Journal of Uncertain Systems
Vol.9, No.1, pp.17-30, 2015
Online at: www.jus.org.uk

Uncertain Dynamic Systems on Time Scales

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Received 28 January 2013; Revised 13 August 2013

Abstract

In this paper we study the existence and uniqueness of the solution for the dynamic systems on time scales with uncertain parameters. For this aim, we introduced the notion of uncertain process on time scales. In addition, the linear dynamic systems on time scales with uncertain parameters are also studied.

Keywords: differential equation, time scale, uncertain measure, uncertain process, existence and uniqueness, finance, uncertain inference

1 Introduction

The uncertain theory was introduced by Liu [14] as an important tool for the study of some real world phenomena which cannot be modeled by fuzziness. The main stages on the development of this theory and some fundamental results can be found in works [7, 14, 15, 16, 22]. The concept of uncertain differential equation was also introduced in [14], but the existence and uniqueness of the solution of this kind of differential equations was obtained by Chen and Liu in the work [6]. The theory of dynamic systems on time scales simultaneously allows us to study continuous and discrete dynamic systems. Since Hilger's initial work [10] there has been a significant growth in the theory of dynamic systems on time scales, covering a variety of different qualitative aspects. The calculus of time scales was initiated by B. Aulbach and S. Hilger in order to create a theory that can unify discrete and continuous analysis. We refer to the books [4, 5], and the papers [1, 2, 21] which are more specific with respect to our target.

Hoffacker [12], Ahlbrandt and Morian [3] are known that have been worked and demonstrated the related ideas to the multivariate case and the study of partial dynamic equations (PDEs), however, these ideas are already known for univariate case of dynamic equations [13]. The uncertain dynamic systems are widely used for solving different problems, and this technique is rather well developed. A lot of scientific works are dedicated to the development of this method [2, 4, 5, 6, 10, 11]. Many interesting and important results have been obtained in this field by different authors. Despite of this fact there still remain a lot of unsolved problems. For filling these gaps, in this article we consider problem of uncertain dynamic systems on time scales.

It is well known that the mathematical models are necessary to be able to study the real world systems. These models involve different parameters whose values are determined by measurements. Even when the measurements are made by modern technologies the end result is that the values of the parameters are different from one measurement to another. Depending on the nature of the values obtained from the measurements, the variation of the parameters can be considered to be a random variable, a fuzzy random variable or a uncertain variable. Moreover, the time variation of the parameters can be continuous, discrete, or may alternate from one time interval to another. In the latter case, the mathematical models on time scales appear as a natural one.

The purpose of this paper is to prove the existence and uniqueness of solution for the dynamic systems on time scales with uncertain parameters. The organization of this paper is as follows. Section 2 presents a few definitions and concepts of time scales. Also, the notion of uncertain process on a time scale is introduced. In Section 3 we prove the existence and uniqueness of solution for the dynamic systems on time scales with

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uncertain parameters. In the last section, we study the linear dynamic systems on time scales with uncertain parameters.

2 Preliminary

2.1 Time Scale

By a *time scale* $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$. Then $\mathbb{T}$ is a complete metric space with the metric defined by $d(t,s) := |t-s|$ for $t,s \in \mathbb{T}$. Since we know that for working into the different connected components of the time scale $\mathbb{T}$, we need the concept of jump operators. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. In this definition we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\sigma(t) > t$, we say t is a *right-scattered point*, while if $\rho(t) < t$, we say t is a *left-scattered point*. Points that are right-scattered and left-scattered at the same time will be called *isolated points*. A point $t \in \mathbb{T}$ such that $t < \sup \mathbb{T}$ and $\sigma(t) = t$, is called a *right-dense point*. A point $t \in \mathbb{T}$ such that $t > \inf \mathbb{T}$ and $\rho(t) = t$, is called a *left-dense point*. Points that are right-dense and left-dense at the same time will be called *dense points*. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}^\kappa = \mathbb{T}\backslash\{m\}$ if $\mathbb{T}$ has a left-scattered maximum m, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. To understand the notions we have to consider some examples for clearing the abstraction of the situation. Given a time scale interval $[a,b]_\mathbb{T} := \{t \in \mathbb{T} : a \le t \le b\}$, then $[a,b]^\kappa_\mathbb{T}$ denoted the interval $[a,b]_\mathbb{T}$ if $a < \rho(b) = b$ and denote the interval $[a,b)_\mathbb{T}$ if $a < \rho(b) < b$. In fact, $[a,b)_\mathbb{T} = [a,\rho(b)]_\mathbb{T}$. Also, for $a \in \mathbb{T}$, we define $[a,\infty)_\mathbb{T} = [a,\infty) \cap \mathbb{T}$. If $\mathbb{T}$ is a bounded time scale, then $\mathbb{T}$ can be identified with $[\inf \mathbb{T}, \sup \mathbb{T}]_\mathbb{T}$.

If $t_0 \in \mathbb{T}$ and $\delta > 0$, then we define the following neighborhoods of t_0: $U_\mathbb{T}(t_0,\delta) := (t_0-\delta, t_0+\delta) \cap \mathbb{T}$, $U^+_\mathbb{T}(t_0,\delta) := [t_0, t_0+\delta) \cap \mathbb{T}$, and $U^-_\mathbb{T}(t_0,\delta) := (t_0-\delta, t_0] \cap \mathbb{T}$.

Let $\mathbb{R}^m$ be the space of m-dimensional column vectors $x = col(x_1, x_2, ..., x_m)$ with a norm $||\cdot||$.

Definition 2. 1 ([4]) *A function $f : \mathbb{T} \to \mathbb{R}^m$ is called* regulated *if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$, and itsleft-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}^m$ is called* rd-continuous *if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. Denote by $C_{rd}(\mathbb{T},\mathbb{R}^m)$ the set of all rd-continuous function from $\mathbb{T}$ into $\mathbb{R}^m$.*

Obviously, a continuous function is rd-continuous, and a rd-continuous function is regulated ([4, Theorem 1.60]).

Definition 2. 2 ([4]) *A function $f : [a,b]_\mathbb{T} \times \mathbb{R}^m \to \mathbb{R}^m$ is called* Hilger continuous *if f is continuous at each point (t,x) where t is right-dense, and the limits*

$$\lim_{(s,y)\to(t^-,x)} f(s,y) \quad and \quad \lim_{y\to x} f(t,y)$$

both exist and are finite at each point (t,x) where t is left-dense.

Definition 2. 3 ([4]) *Let $f : \mathbb{T} \to \mathbb{R}^m$ and $t \in \mathbb{T}^\kappa$. Let $f^\Delta(t) \in \mathbb{R}^m$ (provided it exists) with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\left\| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right\| \le \varepsilon \left| \sigma(t) - s \right| \tag{1}$$

for all $s \in U_\mathbb{T}(t,\delta)$. We call $f^\Delta(t)$ the delta (*or* Hilger) derivative *(*Δ-derivative *for short) of f at t. Moreover, we say that f is* delta differentiable *(*Δ-differentiable *for short) on $\mathbb{T}^\kappa$ provided $f(t)$ exists for all $t \in \mathbb{T}^\kappa$.*

The following result will be very useful.

Proposition 2. 1 ([4, Theorem 1.16]) *Assume that $f : \mathbb{T} \to \mathbb{R}^m$ and $t \in \mathbb{T}^\kappa$.*

(i) If f is Δ-differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If f is Δ-differentiable at t and t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is Δ-differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

It is known [9] that for every $\delta > 0$ there exists at least one partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b)_{\mathbb{T}}$ such that for each $i \in \{1, 2, ..., n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\rho(t_i) = t_{i-1}$. For given $\delta > 0$ we denote by $\mathcal{P}([a,b)_{\mathbb{T}}, \delta)$ the set of all partitions $P : a = t_0 < t_1 < \cdots < t_n = b$ that possess the above property.

Let $f : \mathbb{T} \to \mathbb{R}^m$ be a bounded function on $[a,b)_{\mathbb{T}}$, and let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b)_{\mathbb{T}}$. In each interval $[t_{i-1}, t_i)_{\mathbb{T}}$,where $1 \leq i \leq n$, we choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} (t_i - t_{i-1}) f(\xi_i).$$

We call S a Riemann Δ-sum of f corresponding to the partition P.

Definition 2. 4 ([8]) *We say that f is* Riemann Δ-integrable *from a to b (or on $[a,b)_{\mathbb{T}}$) if there exists a vector $I \in \mathbb{R}^m$ with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|S - I\| < \varepsilon$ for every Riemann Δ-sum S of f corresponding to a partition $P \in \mathcal{P}([a,b)_{\mathbb{T}}, \delta)$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, $i = 1, 2, ..., n$. It is easily seen that such a vector I is unique. The vector $I \in \mathbb{R}^m$ is the* Riemann Δ-integral *of f from a to b, and we will denote it by $\int_a^b f(t)\Delta t$.*

Proposition 2. 2 ([8, Theorem 5.8]) *A bounded function $f : [a,b)_{\mathbb{T}} \to \mathbb{R}^m$ is Riemann Δ-integrable on $[a,b)_{\mathbb{T}}$ if and only if the set of all right-dense points of $[a,b)_{\mathbb{T}}$ at which f is discontinuous is a set of $\Delta-$measure zero.*

Since every regulated function on a compact interval is bounded (see [4, Theorem 1.65]), so, we get that every regulated function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$, is Riemann Δ-integrable from a to b.

Proposition 2. 3 ([11, Theorem 5.8]) *Assume that $a, b \in \mathbb{T}$, $a < b$ and $f : \mathbb{T} \to \mathbb{R}^m$ is rd-continuous. Then the integral has the following properties.*

(i) If $\mathbb{T} = \mathbb{R}$, then *$\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the Riemann integral.*
(ii) If $\mathbb{T}$ consists of isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)_{\mathbb{T}}} \mu(t) f(t).$$

Definition 2. 5 ([4]) *A function $g : \mathbb{T} \to \mathbb{R}^m$ is called a* Δ-antiderivative *of $f : \mathbb{T} \to \mathbb{R}^m$ if $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$.*

Bohner exclaimed that each rd-continuous function has a Δ-antiderivative [4, Theorem 1.74].

Proposition 2. 4 ([8, Theorem 4.1]) *Let $f : \mathbb{T} \to \mathbb{R}^m$ be Riemann Δ-integrable function on $[a,b)_{\mathbb{T}}$. If f has a Δ-antiderivative $g : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$, then $\int_a^b f(t)\Delta t = g(b) - g(a)$. In particular, $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t)$ for all $t \in [a,b)_{\mathbb{T}}$* (see [4, Theorem 1.75]).

Proposition 2. 5 ([8, Theorem 4.3]) *Let $f : \mathbb{T} \to \mathbb{R}^m$ be a function which is Riemann Δ-integrable from a to b. For $t \in [a,b]_{\mathbb{T}}$, let $g(t) = \int_a^t f(t)\Delta t$. Then g is continuous on $[a,b]_{\mathbb{T}}$. Further, let $t_0 \in [a,b)_{\mathbb{T}}$ and let f be arbitrary at t_0 if t_0 is right-scattered, and let f be continuous at t_0 if t_0 is right-dense. Then g is Δ-differentiable at t_0 and $g^{\Delta}(t_0) = f(t_0)$.*

Lemma 2. 1 ([21]) *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function. If $s, t \in \mathbb{T}$ with $s \leq t$, then*

$$\int_s^t g(\tau)\Delta\tau \leq \int_s^t g(\tau)d\tau.$$

Several of the properties of the Riemann Δ-integral are discussed in [1, 4].

2.2 Uncertain Process on Time Scales

Let Γ be an nonempty set and let $\mathcal{L}$ be a σ-algebra of sets of Γ. A mapping $\mathcal{M} : \mathcal{L} \to [0,1]$ is called *uncertain measure* if it satisfies the following axioms:

(A1) $\mathcal{M}(\Gamma) = 1$;

(A2) $\mathcal{M}(A) \leq \mathcal{M}(B)$ for all $A, B \in \mathcal{L}$ with $A \subset B$;

(A3) $\mathcal{M}(A) + \mathcal{M}(A^c) = 1$ for all $A \in \mathcal{L}$, where $A^c := \Gamma \backslash A$;

(A4) For every countable sequence $\{A_n\}$ of elements of $\mathcal{L}$, we have

$$\mathcal{M}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathcal{M}\left(A_n\right).$$

Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra on Γ, and $\mathcal{M}$ an uncertain measure. T he triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertain space*.

Example 2. 1 *Let us consider* $\Gamma = (0,1)$, $\mathcal{L}$ *the* σ*-algebra of all Borel subsets of* Γ. *Let* $\lambda : (0,1) \to \mathbb{R}_+$ *be defined by*

$$\lambda(x) = \left| x - \frac{1}{2} \right|, \ x \in (0,1).$$

Then the mapping $\mathcal{M} : \mathcal{L} \to [0,1]$ *defined by*

$$\mathcal{M}(A) = \begin{cases} \sup\limits_{x \in A} \lambda(x), & \text{if } \sup\limits_{x \in A} \lambda(x) < 1/2 \\ 1 - \sup\limits_{x \in A^c} \lambda(x), & \text{if } \sup\limits_{x \in A} \lambda(x) \geq 1/2, \end{cases}$$

is an uncertain measure on $(0,1)$.

Denote by $\mathcal{B}$ the σ-algebra of all Borel subsets of $\mathbb{R}^m$. A function $X(\cdot) : \Gamma \to \mathbb{R}^m$ is called an *uncertain variable* if X is a measurable function from $(\Gamma, \mathcal{F})$ into $(\mathbb{R}^m, \mathcal{B})$, that is, $X^{-1}(B) := \{\gamma \in \Gamma; X(\gamma) \in B\} \in \mathcal{L}$ for all $B \in \mathcal{B}$. A time scale *uncertain process* is a function $X(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ such that $X(t,\cdot) : \Gamma \to \mathbb{R}^m$ is a uncertain variable for each $t \in \mathbb{T}$. For each point $\gamma \in \Gamma$, the function on $\mathbb{T}$ given by $t \mapsto X(t,\gamma)$ is will be called a *sample path* of the time scale uncertain process $X(\cdot,\cdot)$ corresponding to γ. A time scale uncertain process $X(\cdot,\cdot)$ is said to be regulated (rd-continuous, continuous) if the trajectory $t \mapsto X(t,\gamma)$ is a regulated (rd-continuous, continuous) function on $[a,b]_{\mathbb{T}}$ for each $\gamma \in \Gamma$.

Lemma 2. 2 *Let* $X(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ *be a time scale uncertain process. If the sample path* $t \mapsto X(t,\gamma)$ *is Riemann* Δ*-integrable on* $[a,b)_{\mathbb{T}}$ *for every* $\gamma \in \Gamma$, *then the function* $Y(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ *given by*

$$Y(t,\gamma) = \int_a^t X(s,\gamma)\Delta s, \ \ t \in [a,b]_{\mathbb{T}}$$

is a continuous time scale uncertain process.

Proof: From Proposition 2.5, it follows that the function $t \mapsto \int_a^t X(s,\gamma)\Delta s$ is continuous for each $\gamma \in \Gamma$. Since the Riemann Δ-integral is a limit of the finite sum $S(\gamma) = \sum_{i=1}^n (t_i - t_{i-1}) X(\xi_i, \gamma)$ of measurable functions, we have that $\gamma \mapsto \int_a^t X(s,\gamma)\Delta s$ is a measurable function. Therefore, $Y(\cdot,\cdot)$ is a continuous time scale uncertain process.

3 Uncertain Initial Value Problem

In the following, consider an initial value problem of the form

$$\begin{cases} X^{\Delta}(t,\gamma) = F(t, X(t,\gamma),\gamma), \;\; t \in [a,b]^{\kappa}_{\mathbb{T}} \\ X(a,\gamma) = X_0(\gamma), \end{cases} \tag{2}$$

where $X_0 : \Gamma \to \mathbb{R}^m$ is a uncertain variable and $F : [a,b]^{\kappa}_{\mathbb{T}} \times \mathbb{R}^m \times \Gamma \to \mathbb{R}^m$ satisfies the following assumptions:

(H1) $F(t,x,\cdot) : \Gamma \to \mathbb{R}^m$ is a uncertain variable for all $(t,x) \in [a,b]^{\kappa}_{\mathbb{T}} \times \mathbb{R}^m$,

(H2) for each $\gamma \in \Gamma$, the function $F(\cdot,\cdot,\gamma) : [a,b]^{\kappa}_{\mathbb{T}} \times \mathbb{R}^m \to \mathbb{R}^m$ is a Hilger continuous function at every point $(t,x) \in [a,b]^{\kappa}_{\mathbb{T}} \times \mathbb{R}^m$.

By a *solution* of (2) we mean a time scale uncertain process $X(\cdot,\cdot) : [a,b]^{\kappa}_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ that satisfies conditions in (2).

Remark 1: Consider the uncertain differential Equation (2) as a family (with respect to parameter γ) of deterministic differential equations, namely

$$\begin{cases} X^{\Delta}(t,\gamma) = F(t, X(t,\gamma),\gamma), \;\; t \in [a,b]^{\kappa}_{\mathbb{T}}, \, \gamma \in \Gamma, \\ X(a,\gamma) = X_0(\gamma). \end{cases} \tag{3}$$

Then is not correct to solve each problem (3) to obtain the solutions of (2). Let us give two examples.

Example 3. 1 *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space. Consider an initial value problem of the form*

$$\begin{cases} X^{\Delta}(t,\gamma) = K(\gamma) X^2(t,\gamma), \;\; t \in [0,\infty)_{\mathbb{R}}, \, \gamma \in \Gamma, \\ X(0,\gamma) = 1, \end{cases} \tag{4}$$

where $K : \Gamma \to (0,\infty)$ is a uncertain variable. It is easy to see that, for each $\gamma \in \Gamma$, $X(t,\gamma) = 1/(1-K(\gamma)t)$ is a solution of (4) on the interval $[0, 1/K(\gamma)]$. Since for each $a \geq 0$ we have that $\mathcal{M}(1/K(\gamma) > a) < 1$, it follows that not all solutions $X(\cdot,\gamma)$ are well defined on some common interval $[0,a)$.

Example 3. 2 *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space and let $\Gamma_0 \notin \mathcal{L}$. It is easy to check that, for each $\gamma \in \Gamma$, the function $X(\cdot,\cdot) : [0,1]_{\mathbb{R}} \times \Gamma \to \mathbb{R}$, given by*

$$X(t,\gamma) = \begin{cases} 0 \;\, if \; \gamma \in \Gamma_0 \\ t^{3/2} \;\, if \; \gamma \in \Gamma \backslash \Gamma_0, \end{cases}$$

is a solution of the initial value problem

$$\begin{cases} X^{\Delta}(t,\gamma) = \frac{3}{2} X(t,\gamma), \;\; t \in [0,\infty)_{\mathbb{R}}, \, \gamma \in \Gamma, \\ X(0,\gamma) = 0. \end{cases}$$

But $X(\cdot,\cdot)$ is not a uncertain process. Indeed, we have that

$$\{\gamma \in \Gamma; X(1,\gamma) \in [-\frac{1}{2}, \frac{1}{2}]\} = \Gamma_0 \notin \mathcal{L},$$

that is, $\gamma \mapsto X(1,\gamma)$ is not a measurable function.

Using the Propositions 4 and 5 and [21, Lemma 2.3], it is easy to prove the following result.

Lemma 3. 3 *A time scale uncertain process $X(\cdot,\cdot) : [a,b]^{\kappa}_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ is the solution of the problem (2) if and only if $X(\cdot,\cdot)$ is a continuous time scale uncertain process and it satisfies the following uncertain integral equation*

$$X(t,\gamma) = X_0(\gamma) + \int_a^t F(s, X(s,\gamma),\gamma) \Delta s, \; t \in [a,b]_{\mathbb{T}}, \; \gamma \in \Gamma. \tag{5}$$

The following results is known as Gronwall's inequality on time scale and will be used in this paper.

Lemma 3. 4 ([21, Lemma 3.1]) *Let rd-continuous time scale uncertain processes* $X(\cdot,\cdot), Y(\cdot,\cdot) : [a,b]_{\mathbb{T}}^{\kappa} \times \Gamma \to \mathbb{R}_+$ *be such that*

$$X(t,\gamma) \leq Y(t,\gamma) + \int_a^t q(s)X(s,\gamma)\Delta s,\ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma,$$

where $1 + \mu(t)p(t) > 0$*, for all* $t \in [a,b]_{\mathbb{T}}$*. Then we have*

$$X(t,\gamma) \leq Y(t,\gamma) + \int_a^t e_p(t,\sigma(s)p(s)Y(s,\gamma)\Delta s,\ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma.$$

Theorem 3. 1 *Let* $F : [a,b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^m \times \Gamma \to \mathbb{R}^m$ *satisfies (H1)-(H2) and assume that there exists a rd-continuous time scale uncertain process* $L(\cdot,\cdot) : [a,b]_{\mathbb{T}}^{\kappa} \times \Gamma \to \mathbb{R}_+$ *such that*

$$\|F(t,x,\gamma) - F(t,y,\gamma)\| \leq L(t,\gamma)\,\|x-y\| \tag{6}$$

for every $t \in [a,b]_{\mathbb{T}}^{\kappa}$*,* $x,y \in \mathbb{R}^m$ *and* $\gamma \in \Gamma$*. Let* $X_0 : \Gamma \to \mathbb{R}^m$ *a uncertain variable such that*

$$\|F(t,X_0(\gamma),\gamma)\| \leq M,\ t \in [a,b]_{\mathbb{T}}^{\kappa},\ \gamma \in \Gamma, \tag{7}$$

where $M > 0$ *is a constant. Then the problem (2) has a unique solution.*

Proof: To prove the theorem we apply the method of successive approximations (see [21]). For this, we define a sequence of functions $X_n(\cdot,\cdot) : [a,b]_{\mathbb{T}}^{\kappa} \times \Gamma \to \mathbb{R}^m$, $n \in \mathbb{N}$, as follows:

$$\begin{aligned} &X_0(t,\gamma) = X_0(\gamma) \\ &X_n(t,\gamma) = X_0(\gamma) + \int_a^t F(s,X_{n-1}(s,\gamma),\gamma)\Delta s,\ \ n \geq 1, \end{aligned} \tag{8}$$

for every $t \in [a,b]_{\mathbb{T}}^{\kappa}$ and every $\gamma \in \Gamma$. First, using (7) and the Lemma 2.1, we observe that

$$\begin{aligned} \|X_1(t,\gamma) - X_0(t,\gamma)\| &\leq \int_a^t \|F(s,X_0(\gamma),\gamma)\|\,\Delta s \leq M(t-a) \\ &\leq M(b-a),\ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma. \end{aligned}$$

We prove by induction that for each integer $n \geq 2$ the following estimate holds

$$\|X_n(t,\gamma) - X_{n-1}(t,\gamma)\| \leq M\widetilde{L}(\gamma)\frac{(t-a)^n}{n!} \leq M\widetilde{L}(\gamma)\frac{(b-a)^n}{n!},\ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma, \tag{9}$$

where $\widetilde{L}(\gamma) = \sup_{[a,b]_{\mathbb{T}}} L(t,\gamma)$. Suppose that (9) holds for $n = k \geq 2$. Then, using (6), (7) and Lemma 2.1, we obtain

$$\|X_{k+1}(t,\gamma) - X_k(t,\gamma)\| \leq \int_a^t \|F(s,X_k(s,\gamma),\gamma) - F(s,X_{k-1}(s,\gamma),\gamma)\|\,\Delta s$$

$$\leq \widetilde{L}(\gamma)\int_a^t \|X_k(s,\gamma) - X_{k-1}(s,\gamma)\|\,\Delta s \leq \widetilde{L}(\gamma)\tfrac{M}{k!}\int_a^t (s-a)^k\Delta s$$

$$\leq \widetilde{L}(\gamma)\tfrac{M}{k!}\int_a^t (s-a)^k ds = M\widetilde{L}(\gamma)\tfrac{(t-a)^{k+1}}{(k+1)!} \leq M\widetilde{L}(\gamma)\tfrac{(b-a)^{k+1}}{(k+1)!},$$

for all $t \in [a,b]_{\mathbb{T}}$ and $\gamma \in \Gamma$. Thus, (9) is true for $n = k+1$ and so (9) holds for all $n \geq 2$. Further, we show that for every $n \in \mathbb{N}$ the functions $X_n(\cdot,\gamma) : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are continuous for each $\gamma \in \Gamma$. Let $\varepsilon > 0$ and $t,s \in [a,b]_{\mathbb{T}}$ be such that $|t-s| < \varepsilon/M$. We have

$$\|X_1(t,\gamma) - X_1(s,\gamma)\| = \left\|\int_a^t F(\tau,X_0(\gamma),\gamma)\Delta\tau - \int_a^s F(\tau,X_0(\gamma),\gamma)\Delta\tau\right\|$$

$$= \left\|\int_s^t F(\tau,X_0(\gamma),\gamma)\Delta\tau\right\| \leq \int_s^t \|F(\tau,X_0(\gamma),\gamma)\|\,\Delta\tau \leq \int_s^t \|F(\tau,X_0(\gamma),\gamma)\|\,d\tau$$

$$\leq M|t-s| < \varepsilon$$

and so $t \mapsto X_1(t,\gamma)$ is continuous on $[a,b]_{\mathbb{T}}$ for each $\gamma \in \Gamma$. Since for each $n \geq 2$

$$\|X_n(t,\gamma) - X_n(s,\gamma)\| \leq \int_s^t \|F(\tau, X_{n-1}(\tau,\gamma),\gamma)\| \Delta\tau \leq \int_s^t \|F(\tau, X_0(\gamma),\gamma)\| \Delta\tau$$
$$+ \int_s^t \|F(\tau, X_{n-1}(\tau,\gamma),\gamma) - F(\tau, X_0(\gamma),\gamma)\| \Delta\tau \leq \int_s^t \|F(\tau, X_0(\gamma),\gamma)\| \Delta\tau$$
$$+ \sum_{k=1}^{n-1} \int_s^t \|F(\tau, X_k(\tau,\gamma),\gamma) - F(\tau, X_{k-1}(\tau,\gamma),\gamma)\| \Delta\tau,$$

then, by induction, we obtain

$$\|X_n(t,\gamma) - X_n(s,\gamma)\| \leq M\left(1 + \sum_{k=1}^{n-1} \frac{\widetilde{L}(\gamma)^{k-1}(b-a)^k}{k!}\right)|t-s| \to 0 \text{ as } s \to t.$$

Therefore, for every $n \in \mathbb{N}$ the function $X_n(\cdot,\gamma) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ is continuous for each $\gamma \in \Gamma$. Now, using Lemma 3.3 and (8), we deduce that the functions $X_n(t,\cdot) : \Gamma \to \mathbb{R}^m$ are measurable. Consequently, it follows that for every $n \in \mathbb{N}$ the function $X_n(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}$ is a time scale uncertain process.

Further, we shall show that the sequence $(X_n(t,\cdot))_{n\in\mathbb{N}}$ is uniformly convergent. Denote

$$Y_n(t,\gamma) = \|X_{n+1}(t,\gamma) - X_n(t,\gamma)\|, \ \ n \in \mathbb{N},\ \gamma \in \Gamma.$$

Since

$$Y_n(t,\gamma) - Y_n(s,\gamma) \leq \widetilde{L}(\gamma) \int_s^t \|X_n(\tau,\gamma) - X_{n-1}(\tau,\gamma)\| \Delta\tau$$

then, reasoning as above, we deduce that the functions $t \mapsto Y_n(t,\gamma)$ are continuous on $[a,b]_{\mathbb{T}}$ for each $\gamma \in \Gamma$. Now, using (9), we obtain

$$\sup_{t\in[a,b]_{\mathbb{T}}} \|X_n(t,\gamma) - X_m(t,\gamma)\| \leq \sum_{k=m}^{n-1} \sup_{t\in[a,b]_{\mathbb{T}}} Y_k(t,\gamma) \leq M \sum_{k=m}^{n-1} \frac{\widetilde{L}(\gamma)^k (b-a)^{k+1}}{(k+1)!}$$

for all $n > m > 0$. Since the series $\sum_{n=1}^{\infty} \widetilde{L}(\gamma)^{n-1}(b-a)^n/n!$ converges, then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{t\in[a,b]_{\mathbb{T}}} \|X_n(t,\gamma) - X_m(t,\gamma)\| \leq \varepsilon \ \text{ for all } n,m \geq n_0 \text{ and } \gamma \in \Gamma. \tag{10}$$

Hence, since $([a,b]_{\mathbb{T}}, |\cdot|)$ is a complete metric space, it follows that the sequence $(X_n(t,\cdot))_{n\in\mathbb{N}}$ is uniformly convergent on $[a,b]_{\mathbb{T}}$. Denote $X(t,\gamma) = \lim_{n\to\infty} X_n(t,\gamma)$, $t \in [a,b]_{\mathbb{T}}$, $\gamma \in \Gamma$. Obviously, $t \mapsto X(t,\gamma)$ is continuous on $[a,b]_{\mathbb{T}}$ for each $\gamma \in \Gamma$. Since, by Lemma 2.2 and (8), the functions $\gamma \to X_n(\cdot,\gamma)$ are measurable and $X(t,\gamma) = \lim_{n\to\infty} X_n(t,\gamma)$ for every $t \in [a,b]_{\mathbb{T}}$ and $\gamma \in \Gamma$, we deduce that $\gamma \to X(t,\gamma)$ is measurable for every $t \in [a,b]_{\mathbb{T}}$. Therefore, $X(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ is a continuous time scale uncertain process. We show that $X(\cdot,\cdot)$ satisfies the uncertain integral equation (5). For each $n \in \mathbb{N}$ we put $G_n(t,\gamma) = F(t, X_n(t,\gamma),\gamma)$, $t \in [a,b]_{\mathbb{T}}$, $\gamma \in \Gamma$. Then $G_n(t,\gamma)$ is rd-continuous time scale uncertain process, and we have that

$$\sup_{t\in[a,b]_{\mathbb{T}}} \|G_n(t,\gamma) - G_m(t,\gamma)\| \leq \widetilde{L}(\gamma) \sup_{t\in[a,b]_{\mathbb{T}}} \|X_n(t,\gamma) - X_m(t,\gamma)\|, \ \ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma,$$

for all $n,m \geq n_0$. Using (10) we infer that the sequence $(G_n(\cdot,\gamma))_{n\in\mathbb{N}}$ is uniformly convergent on $[a,b]_{\mathbb{T}}$ for each $\gamma \in \Gamma$. If we take $m \to \infty$, then for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$\sup_{t\in[a,b]_{\mathbb{T}}} \|G_n(t,\gamma) - F(t, X(t,\gamma),\gamma)\| \leq \widetilde{L}(\gamma) \sup_{t\in[a,b]_{\mathbb{T}}} \|X_n(t,\gamma) - X(t,\gamma)\|, \ \ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma,$$

and so $\lim_{n\to\infty} \|G_n(t,\gamma) - F(t, X(t,\gamma),\gamma)\| = 0$ for all $t \in [a,b]_{\mathbb{T}}$ and $\gamma \in \Gamma$. Also, it easy to see that

$$\sup_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t G_n(s,\gamma)\Delta s - \int_a^t F(s, X(s,\gamma),\gamma)\Delta s \right\| \leq \widetilde{L}(\gamma) \int_a^t \|X_n(s,\gamma) - X(s,\gamma)\| \Delta s, \ \gamma \in \Gamma.$$

Since $X(t,\gamma) = \lim_{n\to\infty} X_n(t,\gamma)$ uniformly on $[a,b]_{\mathbb{T}}$, then it follows that

$$\lim_{n\to\infty} \int_a^t G_n(s,\gamma)\Delta s = \int_a^t F(s, X(s,\gamma),\gamma)\Delta s \text{ for all } t \in [a,b] \text{ and } \gamma \in \Gamma.$$

Now, we have

$$\begin{aligned}
&\sup_{t\in[a,b]_{\mathbb{T}}} \left\| X(t,\gamma) - X_0(\gamma) - \int_a^t F(s, X(s,\gamma),\gamma)\Delta s \right\| \le \sup_{t\in[a,b]_{\mathbb{T}}} \|X(t,\gamma) - X_n(t,\gamma)\| \\
&+ \sup_{t\in[a,b]_{\mathbb{T}}} \left\| X_n(t,\gamma) - X_0(\gamma) - \int_a^t F(s, X_{n-1}(s,\gamma),\gamma)\Delta s \right\| \\
&+ \sup_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t F(s, X_{n-1}(s,\gamma),\gamma)\Delta s - \int_a^t F(s, X(s,\gamma),\gamma)\Delta s \right\|.
\end{aligned}$$

Using the two previous convergences

$$X(t,\gamma) = X_0(\gamma) + \int_a^t F(s, X(s,\gamma),\gamma)\Delta s \text{ for all } t \in [a,b]_{\mathbb{T}} \text{ and } \gamma \in \Gamma,$$

that is, $X(\cdot,\cdot)$ satisfies the uncertain integral Equation (5). Then, by Lemma 3.3, it follows that $X(\cdot,\cdot)$ is the solution of the problem (2). Finally, we show the uniqueness of the solution. For this, we assume that $X(\cdot,\cdot), Y(\cdot,\cdot) : [a,b]_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ are two solutions of (5). Since

$$\|X(t,\gamma) - Y(t,\gamma)\| \le \int_a^t \widetilde{L}(\gamma)\, \|X(s,\gamma) - Y(s,\gamma)\|\, ds,\ \ t \in [a,b]_{\mathbb{T}},\ \gamma \in \Gamma,$$

from Lemma 3.4 it follows that $\|X(t,\gamma) - Y(t,\gamma)\| \le 0,\ \ t \in [a,b]_{\mathbb{T}}$, $\gamma \in \Gamma$, and so, the proof is complete.

Let $\mathbb{T}$ be a upper unbounded time scale. Under suitable conditions we can extend the notion of the solution of (2) from $[a,b]^{\kappa}_{\mathbb{T}}$ to $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$, if we define F on $[a,\infty)_{\mathbb{T}} \times \mathbb{R}^m \times \Gamma$ and show that the solution exists on each $[a,b]_{\mathbb{T}}$ where $b \in (a,\infty)_{\mathbb{T}}$, $a < \rho(b)$.

Theorem 3. 2 *Assume that $F : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^m \times \Gamma \to \mathbb{R}^m$ satisfies the assumptions of the Theorem 3.1 on each interval $[a,b]_{\mathbb{T}}$ with $b \in (a,\infty)_{\mathbb{T}}$, $a < \rho(b)$. If there is a constant $M > 0$ such that $\|F(t,x,\gamma)\| \le M$ for all $(t,x) \in [a,b)_{\mathbb{T}} \times \mathbb{R}^m$ then the problem (2) has a unique solution on $[a,\infty)_{\mathbb{T}}$.*

Proof: Let $X(\cdot,\cdot)$ be the solution of (*2*) which exists on $[a,b)_{\mathbb{T}}$ with $b \in (a,\infty)_{\mathbb{T}}$, $a < \rho(b)$, and the value of b cannot be increased. First, we observe that b is a left-scattered point, then $\rho(b) \in (a,b)_{\mathbb{T}}$ and the solution $X(\cdot,\cdot)$ exists on $[a,\rho(b)]_{\mathbb{T}}$. But then the solution $X(\cdot,\cdot)$ exists also on $[a,b]_{\mathbb{T}}$, namely by putting

$$\begin{aligned}
X(b,\gamma) &= X(\rho(b),\gamma) + \mu(b)X^{\Delta}(\rho(b),\gamma) \\
&= X(\rho(b),\gamma) + \mu(b)F(\rho(b), X(\rho(b),\gamma),\gamma).
\end{aligned}$$

If b is a left-dense point, then their neighborhoods contain infinitely many points to the left of b. Then, for any $t, s \in (a,b)_{\mathbb{T}}$ such that $s < t$, we have

$$\|X(t,\gamma) - X(s,\gamma)\| \le \int_s^t \|F(\tau, X(\tau,\gamma),\gamma)\|\, \Delta\tau \le M\, |t-s|\, .$$

Taking limit as $s,t \to b^-$ and using Cauchy criterion for convergence, it follows $\lim_{t\to b^-} X(t,\gamma)$ exists and is finite. Further, we define $X_b(\gamma) = \lim_{t\to b^-} X(t,\gamma)$ and consider the initial value problem

$$\begin{cases} X^{\Delta}(t,\gamma) = F(\tau, X(\tau,\gamma),\gamma),\ \ t \in [b,b_1]_{\mathbb{T}},\ b_1 > \sigma(b), \\ X(b,\gamma) = X_b(\gamma). \end{cases}$$

By Theorem 3.1, one gets that $X(t,\gamma)$ can be continued beyond b, contradicting our assumptions. Hence every solution $X(t,\gamma)$ of (*2*) exists on $[a,\infty)_{\mathbb{T}}$ and the proof is complete.

4 Uncertain Linear Systems

Let $a : \Gamma \to \mathbb{R}$ be a positively regressive uncertain variable, that is, $1 + \mu(t)a(\gamma) > 0$ for all $\gamma \in \Gamma$. Then, by Lemma 2.2, the function $(t, \gamma) \mapsto e_{a(\gamma)}(t, t_0)$ defined by

$$e_{a(\gamma)}(t, t_0) = \left(\int_{t_0}^{t} \frac{\log(1 + \mu(\tau)a(\gamma))}{\mu(\tau)} \Delta\tau \right),\ t_0, t \in \mathbb{T},\ \gamma \in \Gamma,$$

is a continuous time scale uncertain process. For each fixed $\gamma \in \Gamma$, the sample path $t \mapsto e_{a(\gamma)}(t, t_0)$ is the exponential function on time scales (see [4]). It easy to check that the uncertain process $(t, \gamma) \mapsto e_{a(\gamma)}(t, t_0)$ is a solution of the initial value problem (for deterministic case, see [4, Theorem 2.33]):

$$\begin{cases} X^{\Delta}(t, \gamma) = a(\gamma)X(t, \gamma), & t \in [t_0, b]^{\kappa}_{\mathbb{T}},\ \gamma \in \Gamma, \\ X(t_0, \gamma) = 1. \end{cases} \tag{11}$$

If $a : \Gamma \to \mathbb{R}$ is bounded then, by the Theorems 3.1 and 3.2, it follows that (11) has a unique solution on $[t_0, \infty)_{\mathbb{T}}$.

Let us denote by $M_m(\mathbb{R})$ the space of all $m \times m$ matrices. We recall that $||A|| := \sup\{||Ax||; ||x|| \leq 1\}$ define a norm on $M_m(\mathbb{R})$ and the following inequality $||Ax|| \leq ||A|| \cdot ||x||$ holds for all $A \in M_m(\mathbb{R})$ and $x \in \mathbb{R}^m$. A mapping $A : \Gamma \to M_m(\mathbb{R})$ is called an *uncertain matrix* if all its components $a_{ij} : \Gamma \to \mathbb{R}$, $i, j = 1, 2, ..., m$, are uncertain variables. An uncertain matrix A is said to be *regressive* if $I + \mu(t)A(\gamma)$ is invertible for all $t \in \mathbb{T}$ and $\gamma \in \Gamma$, where I is the $m \times m$ identity matrix. Moreover, the set $\mathcal{R}_m = \mathcal{R}(\Gamma, M_m(\mathbb{R}))$ of all regressive uncertain matrices is a group with respect to the addition operation $\oplus$ define

$$A \oplus B = A + B + \mu(t)AB$$

for all $t \in \mathbb{T}$. The inverse element of $A \in \mathcal{R}_m$ is given by

$$\ominus A = -[I + \mu(t)A^{-1}]A = -A[I + \mu(t)A]^{-1}$$

for all $t \in \mathbb{T}$.

Now consider the following homogeneous linear uncertain initial value problem

$$\begin{aligned} &X^{\Delta}(t, \gamma) = A(\gamma)X(t, \gamma), \quad t \in \mathbb{T},\ \gamma \in \Gamma, \\ &X(t_0, \gamma) = X_0(\gamma). \end{aligned} \tag{12}$$

where $A \in \mathcal{R}_m$. The corresponding nonhomogeneous linear uncertain initial value problem is

$$\begin{aligned} &X^{\Delta}(t, \gamma) = A(\gamma)X(t, \gamma) + H(t, \gamma), \quad t \in \mathbb{T},\ \gamma \in \Gamma, \\ &X(t_0, \gamma) = X_0(\gamma). \end{aligned} \tag{13}$$

where $H : \mathbb{T} \times \Gamma \to \mathbb{R}^m$ is an uncertain process.

Theorem 4. 1 *Suppose that $A : \Gamma \to M_m(\mathbb{R})$ is a regressive and bounded uncertain matrix, $X_0 : \Gamma \to \mathbb{R}^m$ is a bounded uncertain variable, and $H(\cdot, \cdot) : [t_0, \infty)_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ is a rd-continuous time scale uncertain process. If there is a constant $\nu > 0$ such that $\|H(t, \gamma)\| \leq \nu$ for all $t \in [t_0, b)_{\mathbb{T}}$ with $b \in (t_0, \infty)_{\mathbb{T}}$, $t_0 < \rho(b)$, then the initial value problem (13) has a unique solution on $[t_0, \infty)_{\mathbb{T}}$.*

Proof: First, we observe that we put $F(t, x, \gamma) := A(\gamma)x + H(t, \gamma)$, then F satisfies the conditions (H_1) and (H_2). Moreover,

$$\|F(t, x, \gamma) - F(t, y, \gamma)\| \leq \|A(\gamma)\| \, \|x - y\|$$

for every $t \in [t_0, \infty)_{\mathbb{T}}$, $x, y \in \mathbb{R}^m$ and $\gamma \in \Gamma$. Therefore, by the Theorem 3.1, it follows that (13) has a unique solution on $[t_0, b]^{\kappa}_{\mathbb{T}}$. Further, let $X(t, \cdot)$ be the solution of (13) which exists on $[t_0, b)_{\mathbb{T}}$ with $b \in (t_0, \infty)_{\mathbb{T}}$, $t_0 < \rho(b)$. Also, let $N > 0$ be such that $\|A(\gamma)\| \leq N$. Then we have

$$\begin{aligned} \|X(t, \gamma)\| &\leq \|X(t_0, \gamma)\| + \int_{t_0}^{t} \|A(\gamma)X(s, \gamma)\| \, \Delta s + \int_{t_0}^{t} \|H(s, \gamma)\| \, \Delta s \\ &\leq 1 + \nu(t - t_0) + N \int_{t_0}^{t} \|X(s, \gamma)\| \, \Delta s. \end{aligned}$$

Then, by the Corollary 6.8 in [4], it follows that

$$\|X(t,\gamma)\| \le (1+\frac{\nu}{N})e_N(t,t_0) - \frac{\nu}{N} \le (1+\frac{\nu}{N})e_N(b,t_0).$$

Hence $\|F(t,X(t,\gamma),\gamma)\| \le M := \nu + (1+\frac{\nu}{N})e_N(b,t_0)$. Proceeding as in the proof of the Theorem 3.2 it follows that the unique solution of (13) exists on $[t_0,\infty)_{\mathbb{T}}$. □

A mapping $\Psi : \mathbb{T}\times\Gamma \to M_m(\mathrm{R})$ is called an *uncertain matrix process* if all its components $\psi_{ij} : \mathbb{T}\times\Gamma \to \mathbb{R}$, $i,j = 1,2,...,m$, are uncertain process. An uncertain matrix process Ψ is said to be *regressive* if $\Psi(t,\cdot) \in \mathcal{R}_m$ for all $t \in \mathbb{T}$. In the following, suppose that $A : \Gamma \to M_m(\mathbb{R})$ is a regressive and bounded uncertain matrix. An uncertain matrix process Ψ_A is said to be an *uncertain matrix solution* of the the following homogeneous linear uncertain differential equation

$$X^{\Delta}(t,\gamma) = A(\gamma)X(t,\gamma),\ t \in \mathbb{T},\ \gamma \in \Gamma, \tag{14}$$

if each column of Ψ_A satisfies (14). An *uncertain fundamental matrix* of (14) is an uncertain matrix solution Ψ_A of (14) such that $\det \Psi_A(t,\gamma) \neq 0$ for all $t \in \mathbb{T}$ and $\gamma \in \Gamma$. An *uncertain transition matrix* of (14) at initial time $s \in \mathbb{T}$ is an uncertain fundamental matrix Ψ_A such that $\Psi_A(s,\gamma) = I$ for all $\gamma \in \Gamma$. The uncertain transition matrix of (14) at initial time $s \in \mathbb{T}$ will be denoted by $U_A(t,s)$. Therefore, the uncertain transition matrix of (14) at initial time $s \in \mathbb{T}$ is the unique solution of the following uncertain matrix initial value problem

$$\Phi^{\Delta}(t,\gamma) = A(\gamma)\Phi(t,\gamma), \quad \Phi(s,\gamma) = I, \tag{15}$$

and $X(t,\gamma) = U_A(t,s)X(s,\gamma)$, $t \ge s$, is the unique solution of the following uncertain initial value problem

$$\begin{aligned} &X^{\Delta}(t,\gamma) = A(\gamma)X(t,\gamma),\ t \in \mathbb{T},\ \gamma \in \Gamma, \\ &X(s,\gamma) = X_0(\gamma). \end{aligned}$$

The existence and uniqueness of the solution of (15) follows from the Theorem 3.1. The uncertain transition matrix of (14) at initial time $s \in \mathbb{T}$ is also called the *uncertain matrix exponential function* (at s), and it is denoted by $e_{A(\gamma)}(t,s)$ or $e_A(t,s)$.

In the following theorem we give some properties of the uncertain transition matrix. The proof of the theorem is the same that in [4, Theorem 5.21].

Theorem 4. 2 *If $A : \Gamma \to M_m(\mathbb{R})$ is a regressive and bounded uncertain matrix, then*

(1) $U_A(t,t) = I$;

(2) $U_A(\sigma(t),s) = [I + \mu(t)A(\gamma)]U_A(t,s)$;

(3) $U_A^{-1}(t,s) = U^T_{\ominus A^T}(t,s)$;

(4) $U_A(t,s) = U_A^{-1}(s,t) = U^T_{\ominus A^T}(s,t)$;

(5) $U_A(t,s)U_A(s,r) = U_A(t,r)$,

for all $t,s,r \in \mathbb{T}$ with $t > s > r$ and all $\gamma \in \Gamma$.

Theorem 4. 3 *(Variation of Constants). Suppose that the assumptions of the Theorem 4.1 hold. Then the unique solution $X(\cdot,\cdot) : [t_0,\infty)_{\mathbb{T}} \times \Gamma \to \mathbb{R}^m$ of the uncertain initial value problem (13) is given by*

$$X(t,\gamma) = U_A(t,t_0)X_0(\gamma) + \int_{t_0}^{t} U_A(t,\sigma(s))H(s,\gamma)\Delta s,\ t \in [t_0,\infty)_{\mathbb{T}},\ \gamma \in \Gamma. \tag{16}$$

Proof: Indeed, we can rewritten (16) as

$$X(t,\gamma) = U_A(t,t_0)\left[X_0(\gamma) + \int_{t_0}^{t} U_A(t_0,\sigma(s))H(s,\gamma)\Delta s\right].$$

Using the product rule to differentiate $X(t,\cdot)$, we infer

$$\begin{aligned} X^{\Delta}(t,\gamma) &= A(\gamma)U_A(t,t_0)\left[X_0(\gamma)+\int_{t_0}^{t}U_A(t_0,\sigma(s))H(s,\gamma)\Delta s\right] \\ &\quad +U_A(\sigma(t),t_0)U_A(t_0,\sigma(t))H(t,\gamma) \\ &= A(\gamma)X(t,\gamma)+H(t,\gamma) \end{aligned}$$

Obviously, $X(t_0,\gamma)=X_0(\gamma)$. Therefore, $X(t,\gamma)$ is the solution of (16). □

Corollary 4. 1 *Let $X_0:\Gamma\to\mathbb{R}^m$ be a bounded uncertain variable. If $A:\Gamma\to M_m(\mathbb{R})$ is a regressive and bounded uncertain matrix, then the unique solution of the uncertain initial value problem (12) is given by*

$$X(t,\gamma)=U_A(t,t_0)X_0(\gamma),\ t\in[t_0,\infty)_{\mathbb{T}},\ \gamma\in\Gamma.$$

Theorem 4. 4 *(Variation of Constants). Suppose that the assumptions of the Theorem 4.1 hold. Then the unique solution $X(\cdot,\cdot):[t_0,\infty)_{\mathbb{T}}\times\Gamma\to\mathbb{R}^m$ of the following uncertain initial value problem*

$$\begin{cases} X^{\Delta}(t,\gamma)=-A^T(\gamma)X^{\sigma}(t,\gamma)+H(t,\gamma), & t\in[t_0,\infty)_{\mathbb{T}},\ \gamma\in\Gamma, \\ X(t_0,\gamma)=X_0(\gamma), \end{cases} \tag{17}$$

on $[t_0,\infty)_{\mathbb{T}}$ given by

$$X(t,\gamma)=\Psi_{\ominus A^T(\gamma)}(t,t_0)X_0(\gamma)+\int_{t_0}^{t}\Psi_{\ominus A^T(\gamma)}(t,s)H(s,\gamma)\Delta s,\ t\in[t_0,\infty)_{\mathbb{T}},\ \gamma\in\Gamma. \tag{18}$$

Proof: Indeed, we can rewrite (17) as

$$\begin{aligned} X^{\Delta}(t,\gamma) &= -A^T(\gamma)[X(t,\gamma)+\mu(t)X^{\Delta}(t,\gamma)]+H(t,\gamma) \\ &= -A^T(\gamma)X(t,\gamma)-\mu(t)A^T(\gamma)X^{\Delta}(t,\gamma)+H(t,\gamma), \end{aligned}$$

that is,

$$[I+\mu(t)A^T(\gamma)]X^{\Delta}(t,\gamma)=-A^T(\gamma)X(t,\gamma)+H(t,\gamma).$$

Since the matrix $A(\gamma)$ is regressive, then $A^T(\gamma)$ is also regressive, and hence we infer that

$$\begin{aligned} X^{\Delta}(t,\gamma) &= -[I+\mu(t)A^T(\gamma)]^{-1}A^T(\gamma)X(t,\gamma)+[I+\mu(t)A^T(\gamma)]^{-1}H(t,\gamma) \\ &= \ominus A^T(\gamma)X(t,\gamma)+[I+\mu(t)A^T(\gamma)]^{-1}H(t,\gamma), \end{aligned}$$

that is,

$$X^{\Delta}(t,\gamma)=\ominus A^T(\gamma)X(t,\gamma)+[I+\mu(t)A^T(\gamma)]^{-1}H(t,\gamma).$$

Now, using the Theorem 4.3 and the properties of the uncertain transition matrix, we obtain that

$$\begin{aligned} X(t,\gamma) &= U_{\ominus A^T}(t,t_0)X_0(\gamma)+\int_{t_0}^{t}U_{\ominus A^T}(t,\sigma(s))[I+\mu(t)A^T(\gamma)]^{-1}H(s,\gamma)\Delta s \\ &= U_{\ominus A^T}(t,t_0)X_0(\gamma)+\int_{t_0}^{t}U_A^T(t,\sigma(s))[I+\mu(t)A^T(\gamma)]^{-1}H(s,\gamma)\Delta s \\ &= U_{\ominus A^T}(t,t_0)X_0(\gamma)+\int_{t_0}^{t}\{[I+\mu(t)A(\gamma)]^{-1}U_A(\sigma(s),t)\}^TH(s,\gamma)\Delta s \\ &= U_{\ominus A^T}(t,t_0)X_0(\gamma)+\int_{t_0}^{t}U_A(s,t)H(s,\gamma)\Delta s, \end{aligned}$$

that is (18). □

Corollary 4. 2 *Let $X_0:\Gamma\to\mathbb{R}$ be a bounded uncertain variable. If $A:\Gamma\to M_m(\mathbb{R})$ is a regressive and bounded uncertain matrix, then the unique solution of the following uncertain initial value problem*

$$\begin{cases} X^{\Delta}(t,\gamma)=-A(\gamma)X^{\sigma}(t,\gamma), & t\in[t_0,\infty)_{\mathbb{T}},\ \gamma\in\Gamma, \\ X(t_0,\gamma)=X_0(\gamma), \end{cases}$$

is given by

$$X(t,\gamma)=U_{\ominus A^T}(t,t_0)X_0(\gamma),\ t\in[t_0,\infty)_{\mathbb{T}},\ \gamma\in\Gamma.$$

Example 4. 1 *Let us consider $\Gamma = (0,1)$, $\mathcal{L}$ the σ-algebra of all Borel subsets of Γ, $\mathcal{M}$ the uncertain measure on Γ defined in Example 2.1, and the following initial value problem*

$$\begin{cases} X^{\Delta}(t,\gamma) = \gamma X(t,\gamma) + e_{\gamma}(t,0), \ \ t \in [0,\infty)_{\mathbb{T}}, \ \gamma \in \Gamma, \\ X(0,\gamma) = \gamma. \end{cases} \tag{19}$$

Then, by the Theorems 4.1 and 4.3, the initial value problem (19) has a unique solution on $[0,\infty)_{\mathbb{T}}$, given by

$$X(t,\gamma) = \gamma e_{\gamma}(t,0) + \int_0^t e_{\gamma}(t,\sigma(s)) e_{\gamma}(s,0)\Delta s,$$

that is,

$$X(t,\gamma) = e_{\gamma}(t,0)\left(\gamma + \int_0^t \frac{1}{1+\mu(s)\gamma}\Delta s\right), \ \ t \in [0,\infty)_{\mathbb{T}}.$$

Next, consider two particular cases.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ for all $t \in \mathbb{N}$, and $e_{\gamma}(t,0) = e^{\gamma t}$. Moreover, in this case we have

$$\int_0^t \frac{1}{1+\mu(s)\gamma}\Delta s = \int_0^t ds = t.$$

It follows that the initial value problem

$$\begin{cases} X^{\Delta}(t,\gamma) = \gamma X(t,\gamma) + e^{\gamma t}, \ \ t \in [0,\infty) \\ X(0,\gamma) = \gamma, \end{cases}$$

has the solution $X(t,\gamma) = (\gamma + t)e^{\gamma t}, \ \ t \in [0,\infty)$.

If $\mathbb{T} = \mathbb{N}$, then $\mu(n) = 1$ for all $n \in \mathbb{N}$, and $e_{\gamma}(n,0) = (1+\gamma)^n$. Moreover, in this case we have

$$\int_0^t \frac{1}{1+\mu(s)\gamma}\Delta s = \sum_{s\in[0,n)} \frac{1}{1+\gamma} = \frac{n}{1+\gamma}.$$

It follows that the difference initial value problem

$$\begin{cases} X_{n+1}(\gamma) = (1+\gamma)X_n(\gamma) + (1+\gamma)^n, \ \ n \in \mathbb{N} \\ X_0(\gamma) = \gamma, \end{cases}$$

has the solution

$$X_n(\gamma) = (\gamma + \frac{n}{1+\gamma})(1+\gamma)^n, \ n \in \mathbb{N}.$$

Example 4. 2 *Let us consider $\Gamma = (0,1)$,$\mathcal{L}$ the σ-algebra of all Borel subsets of Γ, $\mathcal{M}$ the uncertain measure on Γ defined in Example 2.1, and the following initial value problem*

$$\begin{cases} X^{\Delta}(t,\gamma) = -\gamma X^{\sigma}(t,\gamma) + e_{\ominus\gamma}(t,t_0), \ \ t \in [0,\infty)_{\mathbb{T}}, \ \gamma \in \Gamma, \\ X(0,\gamma) = \gamma. \end{cases} \tag{20}$$

The initial value problem (20) has a unique solution on $[t_0,\infty)_{\mathbb{T}}$, given by

$$X(t,\gamma) = \gamma e_{\ominus\gamma}(t,t_0) + \int_0^t e_{\ominus\gamma}(t,s) e_{\ominus\gamma}(s,0)\Delta s,$$

that is,

$$X(t,\gamma) = (\gamma + t)\, e_{\ominus\gamma}(t,0), \ \ t \in [0,\infty)_{\mathbb{T}}, \ \ \gamma \in (0,1).$$

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ for all $t \in \mathbb{R}$, and $e_{\ominus\gamma}(t,0) = e^{-\gamma t}$. It follows that $X(t,\gamma) = (\gamma + t) = e^{-\gamma t}$, $t \in [0,\infty)_{\mathbb{T}}$, $\gamma \in (0,1)$.

If $\mathbb{T} = h\mathbb{N}$ with $h > 0$, then $\mu(t) = h$ for all $t \in h\mathbb{N}$, and $e_{\ominus\gamma}(t, 0) = (1 + \gamma h)^{-t/h}$. It follows that the h-difference initial value problem

$$\begin{cases} X_{t+h}(\gamma) = \frac{1}{1+\gamma h} X_t(\gamma) + h(1 + \gamma h)^{-t/h-1}, & t \in h\mathbb{N} \\ X_0(\gamma) = \gamma, \end{cases}$$

has the unique solution $X_t(\gamma) = (\gamma + t)\,(1 + \gamma h)^{-t/h}$, $t \in h\mathbb{N}$.

If $\mathbb{T} = 2^{\mathbb{N}}$, then $\mu(t) = t$ for all $t \in 2^{\mathbb{N}}$, and $e_{\ominus\gamma}(t, 0) = \prod_{s\in[0,t)} (1 + \gamma s)^{-1}$. It follows that the 2-difference initial value problem

$$\begin{cases} X_t(\gamma) = (1 + \gamma t)X_{2t}(\gamma) - t \prod_{s\in[1,t)} (1 + \gamma s)^{-1}, & t \in 2^{\mathbb{N}} \\ X_1(\gamma) = \gamma, \end{cases}$$

has the unique solution $X_t(\gamma) = (\gamma + t) \prod_{s\in[1,t)} (1 + \gamma s)^{-1}$, $t \in 2^{\mathbb{N}}$.

Example 4. 3 *Let us consider $\Gamma = (0, 1)$, $\mathcal{L}$ the $\sigma-$algebra of all Borel subsets of Γ, $\mathcal{M}$ the uncertain measure on Γ defined in Example 2.1, and the following initial value problem*

$$X^{\Delta}(t, \gamma) = \begin{bmatrix} 1 & \gamma \\ 0 & -1 \end{bmatrix} X(t, \gamma), \quad X(0, \gamma) = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}, \quad t \in [0, \infty)_{\mathbb{T}}, \tag{21}$$

where $1 - \mu(t) \neq 0$ for $t \in [0, \infty)_{\mathbb{T}}$. The matrix $A = \begin{bmatrix} 1 & \gamma \\ 0 & -1 \end{bmatrix}$ has the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$ with the corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -\frac{\gamma}{2} \end{bmatrix}$, respectively. Then

$$\Psi_A(t, \gamma) = \begin{bmatrix} e_{-1}(t, 0) & e_1(t, 0) \\ 0 & -\frac{2}{\gamma} e_1(t, 0) \end{bmatrix}, \quad \Psi_A^{-1}(s, \gamma) = \begin{bmatrix} e_1(s, 0) & \frac{\gamma}{2} e_1(s, 0) \\ 0 & \frac{\gamma}{2} e_{-1}(s, 0) \end{bmatrix}$$

and therefore, the uncertain transition matrix for (21) is given by

$$U_A(t, s) = \begin{bmatrix} e_{-1}(t, s) & \frac{\gamma}{2} e_{-1}(t, s) - \frac{\gamma}{2} e_1(t, s) \\ 0 & e_{-1}(t, s) \end{bmatrix}.$$

It follows that the solution of the initial value problem (21) is given by

$$X(t, \gamma) = U_A(t, 0)X(0, \gamma) = \begin{bmatrix} \frac{3\gamma}{2} e_{-1}(t, 0) - \frac{\gamma}{2} e_1(t, 0) \\ e_{-1}(t, 0) \end{bmatrix} t \in [0, \infty)_{\mathbb{T}}, \ \gamma \in (0, 1).$$

5 Conclusion

The theory of dynamical systems with uncertain parameters can be successfully applied to modeling many real-world phenomena such as in biology to study the population growth, in especially of microorganisms such as bacteria, in physical systems where heterogeneous micro-scale structures are present, in the propagation of electromagnetic waves through a dielectric material with variability in the relaxation time and so on. Some of these applications will be the subject of future works.

Acknowledgments

This work was supported by Higher Education Commission of Pakistan.

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