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A Hybrid Optimal Control Model

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Abstract

A hybrid variable is a mathematical notion used to describe a situation in which randomness and fuzziness simultaneously appear in a system. Based on this concept, a hybrid optimal control problem is presented and investigated. In order to examine this hybrid optimal control problem, we first derive the Bellman's Optimality Principle. The principle is then used to prove a fundamental result called the equation of optimality for hybrid optimal control. This last result is applied to solve a portfolio selection problem in which the price process for a stock is described by a hybrid differential equation.

Keywords: hybrid variable, hybrid process, hybrid optimal control, equation of optimality, portfolio selection

1 Introduction

In recent years it has emerged that, in addition to randomness, various other forms of indeterminacy are inherent in most of the events and processes that we encounter on a day-to-day basis. However, the main forms of indeterminacy are randomness, fuzziness, uncertainty and their interaction effects. For more details on these forms of uncertainty the reader is referred to [4, 7, 10, 15], and references therein. Randomness is an attribute of a phenomenon whose evolution obeys probabilistic laws. On the other hand fuzziness is a notion that describes processes or events whose measurement is intrinsically dim and imperfect.

However, it is common, for example, to hear traders stating that there is a high probability that the market price of a certain stock will be good at the opening of the next trading day. For illustrative purposes let us assume that the current price of the share under discussion is \$5.00 and consider that the price of the same stock on the opening of the next trading day can take on one of the possible qualitative values "not good", "good", "very good". By consulting financial economics experts or by analysing historical data for the stock, it is possible to estimate the probability distribution of the variable, " price of stock at opening of market on the following day". Suppose that P(not good) = 1/4, P(good) = 1/2, P(very good) = 1/4. The challenge is to determine where we put the demarcation between, for example, "not good" and "good". Similarly, it is not unambiguously clear where the line between "good" and "very good" would be placed. The demarcation between any two of the above scenarios cannot be placed at a definitive position. For example, if the value of the stock falls to \$4,99 will, this be a good price? What about \$5,00, \$5,02, \$5,50 or \$8,00? So, the variable, "price of stock at opening on next day," is a hybrid variable. It is a random variable taking fuzzy 'variable' values. Fuzzy random variables were introduced by Kwakernaak [7], as a random element taking fuzzy variable values. Fuzzy random variables were also studied by Kruse and Meyer [6], Liu and Liu [11], Negoita and Ralescu [13] and Puri and Ralescu [15]. More rigorously, the term hybrid variable was proposed by Liu [8] to describe the phenomena with both fuzziness and randomness. A hybrid variable is a measurable function from a chance space to the set of real numbers. Chance theory, the counterpart of probability theory and fuzzy theory, is employed to investigate the evolution of phenomena with fuzziness and randomness.

The concept of control can be described as the process of influencing the behaviour of a dynamical system to achieve a desired goal. If the goal is to optimise some payoff function (or cost function) which depends

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on the control inputs to the system, then the problem is one of optimal control. Optimal control theory has achieved extensive developments with application in many fields such as production engineering, programming, economy, finance and management.

Merton gave a remarkable impetus to the study of stochastic optimal control problems for finance in the 1970's [12], and since then, the mathematical branch of stochastic optimal control theory has attracted a lot of attention from many scientists. In Merton's portfolio selection problem [12] the price process of a stock is modelled using a geometric Brownian motion. This paper is partly motivated by the need to examine the portfolio selection problem where the price process of the stock is driven by both randomness and fuzziness. Fleming and Rishel [2], Harrison [3] and Karatzas [5] studied the optimal control problem of Browian motion or stochastic differential equations and applications in finance. Dixit and Pindyck [1] studied the use of dynamic programming in optimization over Ito processes. Oksendal and Sulem [14] presented the theory of applied stochastic control for jump diffusions. Recently, Zhu [17] introduced the notion of optimal control for fuzzy processes by considering a dynamical system whose evolution is driven by a fuzzy differential equation. Zhu derived the optimality principle for fuzzy control and subsequently proved the equation of optimality. To illustrate the applicability of fuzzy control theory, Zhu [17] applied the equation of optimality to solve a two-asset portfolio selection problem where one of the assets is a risk-free bank account and the second one is a stock whose price process is driven a geometric C-process expressed as a fuzzy differential equation.

The major motivation of this paper lies in the need to extend Zhu's geometric fuzzy model of the stock price to the hybrid case. From a more general perspective, this paper has been motivated by the need to generalise fuzzy optimal control presented in [17] to hybrid optimal control by considering processes whose evolution is driven by both the Brownian motion and the C-process. As explained in paragraph two of this paper, the notion of a hybrid process constitutes a more realistic mathematical description of various phenomena that occur in nature and society.

The use of a hybrid differential equation in mathematical control problem constitutes the novelty of the paper. Using the hybrid differential equation we derive the optimality principle and also proceed to prove an optimality equation for hybrid control. In our case, the optimality equation is a non-linear second order partial differential equation. This is unlike in Zhu's [17] case where the optimality equation is a first-order partial differential equation. After proving the non-linear second order partial differential equation of optimality for hybrid control we applied it to investigate a portfolio selection problem. Two assets, a risk-free bank account and a risky asset whose dynamics are driven by a Brownian motion and a C-process, are considered. No earlier research has examined a hybrid optimal control. This paper seeks to address that gap.

The paper is organised as follows. In Section 2, we review some notations and concepts, such as chance space, hybrid variable, expected value of a hybrid variable, hybrid process, and hybrid differential equation. In Section 3, we derive the principle of optimality for hybrid control based on the Bellman's Optimality Principle. The principle is then used to prove a fundamental result called the equation of optimality for hybrid optimal control in Section 4, and this last result is applied to solve a portfolio selection problem in which the price process for a stock is described by a hybrid differential equation.

2 Preliminary

We start by providing some essential concepts and theorems related to hybrid processes. Let Θ be a nonempty set and $\mathcal{P}$ the power set of Θ, and let Ω be a nonempty set, and $\mathcal{A}$ a σ-algebra over Ω. If Ω is countable, usually $\mathcal{A}$ is the power set of Ω. If Ω is uncountable, for example $\Omega = [0, 1]$, usually $\mathcal{A}$ is the Borel $\sigma-$algebra on Ω.

Definition 2.1 ([8]) *Let $(\Theta, \mathcal{P}, Cr)$ be a credibility space and $(\Omega, \mathcal{A}, Pr)$ be a probability space. The product $(\Theta, \mathcal{P}, Cr) \times (\Omega, \mathcal{A}, Pr)$ is called a chance space.*

It can be easily observed that the cross product $\Theta \times \Omega$ is the universal set that consists of elements of the form (θ, ω), where $\theta \in \Theta$ and $\omega \in \Omega$.

Definition 2.2 ([9]) *Let $(\Theta, \mathcal{P}, Cr) \times (\Omega, \mathcal{A}, Pr)$ be a chance space. A subset $\Lambda \subset \Theta \times \Omega$ is called an event if $\Lambda(\theta) = \{\omega \in \Omega | (\theta, \omega) \in \Lambda\} \in \mathcal{A}$ for each $\theta \in \Theta$.*

Definition 2.3 ([9]) *Let $(\Theta, \mathcal{P}, Cr) \times (\Omega, \mathcal{A}, Pr)$ be a chance space, and $\mathcal{L}$ a collection of all events i.e., $\mathcal{L} = \{\Lambda \subset \Theta \times \Omega | \Lambda(\theta) \in \mathcal{A}, \forall \theta \in \Theta\}$. Then $\mathcal{L}$ is a σ-algebra over $\Theta \times \Omega$.*

Definition 2.4 ([8]) *A hybrid variable is a measurable function from a chance space* $(\Theta, \mathcal{P}, Cr) \times (\Omega, \mathcal{A}, Pr)$ *to the set of real numbers, i.e., for any Borel set* B *of real numbers the set*

$$\{\xi \in B\} = \{(\theta, \omega) \in \Theta \times \Omega | \xi(\theta, \omega) \in B\}$$

is an event.

Definition 2.5 *The chance distribution* $\phi : \mathbb{R} \to [0, 1]$ *of a hybrid variable* ξ *is defined by*

$$\phi(x) = Ch\{(\theta, \omega) \in \Theta \times \Omega | \xi(\theta, \omega) \leq x\}.$$

Definition 2.6 *Let* ξ *be a hybrid variable. Then the expected value of* ξ *is defined by*

$$E[\xi] = \int_0^{+\infty} Ch\{\xi \geq r\}dr - \int_{-\infty}^0 Ch\{\xi \geq r\}dr$$

provided that at least one of the two integrals is finite.

Definition 2.7 *Let* ζ *be a random variable and* η *be a fuzzy variable with finite expected values. Then the hybrid variable* $\xi = \zeta + \eta$ *has expected value* $E[\xi] = E[\zeta] + E[\eta]$.

Definition 2.8 ([10]) *A hybrid process* X_t *is said to have independent increments if*

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$$

are independent hybrid variables for any times $t_0 < t_1 < \cdots < t_k$. *A hybrid process* X_t *is said to have stationary increments if, for any given* $s > 0$, $X_{t+s} - X_t$ *are identically distributed hybrid variables for all* $t > 0$.

Definition 2.9 ([10]) *Let* B_t *be a Brownian motion and* C_t *a Liu process. Then*

$$D_t = (B_t, C_t)$$

is called a hybrid process. The hybrid process is said to be standard if B_t *and* C_t *are both standard.*

Definition 2.10 ([9]) *Suppose* B_t *is a standard Brownian motion,* C_t *is a standard Liu process, and* μ, σ, γ *are some given functions. Then*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t + \gamma(t, X_t)dC_t \tag{1}$$

is called a hybrid differential equation. A solution is a hybrid process X_t *that satisfies (1) identically in* t.

We assume that $\mu(.,.)$, $\sigma(.,.)$ and $\gamma(.,.)$ satisfy the conditions for the existence and uniqueness of a solution X_t in (1) Using the notion of hybrid differential equation, Liu [9] proposed a basic stock model for hybrid financial markets as a hybrid counterpart of Black-Scholes stock model.

3 Problem of Hybrid Optimal Control

The hybrid optimal control paradigm shift consists in intervening in the evolution of a dynamical system in a manner that produces the best results according to a pre-specified objective function. The central assumption is that the system is governed by a hybrid differential equation. Unless stated otherwise, we assume that B_t is a 1-dimensional standard Brownian motion and C_t is a 1-dimensional standard Liu process. Assume that $\mathcal{U}$ is a compact subset of $\mathbb{R}$. Suppose that if we apply the control process $u = u(t) \in \mathcal{U}$ the state of a system at time t is described by a controlled hybrid process X_t^u of the form

$$dX_t^u = \mu(t, X_t^u, u)ds + \sigma(t, X_t^u, u)dB_t + \gamma(t, X_t^u, u)dC_t, \quad X_0^u = x \tag{2}$$

where $X_t \in \mathbb{R}$, $\mu : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $\gamma : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$. Here $u \in \mathcal{U} \subset \mathbb{R}$ is a parameter whose value we can choose in the given set $\mathcal{U}$ at any instant s in order to control the process X_t. We also assume that for a given control process u, the above equation has a unique solution $X_t^u(.)$.

For any $0 < t < T$, let $V(t, x)$ denote the expected optimal reward obtainable in $[t, T]$ with the condition that at time t we are in state $X_t = x$. In the interest of elegance, we shall drop the superscript u in Equation

(2), so that w have $X^u_t = X_t$ for $t \in [0,T]$. We consider the following hybrid optimization problem: Find the value function V and the optimal control $u^* \in \mathcal{A}$ such that

$$V(t,x) \equiv \sup_u E\Big[\int_t^T f(s,X_s,u)ds + G(X(T),T)\Big] = E\Big[\int_t^T f(s,X_s,u^*)ds + G(X(T),T)\Big] \tag{3}$$

subject to Equation (2).

Below is the principle of optimality for hybrid optimal control.

Theorem 3.1 (Principle of Optimality) *For every $(t,x) \in [0,T]\times\mathbb{R}$ and every $\triangle t > 0$, with $t+\triangle t < T$, we have*

$$V(x,t) = \sup_u E\Big[\int_t^{t+\triangle t} f(s,X_s,u)ds + V(t+\triangle t, x+\triangle X_t)\Big], \tag{4}$$

where $x + \triangle X_t = X_{t+\triangle t}$.

Proof: We denote the right hand side of (4) by $\hat{V}(t,x)$. It follows from definition of $V(t,x)$ that

$$V(t,x) \geq E\left[\int_t^{t+\triangle t} f(s,X_s,u|_{[t,t+\triangle t)})ds + \int_{t+\triangle t}^T f(s,X_s,u|_{[t+\triangle t,T]})ds + G(X(T),T)\right], \tag{5}$$

for any u, where $u|_{[t,t+\triangle t)}$ and $u|_{[t+\triangle t,T]}$ are the values of decision u restricted on $[t,t+\triangle t)$ and $[t+\triangle t,T]$, respectively. Since the hybrid process $D_t = (dB_t, dC_t)$ $(s \in [t,t+\triangle t))$ and $D_t = (dB_t, dC_t)$ $(s \in [t+\triangle t,T])$ are independent, we know that

$$\int_t^{t+\triangle t} f(s,X_s,u|_{[t,t+\triangle t)})ds \quad \text{and} \quad \int_{t+\triangle t}^T f(s,X_s,u|_{[t+\triangle t,T]})ds$$

are independent. Thus

$$V(t,x) \geq E\left[\int_t^{t+\triangle t} f(s,X_s,u|_{[t,t+\triangle t)})ds + \left[\int_{t+\triangle t}^T f(s,X_s,u|_{[t+\triangle t,T]},s)ds + G(X(T),T)\right]\right], \tag{6}$$

by Definition 2.7. Taking the supremum with respect to $u|_{[t+\triangle t,T]}$ first, and then $u|_{[t,t+\triangle t)}$ in (6) gives $V(t,x) \geq \hat{V}(t,x)$.

On the other hand, for all u, we have

$$\begin{aligned}
& E\Big[\int_t^{t+\triangle t} f(s,X_s,u)ds + V(t+\triangle t, x+\triangle X_t)\Big] \\
= \;& E\left\{\int_t^{t+\triangle t} f(s,X_s,u|_{[t,t+\triangle t)})ds + \left[\int_{t+\triangle t}^T f(s,X_s,u|_{[t+\triangle t,T]}ds + G(X(T),T)\right]\right\} \\
\leq \;& E\Big[\int_t^{t+\triangle t} f(s,X_s,u)ds + V(t+\triangle t, x+\triangle X_t)\Big] \\
\leq \;& \hat{V}(t,x).
\end{aligned}$$

Hence, $V(t,x) \leq \hat{V}(t,x)$, and then $V(t,x) = \hat{V}(t,x)$. The theorem is proved.

4 Equation of Optimality

Now assume that $D_t = (B_t, C_t)$ is a standard D process. Consider the hybrid optimal control (2) and (3). We want to derive a fundamental result of optimality in hybrid optimal control

Theorem 4.1 (Equation of Optimality) *Let $V(t,x) : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$ function. Then we have*

$$-V_t(t,x) = \sup_u \Big\{f(t,x,u) + V_x(t,x)\mu(t,x,u) + \frac{1}{2}V_{xx}(t,x)\sigma^2(t,x,u)\Big\}. \tag{7}$$

Proof: Take $\triangle t > 0$ to be small, we observe that

$$\int_t^{t+\triangle t} f(s, X_s.u)ds = f(t, x, u)\triangle t + o(\triangle t). \tag{8}$$

By Taylor series expansion, we have

$$\begin{aligned} V(t+\triangle t, x+\triangle X_t) &= V(t,x) + V_t(t,x)\triangle t + V_x(t,x)\triangle X_t + \frac{1}{2}V_{tt}(t,x)(\triangle t)^2 \\ &\quad + \frac{1}{2}V_{xx}(t,x)(\triangle X_t)^2 + V_{xt}(t,x)\triangle t\triangle X_t + o(\triangle t). \end{aligned} \tag{9}$$

Substituting Equations (8) and (9) into Equation (4) we obtain

$$\begin{aligned} V(t,x) &= \sup_u E\Big\{ f(t,x,u)\triangle t + V(t,x) + V_t(t,x)\triangle t + V_x(t,x)\triangle X_t + \frac{1}{2}V_{tt}(t,x)(\triangle t)^2 \\ &\quad + \frac{1}{2}V_{xx}(t,x)(\triangle X_t)^2 + V_{xt}(t,x)\triangle t\triangle X_t + o(\triangle t)\Big\}, \end{aligned}$$

or

$$\begin{aligned} 0 &= \sup_u E\Big\{ f(t,x,u)\triangle t + V_t(t,x)\triangle t + V_x(t,x)\triangle X_t + \frac{1}{2}V_{tt}(t,x)(\triangle t)^2 + \frac{1}{2}V_{xx}(t,x)(\triangle X_t)^2 \\ &\quad + V_{xt}(t,x)\triangle t\triangle X_t + o(\triangle t)\Big\} \\ &= \sup_u \Big\{ f(t,x,u)\triangle t + V_t(t,x)\triangle t + V_x(t,x)E[\triangle X_t] + \frac{1}{2}V_{xx}(t,x)E[\triangle X_t^2] \\ &\quad + V_{tx}(t,x)E[\triangle t\triangle X_t] + o(\triangle t)\Big\}. \end{aligned} \tag{10}$$

From (1), we can formally write

$$\begin{aligned} \triangle X_t^2 &= \mu^2\triangle t^2 + \sigma^2\triangle B_t^2 + \gamma^2\triangle C_t^2 + 2\sigma\gamma\triangle B_t\triangle C_t + 2\mu\triangle t(\sigma\triangle B_t + \gamma\triangle C_t). \\ \triangle t\triangle X_t &= \mu\triangle t^2 + \sigma\triangle t\triangle B_t + \gamma\triangle t\triangle C_t. \end{aligned}$$

From [10] the multiplication rules of the hybrid calculus are

$$\triangle t\triangle t = \triangle t\triangle B_t = \triangle t\triangle C_t = \triangle B_t\triangle C_t = \triangle C_t\triangle C_t = 0, \quad \triangle B_t\triangle B_t = \triangle t,$$

Equation (10) reduces to

$$\begin{aligned} -V_t(t,x)\triangle t &= \sup_u \Big\{ f(t,x,u)\triangle t + V_x(t,x)\mu(t,x,u)\triangle t + \frac{1}{2}V_{xx}(t,x)\sigma^2\triangle t \\ &\quad + V_x(t,x)E[\sigma\triangle B_t + \gamma\triangle C_t] + o(\triangle t)\Big\} \\ &= \sup_u \Big\{ f(t,x,u)\triangle t + V_x(t,x)\mu(t,x,u)\triangle t + \frac{1}{2}V_{xx}(t,x)\sigma^2\triangle t + o(\triangle t)\Big\}, \end{aligned} \tag{11}$$

since B_t is a standard Brownian motion and C_t is a standard Liu process.

Dividing Equation (11) by $\triangle t$, and letting $\triangle t \to 0$, we obtain the result (7). This completes the proof of the theorem.

Remark The equation of optimality in hybrid optimal control gives a necessary condition for an extremum. If the equation has solutions, then the optimal decision and optimal expected value of the objective function are determined. If function f is convex in its arguments, then the equation will produce a minimum, and if f is concave in its arguments, then it will produce a maximum. We note that the boundary condition for the equation is $V(T, X(T)) = G(T, X(T))$.

5 A Portfolio Selection Model

The problem of portfolio selection has been examined by several researchers over the years. Under the assumption that only two investment alternatives, a safe asset and a stock, exist, Merton [12] formulated and solved this problem using optimal stochastic control. In his work Merton considered that the stock price process is driven by a geometric Brownian motion. If we assume that the risky asset earns a hybrid return, this generalized Merton's model may be solved by a hybrid optimal control.

Now assume we have a financial market which is described by a basic hybrid stock model in which the bond price P_t^0 and the stock price P_t^1 are determined, respectively, by

$$\begin{cases} dP_t^0 = bP_t^0 dt \\ dP_t^1 = eP_t^1 dt + \sigma P_t^1 dB_t + \gamma P_t^1 dC_t, \end{cases}$$

where b is the riskless rate of return, e is the stock drift, σ is a random stock diffusion, γ is a fuzzy stock diffusion, B_t is a standard Brownian motion, and C_t is a standard Liu process.

Let X_t be the wealth of an investor at time t. The investor allocates a fraction u_t of the wealth in a risky asset and the remainder in a sure asset. We denote $X_t^{u,\rho}$ the value of the wealth process when the continuous investment strategy u_t is applied and the consumption rate is $\rho(t)$. Define $Z_t := X_t^{u_t,\rho}$. The dynamics for Z_t is then given by

$$\begin{aligned} dZ_t &= b(1-u_t)Z_t dt + u_t Z_t(edt + \sigma dB_t + \gamma dC_t) - \rho dt \\ &= [bZ_t + (e-b)u_t Z_t - \rho]dt + u_t Z_t(\sigma dB_t + \gamma dC_t). \end{aligned} \tag{12}$$

In Equation (12), the growth rate is reduced by consumption by an amount ρ. Assume that an investor is interested in maximizing the expected utility over a finite time horizon. Then the portfolio selection model is given by

$$\begin{cases} V(t,x) \equiv \sup\limits_{0\le u_t \le 1} E\Big[\displaystyle\int_0^T e^{-\beta t}\frac{\rho(t)^\lambda}{\lambda}dt\Big] \\ \text{subject to} \\ dZ_t = [bZ_t + (e-b)u_t Z_t - \rho]dt + u_t Z_t(\sigma dB_t + \gamma dC_t), \end{cases}$$

where β and λ are given constants such that $\beta > 0$ and $0 < \lambda < 1$. It follows from the equation of optimality (7) that

$$\begin{aligned} -V_t &= \max_{\rho,u}\{e^{-\beta t}\frac{\rho^\lambda}{\lambda} + V_z[bz + (e-b)zu - \rho] + \frac{1}{2}V_{zz}\sigma^2 z^2 u^2\} \\ &= \max_{\rho,u} L(\rho,u), \end{aligned} \tag{13}$$

where $L(\rho,u)$ represents the term enclosed by the braces. The optimal (ρ,u) satisfies

$$\begin{aligned} \tfrac{\partial L(\rho,u)}{\partial \rho} &= e^{-\beta t}\rho^{\lambda-1} - V_z = 0 &\Rightarrow\quad& \rho = [e^{\beta t}V_z]^{\frac{1}{\lambda-1}} \\ \tfrac{\partial L(\rho,u)}{\partial u} &= V_z(e-b)z + \sigma^2 z^2 V_{zz}u = 0 &\Rightarrow\quad& u = \tfrac{(b-e)V_z}{\sigma^2 z V_{zz}}. \end{aligned}$$

Substituting the preceding results into Equation (13), we obtain

$$-V_t = e^{-\beta t}\frac{[e^{\beta t V_z}]^{\frac{\lambda}{\lambda-1}}}{\lambda} + V_z bz - V_z[e^{\beta t V_z}]^{\frac{1}{\lambda-1}} - \frac{V_z^2(b-e)^2}{\sigma^2 V_{zz}}.$$

Multiplying this equation by $e^{\beta t}$ we get

$$-V_t e^{\beta t} = \frac{[e^{\beta t V_z}]^{\frac{\lambda}{\lambda-1}}}{\lambda} + V_z bze^{\beta t} - V_z e^{\beta t}[e^{\beta t V_z}]^{\frac{1}{\lambda-1}} - \frac{V_z^2(b-e)^2}{\sigma^2 V_{zz}}e^{\beta t}. \tag{14}$$

We conjucture that the solution to the proceding nonlinear second-order-differential equation is of the form $V(t,z) = kz^\lambda e^{-\beta t}$. Then $V_t = -k\beta z^\lambda e^{-\beta t}$, $V_z = k\lambda z^{\lambda-1}e^{-\beta t}$ and $V_{zz} = k\lambda(\lambda-1)z^{\lambda-2}e^{-\beta t}$. Sustituting these identities into Equation (14) we get

$$-k\beta z^\lambda = \frac{[k\lambda]^{\frac{\lambda}{\lambda-1}}}{\lambda}z^\lambda + kb\lambda z^\lambda - [k\lambda]^{\frac{\lambda}{\lambda-1}}z^\lambda - \frac{(b-e)^2 k\lambda}{2\sigma^2(\lambda-1)}z^\lambda.$$

Dividing both sides by z^λ, and re-arranging we find

$$[k\lambda]^{\frac{1}{\lambda-1}} = \frac{\beta + b\lambda - \frac{(b-e)^2 k\lambda}{2\sigma^2(\lambda-1)}}{\lambda - 1}.$$

So we conclude that

$$k\lambda = \left\{ \frac{\beta + b\lambda - \frac{(b-e)^2 k\lambda}{2\sigma^2(\lambda-1)}}{\lambda - 1} \right\}^{\lambda-1}. \tag{15}$$

Using Equation (15), we find that the optimal consumption rate and the optimal fraction of investment on risky asset is found, respectively, by

$$\rho(z) = z[k\lambda]^{\frac{1}{\lambda-1}} \quad \text{and} \quad u(z) = \frac{(b-e)}{(\lambda-1)\sigma^2}.$$

We remark that the optimal consumption rate calls for the investor to consume a constant fraction of wealth at each moment, and the optimal fraction of investment on risky asset is independent of total wealth. These conclusions are similar to that in the case of randomness [4] only or fuzziness only [17].

6 Conclusion and Recommendation

Based on the concept of hybrid process, we studied a hybrid optimal control problem: optimizing the expected value of an objective function subject to a hybrid differential equation. By using the Bellman's Principle of Optimality in dynamic programming, we presented the principle of optimality and a fundamental result called equation of optimality for hybrid optimal control. As an application of the equation of optimality, we solved a portfolio selection model. As a line for further research the problem of hybrid optimal control with jumps can be studied. A hybrid process with jumps is a more realistic description of phenomena such as stock price process. The price process of a stock may experience random jumps due to sudden shift in policy by a central bank, war or other natural disasters.

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