem 4.14. *Let (X, f) be a non-zero an (γ, δ)-$\mathcal{N}$-ideal over X, where (γ, δ) is one of the following:*

$$(1)\,(\in, q),\ \ (2)\,(\in, \in \wedge q),\ \ (3)\,(q, \in),\ \ (4)(q, \in \wedge q),\ \ (5)\,(\in \vee q, q),\ \ (6)\,(\in \vee q, \in \wedge q),\ \ (7)(\in \vee q, \in).$$

Then $(X, f) = \chi_{(X_0, f)}$, the $\mathcal{N}$-characteristic function over (X_0, f).

Proof. Assume that there exists $x \in (X_0, f)$ such that $f(x) > -1$. For $\gamma = \in$, choose $t \in [-1, 0)$ such that $\min\{-1 - f(x), f(x), f(0)\} < t$. Then $x_t \delta f$ and $0_t \delta f$, but $(x-0)_{\min\{t,t\}} = x_t \overline{\delta} f$ where $\delta = q$ or $\delta = \in \wedge q$, which is a contradiction. For $\gamma = q$. Then $x_{-1}\delta f$ and $0_{-1}\delta f$, but $(x-0)_{\min\{-1,-1\}} = x_{-1}\overline{\delta} f$ where $\delta = \in$ or $\delta = \in \wedge q$. This is a contradiction. Finally, for $\gamma = \in \vee q$. Then $x_t \delta f$ and $0_{-1}\delta f$, but $(x-0)_{\min\{t,-1\}} = x_t \overline{\delta} f$ where $\delta = q$ or $\delta = \in \wedge q$. This is a contradiction. Note that $x_{-1}\delta f$ and $0_{-1}\delta f$, but $(x-0)_{\min\{-1,-1\}} = x_{-1}\overline{\delta} f$. This is a contradiction. Hence $(X, f) = \chi_{(X_0, f)}$.

□

5 $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-Ideals of Subtraction Algebras

Definition 5.1. An $\mathcal{N}$-structure (X, f) is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X if for all $t, r \in [-1, 0)$ and $x, y \in X$,
(GNI18) $((x-y)_t \overline{\in} f) \Rightarrow (x_t \overline{\in} \vee \overline{q} f)$,
(GNI19) $(\exists\, x \vee y \in X,\ (x \vee y)_{\min\{t,r\}} \overline{\in} f) \Rightarrow (x_t \overline{\in} \vee \overline{q} f$ or $y_r \overline{\in} \vee \overline{q} f)$.

Theorem 5.2. *An $\mathcal{N}$-structure (X, f) is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X if and only if it satisfies the following assertions:*
(GNI20) $(x, y \in X)$ $(min\{(f(x-y), -0.5\} \leq f(x))$,
(GNI21) $(x, y \in X)$ $(\exists (x \vee y) \ \Rightarrow\ min\{f(x \vee y), -0.5\} \leq max\{f(x), f(y)\})$.

Proof. If there exist $x, y \in X$ such that $\min\{(f(x-y), -0.5\} > f(x) = t$, then $t \in (-0.5, 0]$, $(x-y)_t \overline{\in} f$ and $x_t \in f$. It follows that $x_t \overline{\in} \vee \overline{q} f$. Then $(t = f(x)$ and $t + f(x) = -1)$. It follows that $t = -0.5$. This is a contradiction with $t > -0.5$. Hence (GNI18) holds. If there exist $x, y \in X$ and $x \vee y$ such that $\min\{f(x \vee y), -0.5\} > \max\{f(x), f(y)\} = 0.5$, then $t \in (-0.5, 0]$, $(x \vee y)_t \overline{\in} f$ and $x_t \in f, y_t \in f$. It follows that $x_t \overline{\in} \vee \overline{q} f$ or $y_t \overline{\in} \vee \overline{q} f$. Then $(t = f(x)$ and $t + f(x) = -1)$ or $(t = f(y)$ and $t + f(y) = -1)$. It follows that $t = -0.5$. This is a contradiction with $t > -0.5$. Hence (GNI21) holds.

Conversely, let $x, y \in X$ such that $(x-y)_t \overline{\in} f$, then $f(x-y) > t$. Then we have the following:

(1) If $f(x-y) \leq f(x)$, then $f(x) > t$. It follows that $x_t \overline{\in} f$ which implies that $x_t \overline{\in} \vee \overline{q} f$.

(2) If $f(x-y) > f(x)$, then $-0.5 \leq f(x)$. Hence $\min\{f(x-y), -0.5\} \leq f(x)$. Setting $x_t \in f$, then $t \geq f(x) \geq -0.5$. It follows that $x_t \overline{\in} f$ which implies that $x_t \overline{\in} \vee \overline{q} f$.
Similarly,

(3) If $f(x \vee y) \leq \max\{f(x), f(y)\}$, then $\max\{f(x), f(y)\} > \max\ \{t, r\}$ and $f(x) > t$ or $f(y) > s$. It follows that $x_t \overline{\in} f$ or $y_s \overline{\in} f$ which implies that $x_t \overline{\in} \vee \overline{q} f$ or $y_r \overline{\in} \vee \overline{q} f$.

(4) If $f(x \vee y) > \max\{f(x), f(y)\}$, then $-0.5 \leq \max\{f(x), f(y)\}$. Hence

$$\min\{(f(x \vee y), -0.5\} \leq \max\{f(x), f(y)\}.$$

Setting $x_t \in f$, $y_r \in f$ then $t \geq f(x) \geq -0.5$ or $r \geq f(y) \geq -0.5$. It follows that $x_t \overline{\in} f$ or $y_r \overline{\in} f$ which implies that $x_t \overline{\in} \vee \overline{q} f$ or $y_r \overline{\in} \vee \overline{q} f$.

Thus, (X, f) is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X. □

Example 5.3. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be an $\mathcal{N}$-structure defined by:

$$f(0) = -0.3, f(a) = -0.4 \text{ and } f(b) = -0.2.$$

It is easy to check that (X, f) is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X. However, (X, f) is not an $\mathcal{N}$-ideal of X. Since

$$\begin{aligned} f(b \vee b) &= f(0 - ((0-b) - b))) \\ &= f(0) = -0.3 \nleq -0.4 = \max\{-0.4, -0.4, -0.5\} \\ &= \max\{f(b), f(b), -0.5\}. \end{aligned}$$

Theorem 5.4. *An $\mathcal{N}$-structure $(X, f)$is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X if and only if the set $C(f;t)(\neq \emptyset)$ is an ideal of X for all $t \in [-1, -0.5)$.*

Proof. Assume that $C(f;t)(\neq \emptyset)$ is an ideal of X for all $t \in [-1, -0.5)$. If there exist $x, y \in X$ such that $\min\{(f(x-y), -0.5\} > f(x) = s$, then $s \in [-1, -0.5)$, $f(x-y) > s$, $x, y \in C(f;s)$. Since $C(f;s)$ is a bi-ideal of X, we have $x - y \in C(f;s)$ or $f(x-y) \leq s$, which contradicts with $f(x-y) > s$. Assume that there exist $x, y \in X$ and $x \vee y \in X$ such that $\min\{f(x \vee y), -0.5\} > \max\{f(x), f(y)\} = s$, then $s \in [-1, -0.5), f(x \vee y) > s, x, y \in C(f;s)$. Since $C(f;t)$ is a bi-ideal of X, we have $x \vee y \in C(f;s)$ or $f(x \vee y) \leq s$, which contradicts with $f(x \vee y) > s$. Hence (X, f) is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X.

Conversely, Assume that (X, f) is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X. We show that $C(f;t)$ is an ideal of X. Let $t \in [-1, -0.5)$ and $x, y \in X$, Then

$$-0.5 > t > f(x) > \min\{f(x-y), -0.5\} > f(x-y),$$

and there exist $x \vee y \in X$ such that

$$-0.5 > t > \max\{f(x), f(y)\} > \min\{f(x \vee y), -0.5\} > f(x \vee y).$$

This implies $x - y \in C(f;t)$ and $x \vee y \in C(f;t)$. Hence $C(f;t)$ is an ideal of X.

□

From the above discussion, we know that an $\mathcal{N}$-structure (X, f) over X may satisfies the condition that for some $t \in [-1, 0)$, $C(f;t)$ is an ideal of X, but for other $t \in [-1, 0)$, $C(f;t)$ is not a ideal of X.

Remarks 7: Let $K(f;t) = \{t \mid t \in [-1,0), \text{and}\, C(f;t) \text{ is an empty- set or an ideal of } X\}$.

- When $K(f;t) = [-1, 0)$, then (X, f) is an $\mathcal{N}$-ideal of X (Theorem 2.8).
- When $K(f;t) = (-0.5, 0]$, (X, f) is an $(\in, \in \vee\, q)$-$\mathcal{N}$-ideal of X (Theorem 3.27).
- When $K(f;t) = [-1, -0.5)$, $(X, f)$is an $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-$\mathcal{N}$-ideal of X (Theorem 5.4).

An obvious question is: whether (X, f) is a kind of $\mathcal{N}$-ideal or not when $K(f;t) \neq \emptyset$ (*e.g.*, $K_t = (-0.5, 0], \alpha, \beta \in [-1, 0]$ and $\alpha < \beta$)?

To answer the above question, we introduce the concept of an $\mathcal{N}$-ideal of X with thresholds in the following way:

Definition 5.5. Let $\alpha, \beta \in [-1, 0]$ and $\alpha < \beta$. Let (X, f) be an $\mathcal{N}$-structure over X. Then (X, f) is called an $\mathcal{N}$-ideal with thresholds of X if it satisfies the following conditions:
(GNI22) $(x, y \in X)$ $(\min\{f(x - y), \beta\} \le \max\{f(x), \alpha\})$,
(GNI23) $(x, y \in X)$ $(\exists (x \vee y \in X) \;\Rightarrow\; \min\{f(x \vee y), \beta\} \le \max\{f(x), f(y), \alpha\})$.

Now, we characterize $\mathcal{N}$-ideals with thresholds by their closed $(f, t) - cut$ ideals over X.

Theorem 5.6. *An $\mathcal{N}$-structure (X, f) over X is an $\mathcal{N}$-ideal with thresholds (α, β) of X if and only if $C(f;t) (\neq \emptyset)$ is an ideal of X, for all $t \in (\alpha, \beta]$.*

Proof. Let (X, f) be an $\mathcal{N}$-ideal with thresholds of X and $t \in (\alpha, \beta]$. Let $x, y \in C(f;t)$. Then $f(x) \le t$. Now

$$f(x - y) = \min\{(f(x - y), \beta\} \le \max\{f(x), \alpha\} = \max\{t, \alpha\} = t,$$

which implies that $f(x - y) \le t$ and so $x - y \in C(f;t)$. Now, let $x, y \in C(f;t)$. Then $f(x) \le t$ and $f(y) \le t$. There exists $x \vee y \in X$, we have

$$f(x \vee y) = \min\{(f(x \vee y), \beta\} \le \max\{f(x), f(y), \alpha\} = \max\{t, t, \alpha\} = t,$$

which implies that $f(x \vee y) \le t$ and so $x \vee y \in C(f;t)$.

Conversely, let (X, f) be an $\mathcal{N}$-structure over X such that $C(f;t) (\neq \emptyset)$ be an ideal of X and $t \in (\alpha, \beta]$. If there exist $x, y, \in X$ such that

$$\min\{(f(x - y), \beta\} > \max\{f(x), \alpha\} = t,$$

which implies $f(x - y) > t$. But $C(f;t)$ is a an ideal of X, we get $x - y \in C(f;t)$. Hence $f(x - y) \le t$, which is a contradiction with $f(x - y) > t$. Hence (X, f) satisfies (GNI22). Now, let $x, y, \in X$, assume that there exists $x \vee y \in X$ such that

$$\min\{f(x \vee y), \beta\} > \max\{f(x), f(y), \alpha\} = t,$$

then $t \in (\alpha, \beta], f(x \vee y) > t, x \in C(f;t), y \in C(f;t)$. Since $C(f;t)$ is an ideal of X, we get $x \vee y \in C(f;t)$. Hence $f(x \vee y) \le t$, which is a contradiction with $f(x \vee y) > t$. Hence (X, f) satisfies (GNI23).

□

Example 5.7. Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table which is given in Table 1. Let (X, f) be a $\mathcal{N}$-structure defined by:

$$f(0) = -0.3, f(a) = -0.2 \text{ and } f(b) = -0.8.$$

It is clear that (X, f) is not both an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal and an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. Since

$$\begin{aligned}
\min\{f(b \vee b), -0.5\} &= \min\{f(a - ((a - b) - b))), -0.5\} \\
&= \min\{f(0), -0.5\} \\
&= \min\{-0.3, -0.5\} \\
&= -0.5 \\
&\not\leq -0.8 = \max\{-0.8, -0.8\} \\
&= \max\{f(b), f(b)\}.
\end{aligned}$$

Thus (X, f) is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal on X. Also

$$\begin{aligned}
f(b \vee b) &= f(0 - ((0 - b) - b))) \\
&= f(0) \\
&= -0.3 \\
&\not\leq -0.5 = \max\{-0.8, -0.8, -0.5\} \\
&= \max\{f(b), f(b), -0.5\}.
\end{aligned}$$

Thus (X, f) is not an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X. That is, if $K_t = (-0.9, -0.2]$, (X, f) is an $\mathcal{N}$-ideal with thresholds $\alpha = -0.8$ and $\beta = -0.2$ of X. But (X, f) neither are $\mathcal{N}$-ideal and $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X nor is $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X.

Remarks 8: From the Definition 5.5, we have observed following results:

1. Every $\mathcal{N}$-ideal (X, f) of X, with thresholds $\alpha = -1$ and $\beta = 0$ is an $\mathcal{N}$-ideal of X.
2. Every $\mathcal{N}$-ideal (X, f) of X with thresholds $\alpha = -0.5$ and $\beta = 0$ is an $(\in, \in \vee q)$-$\mathcal{N}$-ideal of X.
3. Every $\mathcal{N}$-ideal (X, f) of X with thresholds $\alpha = -1$ and $\beta = -0.5$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal of X.
4. Every $\mathcal{N}$-ideal (X, f) of X with thresholds $\beta < f(x) \leq \alpha$ or $\beta \leq f(x) < \alpha$, for all $x \in X$, is an $\mathcal{N}$-ideal of X.

Remarks 9: Hence, in a subtraction algebra, we have the following relations among falling $\mathcal{N}$-ideal, $(\in, \in \vee q)$-$\mathcal{N}$-ideal, $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal and $\mathcal{N}$-ideal with thresholds:

$$\begin{tikzcd}
& \mathcal{N}\text{-ideal with thresholds} & \\
(\in, \in \vee q)\text{-}\mathcal{N}\text{-ideal} \arrow[ur] & & (\overline{\in}, \overline{\in} \vee \overline{q})\text{-}\mathcal{N}\text{-ideal} \arrow[ul] \\
& \mathcal{N}\text{-ideal} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

6 Conclusion

To obtain a general type of an $\mathcal{N}$-ideal of a subtraction algebra, we have introduced the notion of an $(\in, \in \vee q)$-$\mathcal{N}$- ideal. Moreover, we have provided $(\gamma,\ \delta)$-$\mathcal{N}$-ideal, which is a generalization of $(\in, \in \vee q)$-$\mathcal{N}$- ideal and have studied their related properties. We have provided example which is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-$\mathcal{N}$-ideal but not an $(\in, \in \vee q)$-$\mathcal{N}$-ideal. Further, we have dealt with characterizations of an $(\in, \in \vee q)$-$\mathcal{N}$- ideal and an$(\gamma,\ \delta)$-$\mathcal{N}$-ideal. Finally, the notion of an $\mathcal{N}$-ideal with thresholds are discussed with examples. Some important issues for future work are: (1) To develop strategies for obtaining more valuable results (2) To apply these definitions and results for studying related notions in other $\mathcal{N}$-structures such as $\mathcal{N}$-structure over semigroup, $\mathcal{N}$-structure over BCK/BCI-algebras, etc. (3) To study ($\mathcal{N}$-structure) soft set theoretical aspects based on these notions herein.

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