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Geometric Average Asian Option Pricing for Uncertain Financial Market

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Abstract

Asian options are among the most popular path-dependent options in financial market whose payoffs depend on the average value of underlying asset throughout the life of the option. There are two types of Asian options: geometric average Asian option and arithmetic average Asian option. In this paper, the geometric Asian option pricing problem is investigated under the assumption that the underlying stock price is assumed following an uncertain differential equation, and the geometric average Asian option pricing formulae are derived under this assumption.

Keywords: uncertainty theory, financial market, uncertain differential equation, Asian option

1 Introduction

Asian options are path dependent options that depend on the average value of the underlying asset price over a certain period. Asian options can be classified into two types: one is geometric average Asian option, another is arithmetic average Asian option. As a risk management instrument, Asian options have been widely accepted in financial practice.

Asian options pricing has been investigated by many scholars. The previous options pricing methods are all in the framework of Black-Scholes model. Although Black-Scholes model is used widely in financial theory and practice, it was still challenged by many scholars. Many empirical investigations showed that the distribution of underlying asset has a higher peak and heavier tails than the normal probability distribution which is assumed in Black-Scholes model. This puzzle showed that Black-Scholes model is inconsistent with the empirical phenomena in financial market.

In real financial practice, with the cognitive resources limitations, many investors do not use the databases of extremely large size to infer the parameter estimates, but they usually give belief degrees about financial events which they based on to make their decisions. We argue that investors' belief degrees play an important role in decision making for financial practice.

Uncertainty theory was founded by Liu [5] in 2007, and it has become a branch of axiomatic mathematics for modeling belief degrees. As a branch of axiomatic mathematics to deal with belief degrees, uncertainty theory will play an important role in financial theory and practice. Liu [7] initiated the pioneer work of uncertain finance in 2009. Afterwards, many researchers devoted themselves to study of financial problems by using uncertainty theory. For example, Chen [2] investigated American option pricing problem and derived the pricing formulae for Liu's uncertain stock model. Chen and Gao [3] introduced an uncertain term structure model of interest rate. Besides, based on uncertainty theory, Chen, Liu and Ralescu [4] proposed an uncertain stock model with periodic dividends and investigated some options pricing for this type of model. Peng and Yao [11] proposed an uncertain stock model with mean-reverting process, and some option pricing formulae were investigated on this type of stock model. Yao [12] gave the no-arbitrage determinant theorems on mean-reverting stock model in uncertain market.

In this paper, the option pricing problem of geometric average Asian option is investigated under Liu's uncertain stock model, and the geometric average Asian option pricing formulae are derived under this model.

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2 Preliminary

The following is some useful definitions and theorems as needed.

Definition 2.1 ([6]) *Suppose C_t is a canonical Liu process, and f and g are two functions. Then*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{2.1}$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (2.1) identically in t.

Definition 2.2 ([14]) *Let α be a number with $0 < \alpha < 1$. An uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{2.2}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$dX_t^\alpha = f(t, X_t^\alpha)dt + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt \tag{2.3}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{2.4}$$

Theorem 2.1 ([14]) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \tag{2.5}$$

respectively. Then the solution X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha. \tag{2.6}$$

Theorem 2.2 ([13]) *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \tag{2.7}$$

respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the time integral $\int_0^s J(X_t)dt$ has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(X_t^\alpha)dt. \tag{2.8}$$

3 Geometric Average Asian Option Pricing

Different from Black-Scholes setting, Liu [7] supposed that the stock price follows an uncertain differential equation and presented an uncertain stock model in which the bond price X_t and the stock price S_t are determined by

$$\begin{cases} dX_t = rX_t dt \\ dS_t = \mu S_t dt + \sigma S_t dC_t \end{cases} \tag{3.1}$$

where X_t is the bond price, S_t is the stock price, r is the riskless interest rate, μ is the log-drift, σ is the log-diffusion, and C_t is a canonical Liu process.

Assume an Asian option has a strike price K and an expiration time T, the payoff of geometric average Asian call option is

$$\left[\exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right) - K\right]^+, \tag{3.2}$$

and the payoff of geometric average Asian put option is

$$\left[K - \exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right)\right]^+ \tag{3.3}$$

where S_t denotes the underlying asset price at time t. Considering the time value of money resulted from the bond, the option price should be the expected present value of the payoff.

Definition 3.1 *Assume an Asian option has a strike price K and an expiration time T. Then the geometric average Asian call option price is*

$$f_c = \exp(-rT)E\left[\exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right) - K\right]^+, \tag{3.4}$$

and the geometric average Asian put option price is

$$f_p = \exp(-rT)E\left[K - \exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right)\right]^+. \tag{3.5}$$

Theorem 3.1 *Assume an Asian option for the stock model (3.1) has a strike price K and an expiration time T. Then the geometric average Asian call option price is*

$$f_c = \exp(-rT)\int_0^1 \left(S_0 \exp\left(\frac{\mu T}{2} + \frac{\sigma T\sqrt{3}}{2\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ d\alpha. \tag{3.6}$$

Proof: Since $J(x) = \ln(x)$ is a strictly increasing function, and $dS_t = \mu S_t dt + \sigma S_t dC_t$ has an α-path

$$S_t^\alpha = S_0 \exp\left(\mu t + \sigma\Phi^{-1}(\alpha)t\right) \tag{3.7}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, it follows from Theorem 2.2 that the time integral $\int_0^T \ln(S_t)dt$ has an inverse uncertainty distribution

$$\begin{aligned} \Phi_T^{-1}(\alpha) &= \int_0^T \ln(S_t^\alpha)dt \\ &= T\ln S_0 + \frac{\mu T^2}{2} + \frac{\sigma T^2\sqrt{3}}{2\pi}\ln\frac{\alpha}{1-\alpha}. \end{aligned} \tag{3.8}$$

Since $\left[\exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right) - K\right]^+$ is an increasing function with respect to $\int_0^T \ln(S_t)dt$, it has an inverse uncertainty distribution

$$\begin{aligned} \Psi_T^{-1}(\alpha) &= \left[\exp\left(\frac{1}{T}\Phi_T^{-1}(\alpha)\right) - K\right]^+ \\ &= \left[S_0 \exp\left(\frac{\mu T}{2} + \frac{\sigma T\sqrt{3}}{2\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right]^+. \end{aligned} \tag{3.9}$$

It follows from Definition 3.1 that the geometric average Asian call option price formula is just (3.6).

Theorem 3.2 *Assume an Asian option for the stock model (3.1) has a strike price K and an expiration time T. Then the geometric average Asian put option price is*

$$f_p = \exp(-rT)\int_0^1 \left(K - S_0 \exp\left(\frac{\mu T}{2} + \frac{\sigma T\sqrt{3}}{2\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha. \tag{3.10}$$

Proof: Since $\left[K - \exp\left(\frac{1}{T}\int_0^T \ln(S_t)dt\right)\right]^+$ is a decreasing function with respect to $\int_0^T \ln(S_t)dt$, it is easy to verify that it has an inverse uncertainty distribution

$$\Psi_T^{-1}(\alpha) = \left[K - S_0 \exp\left(\frac{\mu T}{2} + \frac{\sigma T\sqrt{3}}{2\pi} \ln \frac{1-\alpha}{\alpha}\right)\right]^+. \tag{3.11}$$

It follows from Definition 3.1 that the geometric average Asian put option price is

$$f_p = \exp(-rT) \int_0^1 \left(K - S_0 \exp\left(\frac{\mu T}{2} + \frac{\sigma T\sqrt{3}}{2\pi} \ln \frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha. \tag{3.12}$$

The geometric average Asian put option price formula is verified.

4 Conclusion

In this paper, based on the uncertainty theory, the problem of geometric average Asian option pricing was investigated for the Liu's uncertain stock model. By the means of uncertain calculus method, the geometric average Asian option price formulae were derived for uncertain financial market.

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